PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 9, September 2013, Pages 3223–3230
S 0002-9939(2013)11584-4
Article electronically published on June 4, 2013

ON DIFFERENT EXTREMAL BASES FOR $\mathbb{C}$-CONVEX DOMAINS

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(Communicated by Franc Forstneric)

Abstract. We provide a counterexample (a strongly convex domain in $\mathbb{C}^3$) to a statement about certain extremal bases in the papers of J. D. McNeal in 1992 and 1994 which was used to prove estimates for the Bergman kernel and invariant metrics. We also prove that those estimates in fact remain correct for the same bases (other authors had introduced different bases to fix this).

1. Introduction

In order to estimate the Bergman kernel (on the diagonal) and the Carathéodory, Bergman, and Kobayashi metrics on convex (or linearly convex) domains, special coordinates near the boundary were introduced by J.-H. Chen [Che 89] in his Ph.D. dissertation and by J. D. McNeal [McN 92, McN 94]. Many further studies were based on the orthonormal basis introduced in these papers; see e.g. [MS 94, MS 97, Gau 97, McN 01]. We will call this basis a *maximal basis*; a detailed construction, which also justifies the name, will be described later. On the other hand, for the same purpose a *minimal basis* was introduced, for example, in the papers [Hef 02, Con 02, NP 03, Hef 04, DF 06, NPZ 09]. For a general notion of extremal basis see [CD 08]. Looking more carefully at the work based on the maximal basis, it can be seen that from the very beginning a crucial property of that basis is used but never proved. In this note we will present an example showing that exactly this property unfortunately does not hold in general for the maximal basis, but it does hold for the minimal basis. This property may be phrased by saying that certain vectors connected with this basis are orthogonal to certain complex tangent planes; details will be given later. Starting with this observation it should be asked whether the estimates for invariant metrics given with the help of the maximal basis still hold. In this note we show that the answer is positive.

A description of the two extremal bases for a domain $D \subset \mathbb{C}^n$ containing no complex lines. (a) *Maximal basis.* Fix a point $q \in D$. Let $d_D(q;a) := \sup\{r > 0 : q + r\mathbb{D}a \subset D\}$ be the boundary distance of q in the direction of a ($\mathbb{D}$ is the open unit disc in $\mathbb{C}$). We shall assume that $d_D(q;\cdot)$ is a continuous function[1] (it is always lower semicontinuous). Choose a boundary point $p_1 \in \partial D$ such that $m_1 := \|p_1 - q\| = d_D(q)$, where $d_D(q)$ denotes the Euclidean boundary

Received by the editors September 28, 2011 and, in revised form, November 29, 2011 and December 4, 2011.
2010 *Mathematics Subject Classification.* Primary 32F17.
Key words and phrases. Extremal basis, $\mathbb{C}$-convex domain.

[1] This holds if, for example, ∂D is Lipschitz.

distance of the point q. Put $a_1 := (p_1 - q)/\|p_1 - q\|$. Note that the point p_1 is not in general uniquely determined. Denote by H_1 the affine complex hyperplane through q which is orthogonal to the vector a_1, i.e. $H_1 = q + \operatorname{span}\{a_1\}^\perp$. Put $D_2 := D \cap H_1$. Choose a unit vector $a_2 \in \operatorname{span}\{a_1\}^\perp$ and a boundary point $p_2 \in \partial D \cap H_1$ such that $m_2 := \operatorname{dist}(q, a_2, \partial D) = \sup\{d_D(q; a)\}$, where the supremum is taken over all unit vectors a in $\operatorname{span}\{a_1\}^\perp$, and $p_2 = q + m_2 a_2$. In the next step put $H_2 := q + \operatorname{span}\{a_1, a_2\}^\perp$; H_2 is the affine $(n-2)$-dimensional plane through q orthogonal to $\operatorname{span}\{a_1, a_2\}$. Define $D_3 := H_2 \cap D$ and continue this procedure, which finally leads to an orthonormal basis $a_1, \dots, a_n$, which is called a *maximal basis* of D at q, to a sequence of positive numbers $m_2 \geq \cdots \geq m_n$, and to boundary points $p_1, \dots, p_n$ with $p_j = q + m_j a_j$ for all j's. Obviously, in this general context this basis depends on q and is, in general, not uniquely determined.

(b) *Minimal basis.* Let q, $e_1 := a_1$, $s_1 := m_1$, $\widetilde{p}_1 := p_1$, where a_1, p_1, and m_1 are taken from the former construction. Let H_1 also be as above. Define a new boundary point $\widetilde{p}_2 = q + s_2 e_2$, where $e_2 \in \operatorname{span}\{e_1\}^\perp$ is a unit vector and $s_2 = \inf\{d_D(q; a)\}$, where the infimum is taken over all unit vectors a in $\operatorname{span}\{e_1\}^\perp$. Note that in this construction, opposite to the one above, $\widetilde{p}_2$ is chosen to be a nearest boundary point of $\partial D \cap H_1$ to q. Then put $H_2 := q + \operatorname{span}\{e_1, e_2\}^\perp$; H_2 is the affine $(n-2)$-dimensional plane through q orthogonal to $\operatorname{span}\{e_1, e_2\}$. Define $D_3 := H_2 \cap D$ and continue this procedure for H_3 by always taking the nearest boundary point. This finally leads to an orthonormal basis $e_1, \dots, e_n$, which is called a *minimal basis* of D at q, to a sequence of positive numbers $s_1 \leq s_2 \leq \cdots \leq s_n$, and to boundary points $\widetilde{p}_1, \dots, \widetilde{p}_n$ of D with $\widetilde{p}_j = q + s_j e_j$, $1 \leq j \leq n$. As above, this basis depends on q and is, in general, not uniquely determined.

Assume now that, in addition, D is convex and $\mathcal{C}^\infty$-smooth near a boundary point p_1 (of finite type). Let r be its boundary defining function. Then the property indicated in the introduction can be described as follows: Fix $q \in D$ on the inner normal at p_1, sufficiently near to p_1, and take the coordinate system given by the maximal basis at q; i.e. take $q = 0$ and write any point z of $\mathbb{C}^n$ as $z = \sum_{j=1}^n w_j a_j$. Then it is claimed (see, for example, [Che 89, Proposition 2.2 (i)] and [McN 92, Proposition 3.1 (i)]) that

$$\frac{\partial r(p_k)}{\partial w_j} = 0, \quad j = k+1, \dots, n. \tag{*}$$

In the original coordinate system this property reads as

$$\sum_{s=1}^n \frac{\partial r(p_k)}{\partial z_s} a_{j,s} = 0, \quad j = k+1, \dots, n. \tag{1}$$

Therefore, an equivalent form to state (*) is to say that the vectors a_j, $j = k+1, \dots, n$, belong to the complex tangent space $T^{\mathbb{C}}_{p_k}(\partial D)$ or that $T^{\mathbb{C}}_{p_k}(\partial D) \cap \operatorname{span}\{a_1, \dots, a_k\}^\perp = \operatorname{span}\{a_{k+1}, \dots, a_n\}$. We should point out that exactly the property (*) is the basis of the arguments in those papers dealing with maximal bases (minimal bases have this crucial property). But, as the following example will show, (*) is not necessarily true near the boundary of a domain in $\mathbb{C}^3$. Nevertheless, in section 3 it will be proved that the estimates obtained in terms of the maximal basis still hold.

2. An example

We give an example of a smooth strictly convex domain containing points q arbitrarily close to the boundary such that condition (*) fails to hold for the maximal basis at any such point q.

Let β_1 and β_2 be real numbers with $0 < \beta_1 < \beta_2 < 1$. Define

$$D := \{(z, z_3) \in \mathbb{C}^2 \times \mathbb{C} : \rho(z) + |z_3|^2 < 1\},$$

where $\rho(z) = x_1^2 + \beta_1 y_1^2 + x_2^2 + \beta_2 y_2^2$. Note that D is a strictly convex domain with real-analytic boundary. Fix $q = (0, 0, \delta)$, $0 < \delta < 1$. Then following the construction of the maximal basis of D at q leads to $m_1 = 1 - \delta$ and $a_1 = p_1 = (0, 0, 1)$. At the next step the construction gives the domain

$$D_\delta := \{z \in \mathbb{C}^2 : \rho(z) < 1 - \delta^2\}.$$

Note that D_δ is up to a dilatation D_0. So it suffices to study D_0.

If we have a maximal basis, write $a_2 = (\alpha', \alpha'', 0)$ where $\alpha', \alpha'' \in \mathbb{C}$. Then $\operatorname{span}\{a_1, a_2\}^\perp$ is spanned by $(-\overline{\alpha''}, \overline{\alpha'}, 0)$. Put

$$\mathcal{T} := \{b \in \mathbb{C}^2 : -\bar{b}_2 \frac{\partial\rho(b)}{\partial z_1} + \bar{b}_1 \frac{\partial\rho(b)}{\partial z_2} = 0\}.$$

If $p = (b, \delta) \in \mathbb{C}^2 \times \mathbb{C}$ is obtained by the construction and satisfies (*), then $a_2 = \frac{b}{\|b\|}$ and (because of (1) and homogeneity of ρ) $b \in \mathcal{T}$.

Lemma 2.1. $\mathcal{T} = \{b \in \mathbb{C}^2 : b_1 = 0$ *or* $b_2 = 0$ *or* $\operatorname{Im} b_1 = \operatorname{Im} b_2 = 0\}$.

Proof. Simple calculations show that $b \in \mathcal{T}$ if and only if

$$\begin{aligned}(\beta_1 - \beta_2) \operatorname{Im} b_1 \operatorname{Im} b_2 &= 0,\\ (1 - \beta_1) \operatorname{Im} b_1 \operatorname{Re} b_2 &= (1 - \beta_2) \operatorname{Im} b_2 \operatorname{Re} b_1,\end{aligned}$$

from which the statement of the lemma follows. □

The next result shows that the property (*) is, in general, not true for a maximal basis.

Proposition 2.2. *Let $b \in \partial D_0$ be such that*

$$m_2 = d_{D_0}(0; \frac{b}{\|b\|}) = \sup_{a \in \mathbb{C}^2, \|a\|=1} d_{D_0}(0; a).$$

Then $b \notin \mathcal{T}$.

Proof. By homogeneity, it is enough to work with a unit vector b. Observe that $\rho(re^{i\alpha}b) < 1$ for all $\alpha \in \mathbb{R}$ if and only if $r^2 R(b) < 1$, where $R(b) := \max\{\rho(e^{i\alpha}b) : \alpha \in \mathbb{R}\}$. Therefore, $d_{D_0}(0; b) = 1/\sqrt{R(b)}$ will be maximal if and only if $R(b)$ is minimal. Write $b = (e^{i\varphi_1} \cos\Theta, e^{i\varphi_2} \sin\Theta)$, where $0 \le \Theta < \frac{\pi}{2}$ and $0 \le \varphi_1, \varphi_2 \le 2\pi$. According to Lemma 2.1, if $b \in \mathcal{T}$, there are three possibilities for b:

- $\Theta = 0 :$ $\rho(e^{i\alpha}b) = \cos^2(\alpha + \varphi_1) + \beta_1 \sin^2(\alpha + \varphi_1)$.
- $\Theta = \pi/2 :$ $\rho(e^{i\alpha}b) = \cos^2(\alpha + \varphi_2) + \beta_2 \sin^2(\alpha + \varphi_2)$.
- $\varphi_1, \varphi_2 \in \{0, \pi\}$: $\rho(e^{i\alpha}b) = \cos^2\alpha + \sin^2\alpha(\beta_1 \cos^2\Theta + \beta_2 \sin^2\Theta)$.

Hence $R(b) = 1$ in all three cases.

On the other hand, there is a unit vector $b^* \in \mathbb{C}^2$ with $R(b^*) < 1$ which implies that $R(b)$ is never minimal when $b \in \mathcal{T}$, which will prove the proposition. To define b^*, take $\Theta := \pi/4$, $\varphi_1 := 0$, and $\varphi_2 := \pi/2$. Then $2\rho(e^{i\alpha}b^*) = 1+\beta_2+(\beta_1-\beta_2)\sin^2\alpha$. Since $\beta_1 < \beta_2 < 1$, it follows that $R(b^*) = \frac{1+\beta_2}{2} < 1$. □

3. Estimates and localization

Estimates. Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain containing no complex lines; i.e. any non-empty intersection with a complex line is biholomorphic to $\mathbb{D}$ (cf. [APS 04, Hör 94]). Then, for any $z \in D$, the function $1/d(z;\cdot)$ is convex (cf. [NT 10]), and hence $d_D(z;\cdot)$ is a continuous function. Denote by $e_1(z), \dots, e_n(z)$ a minimal basis at z and by $a_1(z), \dots, a_n(z)$ a reordered maximal basis at z, which means that the new $a_1(z)$ is the old one, but then $a_2(z) = a_n$, $a_3(z) = a_{n-1}$, etc. Let $s_1(z) \le \dots \le s_n(z)$ and $m_1(z) \le \dots \le m_n(z)$ be the respective numbers (recall that $s_1(z) = m_1(z) = d_D(z)$). Set $s_D(z) := \prod_{j=1}^n s_j(z)$ and $m_D(z) := \prod_{j=1}^n m_j(z)$. Moreover, denote by $K_D(z)$ and $F_D(z;X)$ the Bergman kernel and the Bergman, the Carathéodory, or the Kobayashi metric of D, respectively. For $X \in \mathbb{C}^n$, set

$$E_D(z;X) := \sum_{j=1}^n \frac{|\langle X, e_j(z)\rangle|}{s_j(z)}, \quad A_D(z;X) := \sum_{j=1}^n \frac{|\langle X, a_j(z)\rangle|}{m_j(z)}.$$

(Observe that these definitions depend a priori on the choice of the bases.)

We shall write $f(z) \lesssim g(z)$ if $f(z) \le cg(z)$ for some constant $c > 0$ depending only on n; $f(z) \sim g(z)$ means that $f(z) \lesssim g(z) \lesssim f(z)$. By [NPZ 09], we know that

$$K_D(z) \sim 1/s_D^2(z), \quad F_D(z;X) \sim E_D(z;X) \sim 1/d_D(z;X)$$

(for weaker versions of these results, see [NP 03, Blu 05, Lie 05]). For short, sometimes we shall omit the arguments z and X. It follows by [NPZ 09, Lemma 15] that

$$K_D \lesssim 1/m_D^2, \quad F_D \lesssim A_D.$$

In particular,

$$1/d_D(z;X) \sim E_D(z;X) \lesssim A_D(z;X).$$

The main consequence of the (wrong) property (*) for maximal bases of a smooth convex bounded domain of finite type is the estimate

$$A_D(z;X) \sim_D 1/d_D(z;X),$$

where the constant in $\sim_D$ depends on D. Using this estimate, it is shown in [Che 89, McN 94, McN 01] that

$$K_D \sim_D 1/m_D^2, \quad F_D \sim_D A_D.$$

The following two propositions imply that these estimates still hold.

The first one is contained in [Hef 04] for the case of a smooth convex bounded domain of finite type. The proof there invokes the estimate $1/d_D(z;X) \sim_D A_D(z;X)$, but, in fact, it uses only the trivial part of this estimate: $1/d_D(z;X) \lesssim_D A_D(z;X)$.

Proposition 3.1. *Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain containing no complex lines. Then $m_j(z) \sim s_j(z)$, $j = 1, \dots, n$, $z \in D$.*

Proof. Fix $z \in D$ and put $m_j = m_j(z)$, $s_j = s_j(z)$. First, we shall prove that $m_j \lesssim s_j$. Since $E_D \lesssim A_D$, it is enough to show that if $E_D \le cA_D$, then $m_j \le c' s_j$, where $c' = n!c$.

Expanding the determinant of the matrix of the unitary transformation between the bases, it follows that $\prod_{j=1}^n |\langle a_j, e_{\sigma(j)}\rangle| \ge 1/n!$ for some permutation σ of $\{1, \dots, n\}$. In particular, $|\langle a_j, e_{\sigma(j)}\rangle| \ge 1/n!$. Then $E_D(z;a_j) \le cA_D(z;a_j)$ implies that $m_j \le c' s_{\sigma(j)}$.

Assume now that $c' s_k < m_k$ for some k. Then

$$c' s_k < m_k \leq m_j \leq c' s_{\sigma(j)}, \quad j \geq k.$$

This shows that $\sigma(j) > k$ for any $j \geq k$, which is a contradiction, since σ is a permutation.

The above arguments show that $\widetilde{s}_j \sim s_j$, where $\widetilde{s}_j$ is the respective number for another minimal basis at z. So we may assume that $e_1 = a_1$. We know that $m_1 = s_1$. It remains to prove that $m_k \gtrsim s_k$ for $k \geq 2$. Choose a unit vector a'_k in $\operatorname{span}(e_k, \dots, e_n)$ orthogonal to $a_{k+1}, \dots, a_n$ ($a'_n = e_n$ if $k = n$). Then a'_k is also orthogonal to $a_1 = e_1$. Hence $m_k \geq d_D(z; a'_k)$ (by construction of a maximal basis). On the other hand, since a'_k is orthogonal to $e_1, \dots, e_{k-1}$, then

$$\frac{1}{d_D(z; a'_k)} \sim E_D(z; a'_k) = \sum_{j=k}^{n} \frac{|\langle a'_k, e_j \rangle|}{s_j} \lesssim \frac{1}{s_k}.$$

So $m_k \geq d_D(z; a'_k) \gtrsim s_k$. □

Proposition 3.2. *Let D be as in Proposition* 3.1. *Then $A_D \sim E_D$.*

Proof. Using the inequality $E_D \lesssim A_D$ and Proposition 3.1, it is enough to show that for any k,

$$\frac{|\langle X, a_k \rangle|}{s_k} \lesssim E_D(z; X).$$

Set $b_{jk} = \langle a_j, e_k \rangle$. Since

$$\frac{1}{s_j} \sim \frac{1}{d_D(z; a_j)} \sim E_D(z; a_j) \geq \frac{|b_{jk}|}{s_k},$$

it follows that $|b_{jk}| \lesssim s_k / s_j$. The unitary matrix $B = (b_{jk})$ transforms the basis $e_1, \dots, e_n$ to the basis $a_1, \dots, a_n$. For the inverse matrix $C = (c_{jk})$ we have

$$|c_{jk}| \leq \sum_{\sigma} |b_{1\sigma(1)} \dots b_{k-1,\sigma(k-1)} b_{k+1,\sigma(k+1)} \dots b_{n,\sigma(n)}|$$

$$\lesssim \sum_{\sigma} \frac{s_{\sigma(1)}}{s_1} \dots \frac{s_{\sigma(k-1)}}{s_{k-1}} \frac{s_{\sigma(k+1)}}{s_{k+1}} \dots \frac{s_{\sigma(n)}}{s_n} = \sum_{\sigma} \frac{s_k}{s_j} = (n-1)! \frac{s_k}{s_j},$$

where σ runs over all bijections from $\{1, \dots, k-1, k+1, \dots, n\}$ to $\{1, \dots, j-1, j+1, \dots, n\}$.

It follows that

$$\frac{|\langle X, a_k \rangle|}{s_k} \leq \sum_{j=1}^{n} |\langle X, e_j \rangle| \frac{|b_{kj}|}{s_k} = \sum_{j=1}^{n} |\langle X, e_j \rangle| \frac{|\overline{c}_{jk}|}{s_k} \lesssim E_D.$$

□

Remark. Replace the construction of a maximal basis by the following: choose "minimal" discs on steps $1, \dots, k$ and "maximal" discs on steps $k+1, \dots, n-1$ (the n-th choice is unique); $k = n-1$ provides a minimal basis, $k = 1$ a maximal basis, and $k = 0$ a basis with no "minimal" discs. Note that Propositions 3.1 and 3.2 remain true with A_D expressed in the new basis. (This construction has an obvious real analog.)

Localization. Let a be a boundary point of a bounded domain $D \subset \mathbb{C}^n$ (such that $d_D(q;\cdot)$ is a continuous function for any $q \in D$). It is easy to see that for any neighborhood U of a one has that $s_D \sim_* s_{D\cap U}$ and that $E_D \sim_* E_{D\cap U}$ near a, where the constant in $\sim_*$ depends on D and U (the same holds for m_D and A_D). Assume that D is $\mathcal{C}^2$-smooth and (weakly) linearly convex near a (cf. [APS 04, Hör 94] for this and other notions of convexity). Then Proposition 3.3 below and the localization principle for the Kobayashi metric κ_D (cf. [JP 93]) imply that $\kappa_D \sim_D E_D$ near a (the constant in $\sim_D$ depends on D). If, in addition, D is pseudoconvex, then the same principle for K_D and the Bergman metric b_D (cf. [JP 93]) implies that $K_D \sim_D 1/s_D^2$ and $b_D \sim_D E_D$ near a. Assume that a is a $\mathcal{C}^\infty$-smooth finite type point (but D not necessarily bounded). Then a is a local holomorphic peak point (see [DF 03]), and strong localization principles (cf. [Nik 02]) imply that $\kappa_D \sim E_D (\sim E_{D\cap U})$ and (if D is pseudoconvex) $K_D \sim 1/s_D^2 (\sim 1/s_{D\cap U}^2)$, $b_D \sim E_D$ near a.

Proposition 3.3. *Let a be a $\mathcal{C}^k$-smooth boundary point $(2 \le k \le \infty)$ of a domain $D \subset \mathbb{C}^n$ with the following property: for any $b \in \partial D$ near a there is a neighborhood U_b such that $D \cap U_b \cap T_b^{\mathbb{C}}(\partial D) = \varnothing$. Then there is a $\mathcal{C}^k$-smooth $\mathbb{C}$-convex domain $G \subset D$ and a neighborhood U of a such that $D \cap U = G \cap U$.*

Proof. We may assume that $a = 0$. Denote by $H_f(z;X)$ the (real) Hessian of a $\mathcal{C}^2$-smooth function f. Set $B_s := \mathbb{B}_n(0,s)$ $(s > 0)$ and

$$r(z) := \begin{cases} -d_D(z), & z \in D, \\ d_D(z), & z \notin D. \end{cases}$$

It follows by the differential inequality for r^2 in the proof of [APS 04, Proposition 2.5.18 (ii)⇒(iii)] that there is an $\varepsilon > 0$ such that r is a $\mathcal{C}^k$-smooth defining function of D in $B_{3\varepsilon}$ and $H_r(z;X) \ge 0$ if $\langle \partial r(z), \overline{X}\rangle = 0$ and $z \in D \cap B_{2\varepsilon}$. Then the proof of [DF 77, Lemma 1] implies that there is a $c > 0$ such that $H_r(z;X) \ge -c|X| \cdot |\langle \partial r(z), \overline{X}\rangle|$, $z \in D \cap B_{2\varepsilon}$. We may assume that $2\varepsilon c \le 1$ and $D \cap B_\varepsilon$ is connected. Now choose a smooth function χ such that $\chi(x) = 0$ if $x \le \varepsilon^2$ and $\chi'(x), \chi''(x) > 0$ if $x > \varepsilon^2$. Set $\theta(z) = \chi(|z|^2)$. We may find a $C > 0$ such that

$$B_{2\varepsilon} \Supset G' := \{z \in B_{2\varepsilon} : 0 > \rho(z) = r(z) + C\theta(z)\} \subset D.$$

Further, the inequalities $2c\varepsilon \le 1$ and $|\langle \partial\theta(z), \overline{X}\rangle| \le \chi'(|z|^2)|z| \cdot |X|$ give $\chi'(|z|^2)|X| > c|\langle \partial\theta(z), \overline{X}\rangle|$ if $z \in B_{2\varepsilon} \setminus \overline{B_\varepsilon}$ and $X \ne 0$. This together with

$$H_r(z;X) \ge -c|X| \cdot |\langle \partial r(z), \overline{X}\rangle|, \quad z \in G',$$
$$H_\rho(z;X) = H_r(z;X) + 2C\chi''(|z|^2)\mathrm{Re}^2\langle z, X\rangle + C\chi'(|z|^2)|X|^2,$$

and the triangle inequality shows that

$$H_\rho(z;X) \ge -c|X| \cdot |\langle \partial\rho(z), \overline{X}\rangle|, \quad z \in \overline{G'}$$

(since $c|z| \le c2\varepsilon \le 1$). Moreover, the last inequality is strict if $z \in \overline{G'} \setminus \overline{B_\varepsilon}$ and $X \ne 0$. This implies that $\partial\rho \ne 0$ on $\partial G' \setminus \overline{B_\varepsilon}$ (otherwise, ρ will attain local minima at some point of this set, which is impossible). So $\partial\rho \ne 0$ on $\partial G'$.

Let G be the connected component of G' containing $D \cap B_\varepsilon$. Then [APS 04, Theorem 2.5.18] (see also [Hör 94, Proposition 4.6.4])[2] implies that G is a $\mathcal{C}^k$-smooth $\mathbb{C}$-convex domain. □

[2]A bounded domain in $\mathbb{C}^n$ $(n > 1)$ with $\mathcal{C}^2$-smooth boundary is $\mathbb{C}$-convex if and only if the (real) Hessian of its defining function is positive semidefinite on the complex tangent space.

Acknowledgments

This note was written during the stay of the third author at the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences, supported by a CNRS-BAS project (September 2009); the stay of the first author at the University of Oldenburg, supported by a DAAD grant (November 2009–January 2010); and the stay of the first and second authors at the Erwin Schrödinger Institute for Mathematical Physics in Vienna (November 2009, programme "The $\bar\partial$-Neumann problem: Analysis, geometry and potential theory"). This research was partially supported by the DFG grant 436POL113/103/0-2. The authors thank H. P. Boas for his careful reading and comments on a previous version of the note. They would also like to thank the referee for carefully examining the paper and for providing them with a number of important comments and remarks.

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