PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 9, September 2013, Pages 3305–3311
S 0002-9939(2013)11600-X
Article electronically published on June 10, 2013

SPECTRAL GEOMETRY OF COSMOLOGICAL AND EVENT HORIZONS FOR KERR-NEWMAN-DE SITTER METRICS

MARTIN ENGMAN AND GERARDO A. SANTANA

(Communicated by Sergei K. Suslov)

For Becky and Mom

Abstract. We study the Laplace spectra of the intrinsic instantaneous metrics on the event and cosmological horizons of a Kerr-Newman-de Sitter space-time and prove that the spectral data from these horizons uniquely determine the metric among all such space-times. This is accomplished by exhibiting formulae relating the parameters of the space-time metric to the traces of invariant and equivariant Green's operators associated with these Laplacians. In particular, an interesting explicit formula for the cosmological constant is found.

1. Introduction

The Kerr-Newman-de Sitter metric exhibits four horizons. Three of these have the following physical interpretations: a Cauchy (inner) horizon, a black-hole event horizon, and a cosmological horizon. In this paper, we study the event and cosmological horizons.

The space-time metric induces on each of these horizons an S^1-invariant Riemannian metric which we call *intrinsic instantaneous metrics*. Each of these metrics' Green's operators yield S^1-invariant and k-equivariant trace formulae. In [4] we proved, for the case of a single horizon in the Kerr metric, that the two trace formulae uniquely determine the two parameters of this space-time. In the present paper there are four parameters which must be determined: the total mass, angular momentum per unit mass, charge, and cosmological constant. The two horizons each provide two distinct types of trace formulae constituting a non-linear system of four equations in the parameters which is then shown to have a unique solution. Each of the parameters is given semi-explicitly in terms of the spectrum, but in the case of the cosmological constant we are, in fact, able to find an explicit formula.

2. Background

The Kerr-Newman-de Sitter family of metrics are stationary, axisymmetric solutions of the source-free (vacuum) Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$$

Received by the editors October 5, 2011 and, in revised form, December 10, 2011.
2010 *Mathematics Subject Classification.* Primary 58J50; Secondary 83C15, 83C57.
The second author thanks María del Rio for her support during the writing of this paper. This work was partially supported by the NSF grants Model Institutes for Excellence and AGMUS Institute of Mathematics at UMET.

with positive cosmological constant Λ. These solutions produce models of space-times depending on Λ as well as three other parameters which determine the mass, angular momentum, and charge. As indicated by the name, these metrics generalize some of the previously discovered solutions including the Kerr solution which, itself, is a generalization of the Schwarzschild solution.

The cosmological constant term in the field equations was proposed by Einstein in order to allow the possibility of a static spatially closed universe on the cosmological scale. But Hubble's observations in 1929 suggested that the universe was expanding (i.e. not static), and so the cosmological constant was removed. It made a come-back in 1968 when (we believe it was) Zel'dovich (see [2] or [20]) realized it was equivalent to the vacuum energy density,

$$\rho_{vac} = \frac{\Lambda}{8\pi}, \tag{2.1}$$

in suitable units. More recently, the discovery of cosmic acceleration in 1998 ([13] and [14]) and the concept of dark energy have suggested the re-introduction of a cosmological constant into the theory.

From a geometric point of view, the most interesting consequence of the presence of a positive Λ is the existence of, in addition to the event horizon, a second *cosmological* horizon. For these models one can think of the horizons as certain, naturally defined, two-dimensional null slices of the space-time which turn out to be Riemannian manifolds, diffeomorphic to S^2, whose rotation invariant metrics are induced from the space-time metric (see [9]). There is little harm in viewing them as two surfaces of revolution (although they may not isometrically embed in euclidean space).

As we will see in the next section, one can define a type of radial coordinate on the space-time, a specific, large value of which corresponds to the cosmological horizon. In astronomical terms, this might represent an "outer" boundary of the observable universe. The event horizon will then be located at another specific, but much smaller, value of this coordinate, which might be interpreted as the boundary of a black hole formed from a gravitationally collapsed star or maybe a super massive black hole.

Our previous work ([5] and [6]) developed two types of trace formulae for each of the invariant and equivariant spectra of surfaces of revolution, and these were found in [4] to uniquely determine the mass and angular momentum parameters of Kerr metrics. As a bonus, thanks to a uniqueness theorem of Robinson ([15]), all non-charged stationary axially symmetric asymptotically flat vacuum space-times with a regular event-horizon are determined by these surface eigenvalues. On the other hand, this spectral data is not sufficient to also specify the charge for Kerr-Newman metrics. In this paper, although to our knowledge there is no similar uniqueness theorem for $\Lambda > 0$, at least we are able to determine all of the parameters due to the presence of twice as many trace formulae.

3. The metric

Explicit expressions for Kerr-Newman-de Sitter metrics can, of course, be found in many references (see, for example, [10]). They are given by

$$ds^2 = -\frac{\Delta_r}{\chi^2\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\Delta_\theta \sin^2\theta}{\chi^2\rho^2}\left[adt - (r^2+a^2)d\phi\right]^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right), \tag{3.1}$$

where $\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2 \cos^2\theta$, $\chi = 1 + \frac{1}{3}\Lambda a^2$, $\Delta_r = (r^2 + a^2)(1 - \frac{1}{3}\Lambda r^2) - 2mr + Q^2$ and $\rho = r^2 + a^2\cos^2\theta$. The parameters (m, a, Q, Λ) (all of which, unless otherwise stated, we assume to be positive) represent, respectively, the total mass, angular momentum per unit mass, the charge, and the cosmological constant.

In general, of course, the equation $\Delta_r = 0$ has four roots. Here we are only interested in two positive real roots, denoted by r_e and r_c, which correspond to the event and cosmological horizons respectively. We assume, of course, that $r_e < r_c$.

Let r_0 denote either r_e or r_c. To obtain the intrinsic instantaneous metrics on these surfaces we pull back the space-time metric (3.1) to the surface defined by $r = r_0$ (so that $dr = 0$) and $dt = 0$ to obtain two-dimensional Riemannian metrics on each of the event and cosmological horizons. Both take the form

$$ds^2 = \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta (r_0^2 + a^2)^2}{\chi^2 \rho^2} \sin^2\theta d\phi^2. \tag{3.2}$$

Following the notation of [16], we define *the scale parameter* by $\eta = \sqrt{r_0^2 + a^2}$ and *the distortion parameter* by $\beta = \frac{a}{\sqrt{r_0^2+a^2}}$. We also define a new parameter:

$$\xi = \frac{\Lambda \frac{a^2}{3}}{1 + \Lambda \frac{a^2}{3}}. \tag{3.3}$$

With the change of variable $x = -\cos\theta$ one finds that the horizon metric is

$$ds_{r_0}^2 = \eta^2 (1 - \xi) \left(\frac{1}{f(x)} dx^2 + f(x) d\phi^2 \right), \tag{3.4}$$

where $(x, \phi) \in (-1, 1) \times [0, 2\pi)$ and

$$f(x) = \frac{1 - \xi(1 - x^2)}{1 - \beta^2(1 - x^2)} (1 - x^2). \tag{3.5}$$

The area of this metric is $A = 4\pi\eta^2(1 - \xi)$ (see [3]). It is well known that the Gauss curvature of such a metric takes the form $K(x) = -f''(x)/(2\eta^2(1 - \xi))$ so that in this case from (3.5) the curvature is

$$K(x) = \frac{1}{\eta^2(1 - \xi)} \left[\frac{\xi}{\beta^2} + \left(1 - \frac{\xi}{\beta^2}\right) \frac{1 - \beta^2(1 + 3x^2)}{(1 - \beta^2(1 - x^2))^3} \right]. \tag{3.6}$$

The special case $\beta^2 = \xi$ gives a constant curvature metric on a horizon but yields only one horizon corresponding to a positive solution of $\Delta_r = 0$.

4. Spectrum of S^1 invariant metrics on S^2

In this section, for the purpose of making the discussion self-contained, we reproduce some of the first author's previous work on the properties of the spectrum of the Laplace operator for S^1-invariant metrics on S^2. This section is similar in content to the summary found in section III of [4], and detailed discussions can be found in [5], [6], [7], and [19].

Let $(x^1, x^2, \cdots, x^n)$ be the coordinate functions on a chart of a Riemannian manifold with metric (g_{ij}). We recall that the *the Laplacian*, in these local coordinates, is given by

$$\Delta_g = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right).$$

Here we study metrics of the form

$$dl^2 = \frac{1}{f(x)}dx^2 + f(x)d\phi^2, \tag{4.1}$$

where $(x, \phi) \in (-1, 1) \times [0, 2\pi)$ and $f(x)$ satisfies $f(-1) = 0 = f(1)$ and $f'(-1) = 2 = -f'(1)$. The coordinates (x, ϕ) are sometimes called *action-angle coordinates*. The Gauss curvature in these coordinates is given by $K(x) = (-1/2)f''(x)$, the area of the metric is 4π, and the canonical (i.e. constant curvature 1) metric is obtained by setting $f(x) = 1 - x^2$.

The Laplacian, in action-angle coordinates, for the metric (4.1) is given by

$$\Delta_{dl^2} = -\frac{\partial}{\partial x}\left(f(x)\frac{\partial}{\partial x}\right) - \frac{1}{f(x)}\frac{\partial^2}{\partial \phi^2}.$$

For any eigenvalue λ of $-\Delta_{dl^2}$ we denote by E_λ and $\dim E_\lambda$ the eigenspace for λ and its multiplicity (degeneracy) respectively. We will use the symbol λ_m to denote the mth <u>distinct</u> eigenvalue, and we adopt the indexing convention $\lambda_0 = 0$. Since S^1 (parametrized here by $0 \le \phi < 2\pi$) acts on (M, g) by isometries, we can separate variables, and because $\dim E_{\lambda_m} \le 2m + 1$ (see [7] for the proof), the orthogonal decomposition of E_{λ_m} has the special form

$$E_{\lambda_m} = \bigoplus_{k=-m}^{k=m} e^{ik\phi} W_k,$$

in which $W_k (= W_{-k})$ is the "eigenspace" (it might contain only 0) of the ordinary differential operator

$$L_k = -\frac{d}{dx}\left(f(x)\frac{d}{dx}\right) + \frac{k^2}{f(x)}$$

with suitable boundary conditions. It should be observed that $\dim W_k \le 1$, a value of zero for this dimension occurring when λ_m is not in the spectrum of L_k.

The discrete set of positive eigenvalues is given by $Spec(dl^2) = \bigcup_{k\in\mathbb{Z}} SpecL_k$, and consequently the non-zero part of the spectrum of $-\Delta$ can be studied via the spectra $SpecL_k = \{0 < \lambda_k^1 < \lambda_k^2 < \cdots < \lambda_k^j < \cdots\} \forall k \in \mathbb{Z}$. The eigenvalues λ_0^j in the case $k = 0$ above (which, in fact, correspond to 1-form eigenvalues in order to avoid the zero eigenvalue) are called the S^1-*invariant eigenvalues* since their eigenfunctions are invariant under the action of the S^1 isometry group. If $k \neq 0$ the eigenvalues are called k *equivariant* or simply *of type* $k \neq 0$. Each L_k has a Green's operator, $\Gamma_k : (H^0(M))^\perp \to L^2(M)$, whose spectrum is $\{1/\lambda_k^j\}_{j=1}^\infty$ and whose trace is defined by

$$\gamma_k(dl^2) \equiv \sum_{j=1}^{\infty} \frac{1}{\lambda_k^j}. \tag{4.2}$$

The formulas of present interest were derived in [5] and [6] and are given by

$$\gamma_0(dl^2) = \frac{1}{2}\int_{-1}^{1} \frac{1 - x^2}{f(x)}dx \tag{4.3}$$

and

$$\gamma_k(dl^2) = \frac{1}{|k|} \quad \text{if } k \neq 0. \tag{4.4}$$

5. Spectral determination of the space-time

In case $f(x)$ is given by (3.5), the metric (3.4) is related to (4.1) via the homothety $ds^2_{r_0} = \eta^2(1-\xi)dl^2$, and it is well known that

$$\lambda \in Spec(dl^2) \quad \text{if and only if} \quad \frac{\lambda}{\eta^2(1-\xi)} \in Spec(\eta^2(1-\xi)dl^2)$$

so that, after an elementary integration and some algebra, the trace formulae (obtained by multiplying (4.3) and (4.4) by $\eta^2(1-\xi)$) for either horizon take the form

$$\gamma_0(\eta^2(1-\xi)dl^2) = \eta^2\left[1-\beta^2+\left(\xi-\beta^2\right)\frac{\left(\sqrt{\frac{1-\xi}{\xi}}\arctan\sqrt{\frac{\xi}{1-\xi}}\right)-1}{\xi}\right] \tag{5.1}$$

and

$$\gamma_k(\eta^2(1-\xi)dl^2) = \frac{\eta^2(1-\xi)}{|k|} \qquad \forall k \neq 0. \tag{5.2}$$

For the remainder of the paper we will suppress the dependence of γ_k on the metric.

An immediate consequence of (5.2) is that the area of the metric (as is the case in [4]) has a representation for each $k \in \mathbb{N}$ given by

$$A = 4\pi k \gamma_k. \tag{5.3}$$

Defining

$$g(\xi) = \frac{\left(\sqrt{\frac{1-\xi}{\xi}}\arctan\sqrt{\frac{\xi}{1-\xi}}\right)-1}{\xi} \tag{5.4}$$

brings equation (5.1) into the form

$$\gamma_0 = \eta^2\left[1-\beta^2+\left(\xi-\beta^2\right)g(\xi)\right]. \tag{5.5}$$

One now has, for each of the two physical horizons $r = r_e$ and $r = r_c$, corresponding triples of parameters and traces, denoted by $(\eta_e, \beta_e, \gamma_k^e)$ and $(\eta_c, \beta_c, \gamma_k^c)$ respectively, each of which must satisfy a non-linear system (5.2), (5.5).

We can now prove:

Theorem 5.1. *If $a \neq 0$, $\Lambda > 0$, and $r_c \neq r_e$, then*

$$\Lambda = \frac{3(\gamma_0^e - \gamma_1^e + \gamma_1^c - \gamma_0^c)}{\gamma_1^c\gamma_0^e - \gamma_1^e\gamma_0^c}. \tag{5.6}$$

Proof. The proof is elementary and is accomplished by simply combining the expressions for the traces consisting of (5.2) (with $k = 1$) and (5.5) for both horizons and using the definitions of the parameters found in section 3. □

Encouraged by this result, one might hope that the other parameters are also spectrally determined, and indeed this is the case. This implies:

Theorem 5.2. *A metric in the 4-parameter family of all Kerr-Newman-de Sitter space-times is uniquely determined by the union of the spectra of its cosmological and event horizons.*

Proof. After defining

$$h(\xi) = \frac{1+\xi g(\xi)}{1-\xi}, \tag{5.7}$$

the pair of equations (5.2) and (5.5) for each horizon yields

$$h(\xi) = \frac{\gamma_0^e - \gamma_0^c}{\gamma_1^e - \gamma_1^c}. \tag{5.8}$$

Once it is verified that the function h is invertible, we obtain

$$\xi = h^{-1}\left(\frac{\gamma_0^e - \gamma_0^c}{\gamma_1^e - \gamma_1^c}\right). \tag{5.9}$$

From the definition (3.3), the angular momentum parameter is given by

$$a^2 = \frac{h^{-1}\left(\frac{\gamma_0^e-\gamma_0^c}{\gamma_1^e-\gamma_1^c}\right)}{1-h^{-1}\left(\frac{\gamma_0^e-\gamma_0^c}{\gamma_1^e-\gamma_1^c}\right)} \cdot \frac{\gamma_1^c\gamma_0^e - \gamma_1^e\gamma_0^c}{(\gamma_0^e - \gamma_1^e + \gamma_1^c - \gamma_0^c)}. \tag{5.10}$$

From (5.2) ($k=1$), (5.9), and (5.10) one can solve for r_e and r_c respectively. The resulting equations are

$$r_{e,c}^2 = \frac{\gamma_1^{e,c}}{1-h^{-1}\left(\frac{\gamma_0^e-\gamma_0^c}{\gamma_1^e-\gamma_1^c}\right)} - \frac{h^{-1}\left(\frac{\gamma_0^e-\gamma_0^c}{\gamma_1^e-\gamma_1^c}\right)}{1-h^{-1}\left(\frac{\gamma_0^e-\gamma_0^c}{\gamma_1^e-\gamma_1^c}\right)} \cdot \frac{\gamma_1^c\gamma_0^e - \gamma_1^e\gamma_0^c}{(\gamma_0^e - \gamma_1^e + \gamma_1^c - \gamma_0^c)}, \tag{5.11}$$

and we have Λ, a^2, r_e and r_c in terms of the traces. Finally, after substituting these (distinct) values of r into the equation $\Delta_r = 0$, a non-singular linear system in the variables m and Q^2 is obtained and, therefore, m and Q^2 are uniquely determined. □

6. Comments

In this paper, as in [4], we have studied the spectra of the Laplacians on the pullbacks of Kerr-Newman (de Sitter) metrics to their horizons. To our knowledge, these are the only papers on the subject. On the other hand, the spectrum associated with the wave equation defined on the four-dimensional space-time (the Teukolsky master equation) has received quite a bit of attention (see [1], [9], [11], [18], [12], and [17] among many others). Separation of variables for the Teukolsky equation yields (in general complex) quasinormal mode frequencies for scalar, electromagnetic, and gravitational radiation of black holes, and, notably, [11] and [17] study the case with a non-zero cosmological constant. One can consider these to be characteristic sounds of the black hole space-time and are observable, in an astrophysical sense, as the ringdown of gravitational waves. In the $\Lambda = 0$ case, the angular operator (and respectively its separation constants) coming from separation of variables in the Teukolsky equation is similar to the Laplacian (and respectively its eigenvalues) on the event horizon in the sense that, for scalar fields, they both reduce to the Laplacian (and corresponding eigenvalues) on the constant curvature S^2 for the Schwarzschild ($a = 0$) case. For $\Lambda > 0$ the differences between the quasinormal modes spectrum and the Laplace spectra become more pronounced. Perhaps some insight into the physical interpretation of the horizon spectra will come from combining equations (2.1) and (5.6) and trying to understand the individual frequencies by comparison with a sum over all quantum fields (quarks,

leptons, gauge fields, etc.) expression for the vacuum energy found in [8]. Such calculations are, however, fraught with many difficulties and are outside our area of expertise, so we will not attempt them here.

In conclusion, although a physical interpretation of the Laplace spectra of the two horizons is difficult to come by, they clearly play an important role in the geometry of the space-time manifold.

References

1. Batic, D., Schmid, H., *Heun equation, Teukolsky equation, and type D metrics*, J. Math. Phys. **48**, 042502, 2007. MR2319887 (2008k:81219)
2. Carroll, S., *The cosmological constant*, Living Rev. Relativity **4** (2001), 1. MR1810924 (2002a:83101)
3. Davies, P. C. W., *Thermodynamic phase transitions of Kerr-Newman black holes in de Sitter space*, Class. Quantum Grav. **6** (1989), 1909-1914. MR1028391 (91e:83067)
4. Engman, M., Cordero-Soto, R., *Intrinsic spectral geometry of the Kerr-Newman event horizon*, J. Math. Phys. **47**, 033503, 2006. MR2219796 (2007i:83105)
5. Engman, M., *New spectral characterization theorems for S^2*, Pacific J. Math., Vol. 154, No. 2, 1992, 215-229. MR1159507 (93d:58173)
6. Engman, M., *Trace formulae for S^1 invariant Green's operators on S^2*, Manuscripta Math. **93** (1997), 357-368. MR1457734 (99c:58160)
7. Engman, M., *Sharp bounds for eigenvalues and multiplicities on surfaces of revolution*, Pacific J. Math., Vol. 186, No. 1, 1998, 29-37. MR1665055 (99m:58193)
8. Frieman, J., et al., *Dark energy and the accelerating universe*, Annual Review of Astronomy and Astrophysics, Vol. 46, 385-432 (2008).
9. Frolov, V., Novikov, I, *Black Hole Physics: Basic Concepts and New Developments*, Kluwer Academic Publishers, Dordrecht, 1998. MR1668599 (99m:83110)
10. Gibbons, G., Hawking, S., *Cosmological event horizon, thermodynamics, and particle creation*, Phys. Rev. **D13**, 2738 (1977). MR0459479 (56:17671)
11. Konoplya, R., Zhidenko, A., *Decay of a charged scalar and Dirac fields in the Kerr-Newman-de Sitter background*, Phys. Rev. D. **76**, 084081 (2007). MR2366001 (2008k:83096)
12. Nollert, H., *Quasinormal modes: The characteristic 'sound' of black holes and neutron stars*, Class. Quantum. Grav. **16**, R 159 (1999). MR1730765 (2000h:83036)
13. Perlmutter, S., Aldering, G., Goldhaber, G., Knop, R.A., Nugent, P., et al., *Measurements of Ω and Λ from 42 High-Redshift Supernovae*, 1999. Astrophys. J. 517:565–586.
14. Riess, A.G., Filippenko, A.V., Challis, P., Clocchiatti, A., Diercks, A., et al., *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*, 1998. Astron. J. 116:1009–1038.
15. Robinson, D. C., *Uniqueness of the Kerr black hole*, Phys. Rev. Lett. **34**, 905, 1975.
16. Smarr, L., *Surface geometry of charged rotating black holes*, Phys. Rev. **D7** 289 (1973).
17. Suzuki, H., Takasugi, E., Umetsu, H., *Perturbations of Kerr-de Sitter black hole and Heun's equations*, Prog. Theor. Phys. **100** (1998), 491-505. MR1664413 (99j:83049)
18. Teukolsky, S., *Rotating black holes: Separable wave equations for gravitational and electromagnetic perturbations*, Phys. Rev. Lett. **29**, 1114 (1972).
19. Zelditch, S., *The inverse spectral problem for surfaces of revolution*, J. Differential. Geom. **49** (1998), no. 2, 207-264. MR1664907 (99k:58188)
20. Zel'dovich, Y.B. (1967), *Cosmological constant and the theory of elementary particles*, Sov. Phys. Usp. **11**, 381-393 (1968).

Departamento de Ciencias y Tecnología, Universidad Metropolitana, San Juan, Puerto Rico 00928

E-mail address: `um_mengman@suagm.edu, mathengman@yahoo.com`

Departamento de Ciencias y Tecnología, Universidad Metropolitana, San Juan, Puerto Rico 00928

E-mail address: `gerardo.appmath@gmail.com`