A NOTE ON THE SPACES OF VARIABLE INTEGRABILITY
AND SUMMABILITY OF ALMEIDA AND HÄSTÖ

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Abstract. We address an open problem posed recently by Almeida and Hästö. They defined the spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ of variable integrability and summability and showed that $\| \cdot \|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ is a norm if $q \geq 1$ is constant almost everywhere or if $1/p(x)+1/q(x) \leq 1$ for almost every $x \in \mathbb{R}^n$. Nevertheless, the natural conjecture (expressed also by Almeida and Hästö) is that the expression is a norm if $p(x), q(x) \geq 1$ almost everywhere. We show that $\| \cdot \|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ is an norm if $1 \leq q(x) \leq p(x)$ for almost every $x \in \mathbb{R}^n$. Furthermore, we construct an example of $p(x)$ and $q(x)$ with $\min(p(x), q(x)) \geq 1$ for every $x \in \mathbb{R}^n$ such that the triangle inequality does not hold for $\| \cdot \|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$.

1. Introduction

For the definition of the spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ we closely follow [1]. Spaces of variable integrability $L_{p(\cdot)}$ and variable sequence spaces $\ell_{q(\cdot)}$ were first considered in 1931 by Orlicz [5], but the modern development started with the paper [4]. We refer to [3] for an excellent overview of the vastly growing literature on the subject.

First of all we recall the definition of the variable Lebesgue spaces $L_{p(\cdot)}(\Omega)$, where $\Omega$ is a measurable subset of $\mathbb{R}^n$. A measurable function $p : \Omega \to (0, \infty]$ is called a variable exponent function if it is bounded away from zero. For a set $A \subset \Omega$ we denote $p_A^+ = \operatorname{ess-sup}_{x \in A} p(x)$ and $p_A^- = \operatorname{ess-inf}_{x \in A} p(x)$; we use the abbreviations $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$. The variable exponent Lebesgue space $L_{p(\cdot)}(\Omega)$ consists of all measurable functions $f$ such that there exists an $\lambda > 0$ such that the modular

$$\varrho_{L_{p(\cdot)}(\Omega)}(f/\lambda) = \int_{\Omega} \varphi_{p(x)} \left( \frac{|f(x)|}{\lambda} \right) dx$$

is finite, where

$$\varphi_{p}(t) = \begin{cases} t^p & \text{if } p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \leq 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$
This definition is nowadays standard and was also used in [1] Section 2.2 and [3] Definition 3.2.1. If we define \( \Omega_\infty = \{ x \in \Omega : p(x) = \infty \} \) and \( \Omega_0 = \Omega \setminus \Omega_\infty \), then the Luxemburg norm of a function \( f \in L_{p(\cdot)}(\Omega) \) is given by
\[
\| f \|_{L_{p(\cdot)}(\Omega)} = \inf\{\lambda > 0 : g_{L_{p(\cdot)}(\Omega)}(f/\lambda) \leq 1\}
\]
\[
= \inf\left\{ \lambda > 0 : \int_{\Omega_0} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \text{ and } |f(x)| \leq \lambda \text{ for a.e. } x \in \Omega_\infty \right\}.
\]
If \( p(\cdot) \geq 1 \), then it is a norm, but it is always a quasi-norm if at least \( p^{-} > 0 \); see [4] for details. We denote the class of all measurable functions \( p : \mathbb{R}^n \to (0, \infty] \) such that \( p^{-} > 0 \) by \( \mathcal{P}(\mathbb{R}^n) \) and the corresponding modular is denoted by \( g_{p(\cdot)} \) instead of \( g_{L_{p(\cdot)}(\mathbb{R}^n)} \).

To define the mixed spaces \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) we have to define another modular. For \( p, q \in \mathcal{P}(\mathbb{R}^n) \) and a sequence \( (f_\nu)_{\nu \in \mathbb{N}_0} \) of \( L_{p(\cdot)}(\mathbb{R}^n) \) functions we define
\[
g_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu = 0}^{\infty} \inf\left\{ \lambda_\nu > 0 : g_{p(\cdot)} \left( \frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}} \right) \leq 1 \right\},
\]
where we put \( \lambda^{1/\infty} := 1 \). The (quasi-) norm in the \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) spaces is defined as usual by
\[
\| f_\nu \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} = \inf\{\mu > 0 : g_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu/\mu) \leq 1\}.
\]
This (quasi-)norm was used in [1] to define the spaces of Besov type with variable integrability and summability. Spaces of Triebel-Lizorkin type with variable indices have recently been considered in [2]. The appropriate \( L_{p(\cdot)}(\ell_{q(\cdot)}) \) space is a normed space whenever \( \text{ess-inf}_{x \in \mathbb{R}^n} \min(p(x), q(x)) \geq 1 \). This was the expected result and coincides with the case of constant exponents.

As pointed out in the remark after Theorem 3.8 in [1], the same question is still open for the \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) spaces.

2. When does \( \| : \ell_{q(\cdot)}(L_{p(\cdot)}) \| \) define a norm?

In Theorem 3.6 of [1] the authors proved that if the condition \( \frac{1}{p(x)} + \frac{1}{q(x)} \leq 1 \) holds for almost every \( x \in \mathbb{R}^n \), then \( \| : \ell_{q(\cdot)}(L_{p(\cdot)}) \| \) defines a norm. They also proved in Theorem 3.8 that \( \| : \ell_{q(\cdot)}(L_{p(\cdot)}) \| \) is a quasi-norm for all \( p, q \in \mathcal{P}(\mathbb{R}^n) \). Furthermore, the authors of [1] posed a question if the (rather natural) condition \( p(x), q(x) \geq 1 \) for almost every \( x \in \mathbb{R}^n \) ensures that \( \| : \ell_{q(\cdot)}(L_{p(\cdot)}) \| \) is a norm.

We give (in Theorem 1) a positive answer if \( 1 \leq q(x) \leq p(x) \leq \infty \) almost everywhere on \( \mathbb{R}^n \). Furthermore, in Theorem 2 we construct two functions \( p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) such that \( \text{inf}_{x \in \mathbb{R}^n} \min(p(x), q(x)) \geq 1 \), but the triangle inequality does not hold for \( \| : \ell_{q(\cdot)}(L_{p(\cdot)}) \| \).

2.1. Positive results. We summarize in the following theorem all the cases when the expression \( \| : \ell_{q(\cdot)}(L_{p(\cdot)}) \| \) is known to be a norm. We include the proof of the case discussed already in [1] for the sake of completeness.

**Theorem 1.** Let \( p, q \in \mathcal{P}(\mathbb{R}^n) \) such that either \( p(x) \geq 1 \) and \( q \geq 1 \) is constant almost everywhere or \( 1 \leq q(x) \leq p(x) \leq \infty \) for almost every \( x \in \mathbb{R}^n \), or \( 1/p(x) + 1/q(x) \leq 1 \) for almost every \( x \in \mathbb{R}^n \). Then \( \| : \ell_{q(\cdot)}(L_{p(\cdot)}) \| \) defines a norm.
Proof. If \( p(x) \geq 1 \) and \( q \geq 1 \) is constant almost everywhere, then the proof is trivial.

In the remaining cases, we want to show that
\[
\|f_\nu + g_\nu \ell_{q(x)}(L_{p(x)})\| \leq \|f_\nu \ell_{q(x)}(L_{p(x)})\| + \|g_\nu \ell_{q(x)}(L_{p(x)})\|
\]
for all sequences of measurable functions \( \{f_\nu\}_{\nu \in \mathbb{N}_0} \) and \( \{g_\nu\}_{\nu \in \mathbb{N}_0} \). Let \( \mu_1 > 0 \) and \( \mu_2 > 0 \) be given with
\[
\varrho_{\ell_{q(x)}(L_{p(x)})}(f_\nu) \leq 1 \quad \text{and} \quad \varrho_{\ell_{q(x)}(L_{p(x)})}(g_\nu) \leq 1.
\]
We want to show that
\[
\varrho_{\ell_{q(x)}(L_{p(x)})}(f_\nu + g_\nu) \leq 1.
\]
For every \( \varepsilon > 0 \), there exist sequences of positive numbers \( \{\lambda_\nu\}_{\nu \in \mathbb{N}_0} \) and \( \{\Lambda_\nu\}_{\nu \in \mathbb{N}_0} \) such that
\[
(1) \quad \varrho_{\ell(x)} \left( \frac{f_\nu(x)}{\mu_1 \lambda_\nu^{1/q(x)}} \right) \leq 1 \quad \text{and} \quad \varrho_{\ell(x)} \left( \frac{g_\nu(x)}{\mu_2 \Lambda_\nu^{1/q(x)}} \right) \leq 1,
\]
together with
\[
\sum_{\nu=0}^{\infty} \lambda_\nu \leq 1 + \varepsilon \quad \text{and} \quad \sum_{\nu=0}^{\infty} \Lambda_\nu \leq 1 + \varepsilon.
\]
We set
\[
A_\nu := \frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2}, \quad \text{i.e.} \quad \sum_{\nu=0}^{\infty} A_\nu \leq 1 + \varepsilon.
\]
We shall prove that
\[
(2) \quad \varrho_{\ell(x)} \left( \frac{f_\nu(x) + g_\nu(x)}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \right) \leq 1 \quad \text{for all} \quad \nu \in \mathbb{N}_0.
\]
Let \( \Omega_0 := \{x \in \mathbb{R}^n : p(x) < \infty\} \) and \( \Omega_\infty := \{x \in \mathbb{R}^n : p(x) = \infty\} \). We put for every \( x \in \Omega_0 \)
\[
F_\nu(x) := \left( \frac{|f_\nu(x)|}{\mu_1 \lambda_\nu^{1/q(x)}} \right)^{p(x)} \quad \text{and} \quad G_\nu(x) := \left( \frac{|g_\nu(x)|}{\mu_2 \Lambda_\nu^{1/q(x)}} \right)^{p(x)}.
\]
Then (1) may be reformulated as
\[
(3) \quad \int_{\Omega_0} F_\nu(x) dx \leq 1 \quad \text{and} \quad \text{ess-sup}_{x \in \Omega_\infty} \frac{|f_\nu(x)|}{\mu_1 \lambda_\nu^{1/q(x)}} \leq 1
\]
and
\[
(4) \quad \int_{\Omega_0} G_\nu(x) dx \leq 1 \quad \text{and} \quad \text{ess-sup}_{x \in \Omega_\infty} \frac{|g_\nu(x)|}{\mu_2 \Lambda_\nu^{1/q(x)}} \leq 1.
\]
Our aim is to prove (2), which reads
\[
(5) \quad \int_{\Omega_0} \left( \frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \right)^{p(x)} dx \leq 1 \quad \text{and} \quad \text{ess-sup}_{x \in \Omega_\infty} \frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \leq 1.
\]
We first prove the second part of (5). First we observe that (3) and (4) imply that
\[ |f_\nu(x)| \leq \mu_1 \lambda_{\nu}^{1/q(x)} \quad \text{and} \quad |g_\nu(x)| \leq \mu_2 \Lambda_{\nu}^{1/q(x)} \]
hold for almost every \( x \in \Omega_\infty \). Using \( q(x) \geq 1 \) and Hölder’s inequality in the form
\[ \frac{\mu_1 \lambda_{\nu}^{1/q(x)} + \mu_2 \Lambda_{\nu}^{1/q(x)}}{\mu_1 + \mu_2} \leq \left( \frac{\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu}}{\mu_1 + \mu_2} \right)^{1/q(x)}, \]
we get
\[ \frac{|f_\nu(x) + g_\nu(x)|}{A_{\nu}^{1/q(x)}(\mu_1 + \mu_2)} \leq 1. \]
If \( q(x) = \infty \), only notational changes are necessary.
Next we prove the first part of (5). Let \( 1 < q(x) \leq p(x) < \infty \) for almost all \( x \in \Omega_0 \). Then we use Hölder’s inequality in the form
\[ F_\nu(x)^{1/p(x)} \lambda_{\nu}^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_{\nu}^{1/q(x)} \mu_2 \]
\[ \leq (\mu_1 + \mu_2)^{-1-1/q(x)}(\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu})^{1/q(x)} \mu_1 + G_\nu(x) \Lambda_{\nu} \mu_2 \]
If \( 1/p(x) + 1/q(x) \leq 1 \) for almost every \( x \in \Omega_0 \), then we replace (6) by
\[ F_\nu(x)^{1/p(x)} \lambda_{\nu}^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_{\nu}^{1/q(x)} \mu_2 \]
\[ \leq (\mu_1 + \mu_2)^{-1-1/p(x)-1/q(x)}(\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu})^{1/q(x)} \mu_1 + G_\nu(x) \mu_2 \]
Using (6), we may continue:
\[ \int_{\Omega_0} \left( \frac{|f_\nu(x) + g_\nu(x)|}{A_{\nu}^{1/q(x)}(\mu_1 + \mu_2)} \right)^{p(x)} \, dx \]
\[ = \int_{\Omega_0} \left( \frac{F_\nu(x)^{1/p(x)} \lambda_{\nu}^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_{\nu}^{1/q(x)} \mu_2}{\mu_1 + \mu_2} \right)^{p(x)} \cdot \left( \frac{\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu}}{\mu_1 + \mu_2} \right)^{-\frac{p(x)}{q(x)}} \, dx \]
\[ \leq \int_{\Omega_0} \frac{\mu_1 \lambda_{\nu}}{\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu}} \int_{\Omega_0} F_\nu(x) \, dx + \frac{\mu_2 \Lambda_{\nu}}{\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu}} \int_{\Omega_0} G_\nu(x) \, dx \leq 1, \]
where we also used (3) and (4). If we start with (7) instead, we proceed in the following way:
\[ \int_{\Omega_0} \left( \frac{|f_\nu(x) + g_\nu(x)|}{A_{\nu}^{1/q(x)}(\mu_1 + \mu_2)} \right)^{p(x)} \, dx \]
\[ = \int_{\Omega_0} \left( \frac{F_\nu(x)^{1/p(x)} \lambda_{\nu}^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_{\nu}^{1/q(x)} \mu_2}{\mu_1 + \mu_2} \right)^{p(x)} \cdot \left( \frac{\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu}}{\mu_1 + \mu_2} \right)^{-\frac{p(x)}{q(x)}} \, dx \]
\[ \leq \int_{\Omega_0} \frac{\mu_1 \lambda_{\nu}}{\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu}} \int_{\Omega_0} F_\nu(x) \, dx + \frac{\mu_2 \Lambda_{\nu}}{\mu_1 \lambda_{\nu} + \mu_2 \Lambda_{\nu}} \int_{\Omega_0} G_\nu(x) \, dx \leq 1. \]
In both cases, this finishes the proof of (5). \( \square \)
Remark 1. (i) A simpler proof of Theorem 1 is possible (and was proposed to us by the referee) if \( 1 \leq q(x) \leq p(x) \leq \infty \). Namely, if \( 1 \leq q \leq p \leq \infty, \lambda > 0 \) and \( t \geq 0 \), then

\[
\varphi_p \left( \frac{t}{\lambda^{1/q}} \right) = \varphi_{\frac{p}{q}} \left( \frac{\varphi_q(t)}{\lambda} \right),
\]

where we use the convention that \( \frac{p}{q} = 1 \) if \( p = q = \infty \). This allows us to simplify the modular \( \varrho_{\ell_q(\ell_{p(x)})} \) to

\[
\varrho_{\ell_q(\ell_{p(x)})}(f) = \sum_{\nu = 0}^{\infty} \| \varphi_q((|f|/\nu)) \|_{\frac{p}{q}}.
\]

This shows that \( \varrho_{\ell_q(\ell_{p(x)})}(f) \) is a composition of only convex functions. Hence, it is a convex modular, and therefore it induces a norm. Unfortunately, we were not able to find such a simplification for the case \( 1/p(x) + 1/q(x) \leq 1 \). The advantage of our proof of Theorem 1 is that it proves both cases in a unified way.

(ii) Let us observe that (5) loses its sense if \( p < q = \infty \). This shows why (9) (which was already used in [1] for \( q^+ < \infty \)) has to be applied with certain care.

(iii) The method of the proof of Theorem 1 can actually be used to show that under the conditions posed on \( p(\cdot) \) and \( q(\cdot) \) in Theorem 1 \( \varrho_{\ell_q(\ell_{p(\cdot)})} \) is a convex modular, which is a stronger result than the norm property.

2.2. Counterexample.

**Theorem 2.** There exist functions \( p, q \in \mathcal{P}(\mathbb{R}^n) \) with \( \inf_{x \in \mathbb{R}^n} p(x) \geq 1 \) and \( \inf_{x \in \mathbb{R}^n} q(x) \geq 1 \) such that \( \| \cdots \|_{\ell_q(\ell_{p(\cdot)})} \) does not satisfy the triangle inequality.

**Proof.** Let \( Q_0, Q_1 \subset \mathbb{R}^n \) be two disjoint unit cubes, let \( p(x) := 1 \) everywhere on \( \mathbb{R}^n \), and put \( q(x) := \infty \) for \( x \in Q_1 \) and \( q(x) := 1 \) for \( x \not\in Q_1 \). Let \( f_1 = \chi_{Q_0} \) and \( f_2 = \chi_{Q_1} \). Finally, we put \( f = (f_1, f_2, 0, \ldots) \) and \( g = (f_2, f_1, 0, \ldots) \).

We calculate for every \( L > 0 \) fixed

\[
\inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{f_1(x)}{\lambda^{1/q(x)} L} \right) \leq 1 \right\} = \inf \left\{ \lambda > 0 : \frac{1}{\lambda L} \leq 1 \right\} = 1/L
\]

and

\[
\inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{f_2(x)}{\lambda^{1/q(x)} L} \right) \leq 1 \right\} = \inf \left\{ \lambda > 0 : \frac{1}{L} \leq 1 \right\}.
\]

If \( L \geq 1 \), then the last expression is equal to zero; otherwise it is equal to \( \infty \). We obtain

\[
\|f\|_{\ell_q(\ell_{p(\cdot)})} = \inf \{ L > 0 : \varrho_{\ell_q(\ell_{p(\cdot)})}(f/L) \leq 1 \} \quad \text{and} \quad \|g\|_{\ell_q(\ell_{p(\cdot)})} = \inf \{ L > 0 : 1/L + 0 \leq 1 \} = 1,
\]

and the same is also true for \( \|g\|_{\ell_q(\ell_{p(\cdot)})} \). It is therefore enough to show that \( \|f + g\|_{\ell_q(\ell_{p(\cdot)})} > 2 \).
Using the calculation
\[
\inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( f_1(x) + f_2(x) \frac{L}{\lambda^{1/q(x)}} \right) \leq 1 \right\} = \inf \left\{ \lambda > 0 : \int_{Q_0} \frac{1}{L \cdot \lambda} + \int_{Q_1} \frac{1}{L} \leq 1 \right\}
\]
\[
= \inf \left\{ \lambda > 0 : \frac{1}{L \cdot \lambda} + \frac{1}{L} \leq 1 \right\} = \frac{1}{L-1},
\]
which holds for every \( L > 1 \) fixed, we get
\[
\|f + g\|_{L^\varrho(\cdot)}(L_{p(\cdot)}) = \inf \left\{ L > 0 : \varrho_{L\varrho(\cdot)}(L_{p(\cdot)}) \left( \frac{f + g}{L} \right) \leq 1 \right\}
\]
\[
= \inf \left\{ L > 0 : 2 \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{f_1(x) + f_2(x)}{L \cdot \lambda^{1/q(x)}} \right) \leq 1 \right\} \leq 1 \right\}
\]
\[
= \inf \left\{ L > 1 : 2 \cdot \frac{1}{L-1} \leq 1 \right\} = 3.
\]

\[\square\]

Remark 2. Let us observe that \( 1 \leq q(x) \leq p(x) \leq \infty \) holds for \( x \in Q_0 \) and that \( 1/p(x) + 1/q(x) \leq 1 \) is true for \( x \in Q_1 \). It is therefore necessary to interpret the assumptions of Theorem 1 in a correct way, namely, that one of the conditions of Theorem 1 holds for (almost) all \( x \in \mathbb{R}^n \). This is not to be confused with the statement that for (almost) every \( x \in \mathbb{R}^n \) at least one of the conditions is satisfied, which is not sufficient.

Remark 3. A similar calculation (which we shall not repeat in detail) shows that one may also put \( q(x) := q_0 \) large enough for \( x \in Q_1 \) to obtain a counterexample. Hence there is nothing special about the infinite value of \( q \), and the same counterexample may be reproduced with uniformly bounded exponents \( p, q \in \mathcal{P}(\mathbb{R}^n) \).

REFERENCES


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