ON THE KOBAYASHI HYPERBOLICITY
OF CERTAIN TUBE DOMAINS

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Abstract. In an article from 2008 the second author introduced three families of tube domains in \( \mathbb{C}^2 \) with holomorphic automorphism group isomorphic to \( \mathbb{R} \rtimes \mathbb{R}^2 \) and envelope of holomorphy equal to \( \mathbb{C}^2 \). In the present paper we show that every domain in each of these families is Kobayashi-hyperbolic.

1. Introduction

A connected complex manifold \( X \) is called Kobayashi-hyperbolic if the Kobayashi pseudodistance on \( X \) is in fact a distance (see [K] for details). If \( X \) is equipped with a Riemannian metric, the hyperbolicity property can be stated as follows: for any point \( x \in X \) there exist a neighborhood \( U \) of \( x \) and a constant \( M > 0 \) such that for all holomorphic maps \( f : \Delta \to X \) with \( f(0) \in U \) one has \( ||df(0)|| < M \), where \( \Delta \) is the unit disk in \( \mathbb{C} \) (see, e.g., [L]). Verification of hyperbolicity for a particular manifold may be a difficult task. In this paper we show that certain explicitly given tube domains in \( \mathbb{C}^2 \) are hyperbolic.

Recall that a tube domain in \( \mathbb{C}^n \) is a domain of the form \( T_D := D + i\mathbb{R}^n \), where \( D \) is a domain in \( \mathbb{R}^n \) called the base of \( T_D \). By Bochner’s theorem, the envelope of holomorphy of \( T_D \) coincides with \( \hat{T}_D \), where \( \hat{T}_D \) is the convex hull of \( D \) in \( \mathbb{R}^n \) (see, e.g., Section 21 in [V]). For tube domains in \( \mathbb{C}^2 \) with \( \hat{T}_D \neq \mathbb{C}^2 \), a hyperbolicity criterion was given in [L]. However, there is no reasonable sufficient condition for \( T_D \) to be hyperbolic in the case \( \hat{T}_D = \mathbb{C}^2 \). All tube domains considered in this paper fall into this last case. In particular, they do not admit any non-constant bounded holomorphic functions.

We will now introduce three families of domains in \( \mathbb{R}^2 \) as follows:

\[
A_{\alpha,s,t} := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{array}{ll}
  x_2 > t x_1^\alpha & \text{if } x_1 > 0, \\
  x_2 > 0 & \text{if } x_1 = 0, \\
  x_2 > s(-x_1)^\alpha & \text{if } x_1 < 0
\end{array} \right\},
\]

\[
b > 0, \alpha \neq 1, s < 0, t > 0,
\]

\[
B_{s,t} := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{array}{ll}
  x_2 > x_1 \log(t x_1) & \text{if } x_1 > 0, \\
  x_2 > 0 & \text{if } x_1 = 0, \\
  x_2 > x_1 \log(s(-x_1)) & \text{if } x_1 < 0
\end{array} \right\},
\]

\[
s > 0, t > 0,
\]
holomorphic automorphism group of each of these tube domains is isomorphic to $\mathbb{R} \rtimes \mathbb{C}$.

In contrast, ascertaining the hyperbolicity of domains in $\mathbb{R} \times _\rho \mathbb{C}$ was given in [I2] either. In the present paper we address this issue by proving the following theorem.

**Theorem 1.1.** Every domain in each of the families $\{T_{A_{\alpha,s,t}}\}$, $\{T_{B_{s,t}}\}$, $\{T_{C_{\alpha,s,t}}\}$ is hyperbolic.

In addition to supplementing the arguments of [I2], the proof of Theorem 1.1 given in the next section is also of independent interest since it in fact applies to a much larger class of tube domains satisfying the condition $T_D = \mathbb{C}^2$ (see Remark 2.3).

**2. Proof of Theorem 1.1**

As pointed out in [I1], for a tube domain $T_D \subset \mathbb{C}^n$ the hyperbolicity property is equivalent to the following condition: for any point $x \in D$ there exist a neighborhood $U$ of $x$ in $D$ and a constant $M > 0$ such that for all harmonic maps $f : \Delta \to D$ with $f(0) \in U$ one has $|df(0)|| < M$. Hence $T_D$ is not hyperbolic if and only if there exist a point $a \in D$ and a sequence $\{f_k\}$ of harmonic maps from $\Delta$ into $D$ such that $f_k(0) \to a$ and $|df_k(0)|| \to \infty$ as $k \to \infty$.

Now let $D$ be a domain in one of the families $\{A_{\alpha,s,t}\}$, $\{B_{s,t}\}$, $\{C_{\alpha,s,t}\}$. Assuming that $T_D$ is not hyperbolic, we obtain a point $a = (a_1, a_2) \in D$ and a sequence $\{f_k\}$ as above, with $f_k = (u_k, v_k)$, where $u_k$, $v_k$ are real-valued harmonic functions on $\Delta$. In our proof of the theorem we utilize level sets of $u_k$. Some fundamental properties of such sets are given in the following proposition.

**Proposition 2.1.** For every $c \in \mathbb{R}$ one has:

(i) there exists $K \in \mathbb{N}$ such that $c \in u_k(\Delta)$ for all $k \geq K$,

(ii) $\operatorname{dist}(0, L_k(c)) \to 0$ as $k \to \infty$, where $L_k(c) := \{z \in \Delta : u_k(z) = c\}$.

**Proof.** We first prove statement (i). Assuming it is false, we obtain a subsequence $\{u_{k_1}\}$ of the sequence $\{u_k\}$ such that for some $c \neq a_1$ one has either $u_{k_1} < c$ in $\Delta$ (if $a_1 < c$) or $u_{k_1} > c$ in $\Delta$ (if $a_1 > c$) for all $k_1$. Then $f_{k_1}(\Delta)$ is contained in either $D_-(c) := D \cap \{z_1 < c\}$ or $D_+(c) := D \cap \{z_1 > c\}$ for all $k_1$, respectively.

Suppose first that $D$ belongs to the family $\{C_{\alpha,s,t}\}$. In this case the open sets $D_-(c)$ and $D_+(c)$ are disconnected and each of their countably many connected
components is bounded. Clearly, a tube domain having a bounded base is hyperbolic. On the other hand, let \( D'(c) \) be the connected component containing the point \( a \). Then \( f_{k\ell}(\Delta) \subset D'(c) \) for large \( k\ell \), which contradicts the hyperbolicity of \( T_{D'(c)} \).

Suppose next that \( D \) belongs to one of the families \( \{A_{a,s,t}\}, \{B_{s,t}\} \). In this case \( D_-(c) \) and \( D_+(c) \) are connected. We will now show that the tube domains \( T_{D_-(c)} \) and \( T_{D_+(c)} \) are hyperbolic, thus contradicting the fact that \( f_{k\ell}(\Delta) \) is contained in either \( D_-(c) \) or \( D_+(c) \) for all \( k\ell \). We use the following well-known result.

**Lemma 2.2** \([\text{E}]\). Let \( X, Y \) be complex manifolds and \( F : X \to Y \) a holomorphic map. Suppose that \( Y \) is hyperbolic and has an open cover \( \{U_\alpha\} \) such that \( F^{-1}(U_\alpha) \) is hyperbolic for every \( \alpha \). Then \( X \) is hyperbolic.

One now easily observes that \( T_{D_-(c)} \) (resp., \( T_{D_+(c)} \)) is hyperbolic by choosing in Lemma 2.2 the manifold \( Y \) to be \( \{(z_1, 0) \in \mathbb{C}^2 : \text{Re} \, z_1 < c\} \) (resp., \( \{(z_1, 0) \in \mathbb{C}^2 : c - m < \text{Re} \, z_1 < c\} \) (resp., \( \{(z_1, 0) \in \mathbb{C}^2 : c < \text{Re} \, z_1 < c + m\} \), \( m \in \mathbb{N} \), and the map \( F \) to be the projection to the \( z_1 \)-coordinate complex line. This completes the proof of statement (i).

We will now prove statement (ii). Assuming it is false, we obtain a subsequence \( \{f_{k\ell}\} \) of the sequence \( \{f_k\} \) and a disk \( \Delta_r \) of radius \( 0 < r < 1 \) centered at the origin such that for some \( c \neq a_1 \), the set \( f_{k\ell}(\Delta_r) \) is contained in either \( D_-(c) \) (if \( a_1 < c \)) or \( D_+(c) \) (if \( a_1 > c \)) for all \( k\ell \). Considering the sequence \( \{f_{k\ell}\} \) of maps from \( \Delta \) to \( D \) defined by \( f_{k\ell}(z) := f_{k\ell}(rz) \) for \( |z| < 1 \), we obtain a contradiction as in the proof of statement (i) above. The proof of Proposition 2.1 is complete. \( \square \)

In the remaining part of the proof of the theorem we will separately consider two cases.

**Case 1.** Suppose that \( D \) belongs to one of the families \( \{A_{a,s,t}\}, \{B_{s,t}\} \). Fix \( R > 0 \), \( p > 0 \), \( q > 0 \), \( 0 < \varepsilon < 1 \) such that \( pR > a_1 \), \( qR > -a_1 \) and consider the open set

\[
\{z \in \Delta : a_1 - pR < u_k(z) < a_1 + qR, \ |z| < 1 - \varepsilon \}.
\]

For all sufficiently large \( k \) the origin lies in this set, and we denote by \( \Omega_k \) its connected component containing the origin. By the maximum principle for harmonic functions, \( \Omega_k \) is a Jordan simply connected domain, and we have

\[
\partial \Omega_k = \Gamma_k \cup \Gamma_k' \cup \gamma_k,
\]

where \( \Gamma_k \subset L_k(a_1 - pR), \Gamma_k' \subset L_k(a_1 + qR), \) and \( \gamma_k := \partial \Omega_k \cap \{|z| = 1 - \varepsilon\} \).

For a subset \( E \subset \partial \Omega_k \), let \( \omega_k(E) \) be the harmonic measure of \( E \) at the origin associated to \( \Omega_k \). By a well-known estimate (see Theorem IV.6.2 on p. 149 in [GM]) and Proposition 2.1 for any sufficiently large \( k \) one has

\[
\omega_k(\gamma_k) \leq \frac{8}{\pi} \sqrt{\frac{\text{dist}(0, \partial \Omega_k)}{1 - \varepsilon}},
\]

which implies

\[
(2.1) \quad \omega_k(\gamma_k) \to 0 \quad \text{as} \quad k \to \infty.
\]
Next, let \( \mu_k := \omega_k(\Gamma_k) \) and \( \mu'_k := \omega_k(\Gamma'_k) \). We have
\[
 u_k(0) = \int_{\Gamma_k} u_k d\omega_k + \int_{\Gamma'_k} u_k d\omega_k + \int_{\gamma_k} u_k d\omega_k \\
= (a_1 - pR)\mu_k + (a_1 + qR)\mu'_k + \int_{\gamma_k} u_k d\omega_k.
\]
(2.2)
Since on \( \gamma_k \) the function \( u_k \) is bounded from above and below by constants independent of \( k \), from (2.1) we obtain that the last summand in (2.2) tends to zero as \( k \to \infty \). Thus, (2.1) and (2.2) yield
\[
\mu_k + \mu'_k \to 1, \\
(a_1 - pR)\mu_k + (a_1 + qR)\mu'_k \to a_1,
\]
which implies
\[
\mu_k \to \frac{q}{p+q}, \quad \mu'_k \to \frac{p}{p+q} \quad \text{as} \quad k \to \infty.
\]
(2.3)
We will now consider two situations.

Case 1a. Assume that \( D = A_{\alpha,s,t} \) for some \( \alpha > 0, \alpha \neq 1, s < 0, t > 0 \). Then
\[
v_k(z) > s(pR - a_1)^\alpha \quad \text{for} \quad z \in \Gamma_k, \gamma_k,
\]
\[
v_k(z) > t(a_1 + qR)^\alpha \quad \text{for} \quad z \in \Gamma'_k.
\]
Therefore, we have
\[
v_k(0) = \int_{\Gamma_k} v_k d\omega_k + \int_{\Gamma'_k} v_k d\omega_k + \int_{\gamma_k} v_k d\omega_k \\
\geq s(pR - a_1)^\alpha \mu_k + t(a_1 + qR)^\alpha \mu'_k + s(pR - a_1)^\alpha \omega_k(\gamma_k).
\]
(2.4)
Letting in the above inequality \( k \to \infty \) and using (2.1), (2.3), we then obtain
\[
a_2 \geq s(pR - a_1)^\alpha \frac{q}{p+q} + t(a_1 + qR)^\alpha \frac{p}{p+q}
\]
\[
= \frac{R^\alpha}{p+q} \left[ sq \left( p - \frac{a_1}{R} \right)^\alpha + tp \left( q + \frac{a_1}{R} \right)^\alpha \right].
\]
(2.5)
Choosing \( p,q \) such that \( (q/p)^{\alpha-1} > |s|/t \) and letting \( R \to \infty \), we now observe that the right-hand side of (2.5) can be made arbitrarily large. This contradiction concludes the proof of the theorem in the case when \( D \) belongs to the family \( \{A_{\alpha,s,t}\} \).

Case 1b. Assume that \( D = B_{s,t} \) for some \( s > 0, t > 0 \). Then
\[
v_k(z) > (a_1 - pR) \log(s(pR - a_1)) \quad \text{for} \quad z \in \Gamma_k,
\]
\[
v_k(z) > (a_1 + qR) \log(t(a_1 + qR)) \quad \text{for} \quad z \in \Gamma'_k,
\]
\[
v_k(z) > C \quad \text{for} \quad z \in \gamma_k, \text{ where } C \text{ is a constant independent of } k.
\]
Therefore, analogously to (2.4) we have
\[
v_k(0) \geq (a_1 - pR) \log(s(pR - a_1)) \mu_k + (a_1 + qR) \log(t(a_1 + qR)) \mu'_k + C \omega_k(\gamma_k).
\]
Letting in the above inequality \( k \to \infty \) and using (2.1), (2.3), we then obtain
\[
a_2 \geq \frac{(a_1 - pR) \log(s(pR - a_1))}{p + q} + \frac{q}{p + q} + \frac{(a_1 + qR) \log(t(a_1 + qR))}{p + q} = \frac{R}{p + q} \left[ \log(s(pR - a_1)) + \frac{a_1}{R} - \frac{q}{p + q} \log(t(a_1 + qR)) \right].
\]
(2.6)

Choosing \( p, q \) such that \( tq > sp \) and letting \( R \to \infty \), we now observe that the right-hand side of (2.6) can be made arbitrarily large. This contradiction concludes the proof of the theorem in the case when \( D \) belongs to the family \( \{ B_{s,t} \} \).

Case 2. Now suppose that \( D \) belongs to the family \( \{ C_{a,s,t} \} \). Fix \( c > a_1 \), \( 0 < \varepsilon < 1 \) and consider the open set
\[
\{ z \in \Delta : u_k(z) < c, \ |z| < 1 - \varepsilon \}.
\]
For all sufficiently large \( k \) the origin lies in this set, and we denote by \( \Omega_k \) its connected component containing the origin. As in Case 1, \( \Omega_k \) is a Jordan simply connected domain, and we have
\[
\partial \Omega_k = \Gamma_k \cup \gamma_k,
\]
where \( \Gamma_k \subset L_k(c) \) and \( \gamma_k := \partial \Omega_k \cap \{|z| = 1 - \varepsilon\} \).

Recall from the proof of Proposition 2.1 that the open set \( D_-(c) = D \cap \{ x_1 < c \} \) has countably many connected components and each component is bounded. Let \( D'(c) \) be the connected component of \( D_-(c) \) containing \( a \). Clearly, \( f_k(\Omega_k) \subset D'(c) \) for all sufficiently large \( k \). This implies that on \( \gamma_k \) the function \( u_k \) is bounded from below by a constant independent of \( k \) if \( k \) is sufficiently large.

For a subset \( E \subset \partial \Omega_k \), we let \( \omega_k(E) \) be the harmonic measure of \( E \) at the origin associated to \( \Omega_k \) and \( \mu_k := \omega_k(\Gamma_k) \). Arguing as in Case 1, we then see that (2.1) holds; that is, \( \mu_k \to 1 \) as \( k \to \infty \).

Next, we have
\[
u_k(0) = \int_{\Gamma_k} u_k d\omega_k + \int_{\gamma_k} u_k d\omega_k = c\mu_k + \int_{\gamma_k} u_k d\omega_k.
\]
(2.7)

Since on \( \gamma_k \) the function \( u_k \) is bounded from above and below by constants independent of \( k \), from (2.1) we obtain that the last summand in (2.7) tends to zero as \( k \to \infty \). Thus, (2.7) implies \( a_1 = c \), which contradicts our choice of \( c \). This completes the proof of the theorem.

Remark 2.3. The proof of Theorem 1.1 in fact applies to more general domains. Indeed, let \( D \) be a domain of the form \( \{(x_1, x_2) : x_2 > h(x_1)\} \), where \( h \in C(\mathbb{R}) \) and satisfies the following property: for every \( b \in \mathbb{R} \) there exist \( p > 0, q > 0 \) such that
\[
qh(b - pR) + ph(b + qR) \to \infty \quad \text{as } R \to \infty.
\]

Then the argument given for Case 1 yields that \( T_D \) is hyperbolic. Next, let \( D \) be a domain bounded by two general spirals, where a spiral is a curve defined by the equation \( r = g(\varphi) \), with \( g \) being an increasing function of \( \varphi \) such that \( \lim_{\varphi \to -\infty} g(\varphi) = 0 \) and \( \lim_{\varphi \to +\infty} g(\varphi) = \infty \). Then the argument given for Case 2 shows that \( T_D \) is hyperbolic.
Remark 2.4. Before attempting to prove Theorem 1.1 in full generality, we set out to show that the domain $T_{A_3,-1,1}$ is Brody hyperbolic (recall that $A_3,-1,1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > x_1^3\}$). Brody hyperbolicity for a tube domain is equivalent to the non-existence of a non-constant harmonic map $f = (u, v)$ from the plane into the base of the domain (cf. [L]). Regarding this question, F. Nazarov suggested that we consider the connected component $\Omega(R, \rho)$ containing the origin of the open set

$$\{z \in \mathbb{C} : -R < u(z) < 2R, \ |z| < \rho\}$$

(assuming without loss of generality that $u(0) = 0$), where $R > 0$ and $\rho > 0$ are large. Then the harmonic measure at the origin of the portion of $\partial \Omega(R, \rho)$ where $|z| = \rho$ tends to 0 as $\rho \to \infty$, and letting $R \to \infty$, one obtains that $v(0)$ is estimated from below by an arbitrarily large number. With F. Nazarov’s kind permission, we used a similar approach in Case 1 of our proof of Theorem 1.1. We also point out that the idea to consider harmonic measures associated to domains bounded by level sets of $u$ was independently suggested to us by E. Poletsky and L. Kovalev.

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