AN EMBEDDING THEOREM

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(Communicated by Michael Hitrik)

Abstract. We consider a weighted space $W^{(2)}_1(\mathbb{R}, q)$ of Sobolev type:

$$W^{(2)}_1(\mathbb{R}, q) = \left\{ y \in AC^{(1)}_{\text{loc}}(\mathbb{R}) : \|y''\|_{L^1(\mathbb{R})} + \|qy\|_{L^1(\mathbb{R})} < \infty \right\},$$

where $0 \leq q \in L^1_{\text{loc}}(\mathbb{R})$ and

$$\|y\|_{W^{(2)}_1(\mathbb{R}, q)} = \|y''\|_{L^1(\mathbb{R})} + \|qy\|_{L^1(\mathbb{R})}.$$

We obtain a precise condition which guarantees the embedding $W^{(2)}_1(\mathbb{R}, q) \hookrightarrow L^p(\mathbb{R}), p \in [1, \infty)$.

1. Introduction

In the present paper, we consider a weighted function space $W^{(2)}_1(\mathbb{R}, q)$ of Sobolev type:

$$W^{(2)}_1(\mathbb{R}, q) = \left\{ y \in AC^{(1)}_{\text{loc}}(\mathbb{R}) : \|y''\|_{L^1(\mathbb{R})} + \|qy\|_{L^1(\mathbb{R})} < \infty \right\}.$$

Here and in the sequel, $\|f\|_p = \|f\|_{L^p(\mathbb{R})}, p \in [1, \infty)$, $AC^{(1)}_{\text{loc}}(\mathbb{R})$, is the set of functions absolutely continuous together with the derivative on every finite interval.

In this paper we always require that

$$0 \leq q \in L^1_{\text{loc}}(\mathbb{R})$$

and usually do not mention this requirement in the statements.

Our general goal is to find conditions under which there is an embedding

$$W^{(2)}_1(\mathbb{R}, q) \hookrightarrow L^p(\mathbb{R}).$$

To be more precise, we give the following definitions.

Definition 1.1 ([6]). Let $p \in [1, \infty)$. We say that the space $W^{(2)}_1(\mathbb{R}, q)$ is embedded into the space $L^p(\mathbb{R})$ (and write $W^{(2)}_1(\mathbb{R}, q) \hookrightarrow L^p(\mathbb{R})$) if $W^{(2)}_1(\mathbb{R}, q) \subseteq L^p(\mathbb{R})$ and

$$\|y\|_p \leq c(p)\{\|y''\|_1 + \|qy\|_1\}, \quad \forall y \in W^{(2)}_1(\mathbb{R}, q).$$

Our general convention is that the letters $c, c(\cdot)$ stand for absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations.

We now state our main result.
Theorem 1.2. Let \( p \in [1, \infty) \). Then the embedding \( W_{1}^{(2)}(\mathbb{R}, q) \hookrightarrow L_{p}(\mathbb{R}) \) takes place if and only if

\[
\exists a > 0 : \quad q_{0}(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t) dt > 0.
\]

Note that for \( p = 1 \) this assertion was proved in [4] and will not be reconsidered here (see §2).

Example 1.3. Let

\[
q(x) = 1 + \cos(|x|^\theta), \quad x \in \mathbb{R}, \quad \theta \in (0, \infty).
\]

In [3] it was shown that in case \((1.5)\) condition \((1.4)\) holds if and only if \( \theta \geq 1 \). Together with Theorem 1.2, this implies that in case \((1.5)\) the embedding \( W_{1}^{(2)}(\mathbb{R}, q) \hookrightarrow L_{p}(\mathbb{R}), \ p \in [1, \infty) \), takes place if and only if \( \theta \geq 1 \).

2. Preliminaries

In this section, we present some facts needed for the proofs (see §3).

Lemma 2.1 ([1, 2]). Suppose conditions \((1.2)\) and \((2.1)\) hold:

\[
\int_{-\infty}^{x} q(t) dt > 0, \quad \int_{x}^{\infty} q(t) dt > 0, \quad \forall x \in \mathbb{R}.
\]

Then there exists a fundamental system of solutions (FSS) \( \{u, v\} \) of the equation

\[
z''(x) = q(x)z(x), \quad x \in \mathbb{R},
\]

which has the following properties:

\[
u(x) > 0, \quad v(x) > 0, \quad u'(x) \leq 0, \quad v'(x) \geq 0, \quad \forall x \in \mathbb{R},
\]

\[
v'(x)u(x) - u'(x)v(x) = 1, \quad \forall x \in \mathbb{R},
\]

\[
\lim_{x \to -\infty} \frac{v(x)}{u(x)} = \lim_{x \to \infty} \frac{u(x)}{v(x)} = 0.
\]

The solutions \( \{u, v\} \) and the function \( \rho \),

\[
\rho(x) = u(x)v(x), \quad x \in \mathbb{R},
\]

satisfy the relations

\[
\int_{-\infty}^{0} \frac{dt}{\rho(t)} = \int_{0}^{\infty} \frac{dt}{\rho(t)} = \infty,
\]

\[
|\rho'(x)| < 1, \quad \forall x \in \mathbb{R},
\]

\[
\frac{v'(x)}{v(x)} = \frac{1 + \rho'(x)}{2\rho(x)}, \quad \frac{u'(x)}{u(x)} = -\frac{1 - \rho'(x)}{2\rho(x)}, \quad x \in \mathbb{R}.
\]

In addition, if \( G(x, t) \) is the Green function,

\[
G(x, t) = \begin{cases} 
  u(x)v(t), & x \geq t, \\
  u(t)v(x), & x \leq t,
\end{cases}
\]

corresponding to the equation

\[
y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R},
\]
then we have the inequality

\begin{equation}
\sup_{t \in \mathbb{R}} \Phi(t) \leq 1, \quad \Phi(t) = \int_{-\infty}^{\infty} q(\xi) G(t, \xi) d\xi, \quad t \in \mathbb{R}.
\end{equation}

Finally, for every \( p \in [1, \infty) \), equation (2.2) has no solutions \( z \in L_p(\mathbb{R}) \) apart from \( z \equiv 0 \).

**Lemma 2.2** [1]. Suppose conditions (1.2) and (2.1) hold. For a given \( x \in \mathbb{R} \), consider the equation in \( d \geq 0 \):

\begin{equation}
d \int_{x-d}^{x+d} q(t) dt = 2.
\end{equation}

Equation (2.13) has a unique finite positive solution for every \( x \in \mathbb{R} \). Denote this solution by \( d(x) \). The function \( d(x) \) is continuous for \( x \in \mathbb{R} \).

Note that the function \( d \) was introduced by M. Otelbaev (see [6]).

**Theorem 2.3** [1]. Suppose conditions (1.2) and (2.1) hold. Then we have the inequalities

\begin{equation}d(x) \leq \rho(x) \leq \frac{3}{2} d(x), \quad x \in \mathbb{R}.
\end{equation}

Estimates of type (2.14) were first obtained (under some additional requirements to the function \( q \)) by Otelbaev (see [7]) and are therefore called Otelbaev’s inequalities (see also [6]).

Consider equation (2.11) with \( f \in L_p(\mathbb{R}), p \in [1, \infty) \). Below by a solution of (2.11) we mean any function \( y \in AC^{(1)}_{loc}(\mathbb{R}) \) satisfying equality (2.11) almost everywhere on \( \mathbb{R} \).

**Definition 2.4** [3]. We say that equation (2.11) is correctly solvable in the space \( L_p(\mathbb{R}), p \in [1, \infty) \), if assertions I–II hold:

I) for any function \( f \in L_p(\mathbb{R}) \), there is a unique solution \( y \in L_p(\mathbb{R}) \) of (2.11);

II) the solution \( y \in L_p(\mathbb{R}) \) of (2.11) satisfies the estimate

\begin{equation}
\|y\|_p \leq c(p) \|f\|_p, \quad \forall f \in L_p(\mathbb{R}).
\end{equation}

**Theorem 2.5** [3]. Let \( p \in [1, \infty) \). Equation (2.11) is correctly solvable in \( L_p(\mathbb{R}) \) if and only if one of the following two equivalent conditions, A, B, holds:

A) One has inequalities (2.11) and \( d_0 < \infty \). Here

\begin{equation}d_0 = \sup_{x \in \mathbb{R}} d(x).
\end{equation}

B) There is \( a \in (0, \infty) \) such that \( q_0(a) > 0 \):

\begin{equation}q_0(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t) dt.
\end{equation}

In addition, if equation (2.11) is correctly solvable in \( L_p(\mathbb{R}), p \in [1, \infty) \), then its solution \( y \in L_p(\mathbb{R}) \) is of the form

\begin{equation}y(x) = (Gf)(x) \overset{def}{=} \int_{-\infty}^{\infty} G(x,t) f(t) dt, \quad x \in \mathbb{R}.
\end{equation}

**Theorem 2.6** [4]. The embedding \( W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_1(\mathbb{R}) \) takes place if and only if equation (2.11) is correctly solvable in \( L_1(\mathbb{R}) \).
Theorem 2.7 ([5, Ch. XI, §6]). Let $K(x,t)$ be a continuous positive function in variables $x, t \in \mathbb{R}$, and let $K$ be the integral operator

$$
(Kf)(x) = \int_{-\infty}^{\infty} K(x,t)f(t) dt, \quad \forall f \in L_1(\mathbb{R}).
$$

Then for every $p \in (1, \infty)$, we have the equality

$$
\|K\|_{L_1(\mathbb{R}) \to L_p(\mathbb{R})} = \sup_{t \in \mathbb{R}} \left( \int_{-\infty}^{\infty} K(x,t)^p dx \right)^{1/p}.
$$

3. PROOFS

Proof of Theorem 1.2. Necessity.

Lemma 3.1. Let $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_p(\mathbb{R}), p \in [1, \infty)$. Then relations (2.1) hold.

Proof. Assume the contrary, say,

$$
\int_{x_0}^{\infty} q(t) dt = 0,
$$

for some $x_0 \in \mathbb{R}$. Let $\varphi \in C^\infty(\mathbb{R})$ be such that $\text{supp} \varphi = [x_0, \infty)$, $0 \leq \varphi \leq 1$, for $x \in \mathbb{R}$ and $\varphi(x) \equiv 1$ for $x \geq x_0 + 1$. Clearly, $\varphi \in AC^{(1)}_{\text{loc}}(\mathbb{R})$ and, in addition,

$$
\int_{-\infty}^{\infty} |\varphi''(t)| dt = \int_{x_0 + 1}^{\infty} |\varphi''(t)| dt = c < \infty.
$$

Hence $\varphi \in W_1^{(2)}(\mathbb{R}, q)$, and since $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_p(\mathbb{R})$, we have

$$
\begin{align*}
\infty &> c \left\{ \int_{-\infty}^{\infty} |\varphi''(t)| dt + \int_{-\infty}^{\infty} |q(t)\varphi(t)| dt \right\} \\
&\geq \left( \int_{-\infty}^{\infty} |\varphi(t)|^p dt \right)^{1/p} \\
&= \left( \int_{x_0 + 1}^{\infty} 1 dt \right)^{1/p}
\end{align*}
$$

Contradiction. \quad \square

Below, conditions (1.2) and (2.1) are assumed to be valid and do not appear in the statements.

Lemma 3.2. For a given $x \in \mathbb{R}$, consider the equation in $s \geq 0$:

$$
\int_{x-s}^{x+s} \frac{dt}{d(t)} = 2.
$$

Equation (3.1) has a unique finite positive solution for every $x \in \mathbb{R}$. Denote it by $s(x)$. Then for every $t \in [x - s(x), x + s(x)]$ and $x \in \mathbb{R}$, we have the inequalities

$$
c^{-1} \leq \frac{u(t)}{u(x)}, \quad \frac{v(t)}{v(x)} \leq c,
$$

$$
c^{-1}d(x) \leq s(x) \leq cd(x).
$$
Proof. Consider the function
\begin{equation}
F(s) = \int_{x-s}^{x+s} \frac{d\xi}{d(\xi)}, \quad s \geq 0.
\end{equation}
Clearly, \(F(0) = 0\), the function \(F(s)\) is monotone increasing and continuous on \([0, \infty)\), and, in addition, in view of (2.7) and (2.14), we have
\begin{equation}
F(s) = \int_{x-s}^{x+s} \frac{dt}{d(t)} \geq \frac{1}{4} \int_{x-s}^{x+s} \frac{dt}{\rho(t)} \to \infty \quad \text{as} \quad s \to \infty.
\end{equation}
Hence, for a given \(x \in \mathbb{R}\), equation (3.1) has a unique finite positive solution \(s(x)\). Further, from (2.8), (2.9) and (2.14), it follows that
\begin{equation}
\frac{v'(\xi)}{v(\xi)} = 1 + \frac{\rho'(\xi)}{2\rho(\xi)} \leq 1 + \frac{4}{\rho(\xi)} \leq \frac{4}{d(\xi)}, \quad \xi \in \mathbb{R}.
\end{equation}
For \(t \in [x, x+s(x)]\) this implies that
\begin{equation}
\ln \frac{v(t)}{v(x)} \leq 4 \int_x^t \frac{d\xi}{d(\xi)} \leq 4 \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{d(\xi)} = 8.
\end{equation}
According to (2.3), this gives
\begin{equation}
c^{-1} \leq 1 - \frac{v(t)}{v(x)} \leq c \quad \text{for} \quad t \in [x, x+s(x)], \quad x \in \mathbb{R}.
\end{equation}
Similarly, if \(t \in [x-s(x), x]\), then
\begin{equation}
\ln \frac{v(x)}{v(t)} \leq 4 \int_t^x \frac{d\xi}{d(\xi)} \leq 4 \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{d(\xi)} = 8
\end{equation}
\begin{equation}
\Rightarrow c^{-1} \leq 1 - \frac{v(x)}{v(t)} \leq c \quad \text{for} \quad t \in [x-s(x), x], \quad x \in \mathbb{R}.
\end{equation}
From (3.5) and (3.6) we obtain (3.2) for \(v\) (inequalities (3.2) for \(u\) are proved in a similar way). Further, by (3.1), (3.2) and (2.14), we get
\begin{equation}
2 = \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{d(\xi)} = \int_{x-s(x)}^{x+s(x)} \frac{\rho(\xi)}{d(\xi)} \cdot \frac{u(x)}{u(\xi)} \cdot \frac{v(x)}{v(\xi)} \cdot \frac{d(x)}{\rho(x)} \cdot \frac{d\xi}{d(x)} \geq \frac{1}{c} \cdot \frac{s(x)}{d(x)},
\end{equation}
\begin{equation}
2 = \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{d(\xi)} = \int_{x-s(x)}^{x+s(x)} \frac{\rho(\xi)}{d(\xi)} \cdot \frac{u(x)}{u(\xi)} \cdot \frac{v(x)}{v(\xi)} \cdot \frac{d(x)}{\rho(x)} \cdot \frac{d\xi}{d(x)} \leq \frac{s(x)}{d(x)}.
\end{equation}
\[ \square \]
Lemma 3.3. Suppose conditions (1.2) and (2.1) hold. Set
\begin{equation}
f_x(t) = \begin{cases} 
1, & \text{if} \quad t \in [x-s(x), x+s(x)], \\
0, & \text{if} \quad t \notin [x-s(x), x+s(x)], 
\end{cases}
\end{equation}
\begin{equation}
y_x(t) = \int_{-\infty}^{\infty} G(t, \xi)f_x(\xi)d\xi, \quad t \in \mathbb{R} \quad (\text{see} \quad 2.10). 
\end{equation}
Then we have the relations
\begin{equation}
-(y_x(t))'' + q(t)y_x(t) = f(t), \quad t \in \mathbb{R},
\end{equation}
\begin{equation}
\|y_x''\|_1 + \|qy_x\|_1 \leq cs(x), \quad x \in \mathbb{R}.
\end{equation}
Proof. From (2.10) and (3.8) it follows that

\begin{equation}
(3.11) \quad y_x(t) = u(t) \int_{-\infty}^{t} v(\xi)f_x(\xi)d\xi + v(t) \int_{t}^{\infty} u(\xi)f_x(\xi)d\xi.
\end{equation}

According to (3.7), integration in (3.11) is along segments which are contained in the finite segment \([x - s(x), x + s(x)\], and therefore the function \(y_x\) is well-defined. Equality (3.9) can be checked directly using Lemma 2.4. Further, from Fubini’s Theorem and (2.12), it follows that

\begin{equation}
(3.12) \quad \|qy\|_1 = \int_{-\infty}^{\infty} q(t) \left( \int_{-\infty}^{\infty} G(t, \xi)f_x(\xi)d\xi \right) dt = \int_{-\infty}^{\infty} q(t) \int_{-\infty}^{\infty} G(t, \xi)f_x(\xi)d\xi dt
\end{equation}

Estimate (3.10) follows from (3.12), (3.7), (3.9) and the triangle inequality for norms.

\textbf{Corollary 3.4.} Suppose conditions (1.2) and (2.1) hold. Let \(x \in \mathbb{R}\), and let \(y_x(t)\), \(t \in \mathbb{R}\), be the function defined in (3.8). Then we have the inequalities

\begin{equation}
(3.13) \quad y_x(t) \geq c^{-1}s^2(x) \quad \text{if} \quad |t - x| \leq s(x),
\end{equation}

\begin{equation}
(3.14) \quad \|y_x\|_p \geq c^{-1}s(x)^{2+1/p}, \quad x \in \mathbb{R}, \quad p \in [1, \infty).
\end{equation}

Proof. Now we use (3.11), (3.2), (3.3) and (2.14):

\begin{align*}
y_x(t) &= u(t) \int_{x-s(x)}^{t} v(\xi)d\xi + v(t) \int_{t}^{x+s(x)} u(\xi)d\xi \\
&= \rho(x) \left\{ \frac{u(t)}{u(x)} \int_{x-s(x)}^{t} \frac{v(\xi)}{v(x)} d\xi + \frac{v(t)}{v(x)} \int_{t}^{x+s(x)} \frac{u(\xi)}{u(x)} d\xi \right\} \\
&\geq \rho(x)s(x)c^{-1} \geq c^{-1}d(x)s(x) \geq c^{-1}s^2(x) \Rightarrow (3.13).
\end{align*}

Estimate (3.14) follows from (3.13):

\[ \|y_x\|_p \geq \left[ \int_{x-s(x)}^{x+s(x)} |y_x(t)|^p dt \right]^{1/p} \geq c^{-1}s(x)^{2+1/p}. \]

Let us now go to the proof of the theorem. Let \(p \in (1, \infty)\), and let \(W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_p(\mathbb{R})\). According to the facts proven above, we have (2.1), the functions \(d, s\) and \(y_x\) (see (3.8)) are defined, and \(y_x \in W_1^{(2)}(\mathbb{R}, q)\) in view of (3.10). This implies that

\[ cs(x) \geq \|(y_x)''\|_1 + \|qy_x\|_1 \geq c^{-1}\|y_x\|_p \geq c^{-1}s(x)^{2+1/p}. \]

Hence \(s(x) \leq c \leq \infty, x \in \mathbb{R}, \) and therefore \(d_0 \leq \infty\) (see (3.3), (2.16)). Then \(q_0(a) > 0\) for some \(a \in (0, \infty)\) (see Theorem 2.5).

\textbf{Proof of Theorem 1.2} Sufficiency.

\textbf{Lemma 3.5.} Suppose conditions (1.2) and (2.1) hold. Then for every \(x \in \mathbb{R}\), we have the equality

\begin{equation}
(3.15) \quad 4\sqrt{\rho(x)} = \int_{-\infty}^{\infty} \frac{G(x, \xi)}{\rho(\xi)^{3/2}} d\xi.
\end{equation}
Proof. The following elementary relations follow from Lemma 2.1:

\[
\left(\sqrt[2]{\frac{v(\xi)}{u(\xi)}}\right)' = \frac{1}{2} \sqrt[2]{\frac{u(\xi)}{v(\xi)}} \left(\frac{v'(\xi)u(\xi) - u'(\xi)v(\xi)}{v(\xi)^2}\right)
\]

\[
= \frac{1}{2} \sqrt[2]{\frac{1}{u(\xi)}} \frac{1}{u(\xi)} = \frac{1}{2} \frac{v(\xi)}{\rho(\xi)^{3/2}}, \quad \xi \in \mathbb{R}
\]

\[
\Rightarrow \sqrt[2]{\frac{v(x)}{u(x)}} = \frac{\int \rho(x) d\xi}{\int x \rho(x)^{3/2}} = \frac{1}{2} \frac{\int x \rho(x)^{3/2}}{\rho(x) d\xi}, \quad t \leq x
\]

\[
\Rightarrow \sqrt[2]{\rho(x)} = \frac{1}{2} \frac{\int x d\xi}{\rho(\xi)^{3/2}}, \quad x \in \mathbb{R}
\]

Similarly, one can check the equality

\[
\sqrt[2]{\rho(x)} = \frac{1}{2} \int \frac{G(x, \xi)d\xi}{\rho(\xi)^{3/2}}, \quad x \in \mathbb{R}.
\]

The latter two relations imply (3.15). \qed

Let us introduce the notation

\[
(3.16) \quad H_p = \sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} G(x, t)p dt, \quad p \in [1, \infty).
\]

Lemma 3.6. Suppose conditions (1.2) and (2.1) hold. Then we have the inequalities (see (2.16))

\[
(3.17) \quad c^{-1}d_0^2 \leq H_1 \leq cd_0^2.
\]

Proof. Let \( x \in \mathbb{R} \). Now we use Lemmas 2.2 and 3.2 and the estimate (3.13):

\[
\int_{-\infty}^{\infty} G(x, t) dt \geq \int_{x-s(x)}^{x+s(x)} G(x, t) dt = u(x) \int_{x-s(x)}^{x} v(t) dt + v(x) \int_{x}^{x+s(x)} u(t) dt
\]

\[
\geq c^{-1}s^2(x) \geq c^{-1}d^2(x) \quad \Rightarrow \quad H_1 \geq c^{-1}d_0^2.
\]

In particular, if \( d_0 = \infty \), then from the facts proven above it follows that \( H_1 = \infty \), and therefore in this case the upper estimate in (3.17) also holds. Now let \( d_0 < \infty \). In the following relations, we use (3.15) and (2.1):

\[
4\sqrt{d_0} \geq 4\sqrt{d(x)} \geq c^{-1}\sqrt{\rho(x)} = \frac{c^{-1}}{4} \int_{-\infty}^{\infty} \frac{G(x, \xi)d\xi}{\rho(\xi)^{3/2}}
\]

\[
\geq c^{-1} \int_{-\infty}^{\infty} \frac{G(x, \xi)d\xi}{\rho(\xi)^{3/2}} \geq \frac{c^{-1}}{d_0^{3/2}} \int_{-\infty}^{\infty} G(x, \xi)d\xi, \quad x \in \mathbb{R}.
\]

Hence \( H_1 \leq cd_0^2 \), as required. \qed

Corollary 3.7. Suppose conditions (1.2) and (2.1) hold, and let \( p \in (1, \infty) \). Then we have the inequalities

\[
(3.18) \quad c^{-1}(p)d_0^{p+1} \leq H_p \leq c(p)d_0^{p+1}.
\]
Proof. The lower estimate in (3.18) is proved exactly as the lower estimate in (3.17). Here, if \( d_0 = \infty \), then also \( H_p = \infty \), and then the upper estimate in (3.18) also holds. Now let \( d_0 < \infty \). The following relations are based on Lemma 2.1 and (3.17):

\[
\int_{-\infty}^{\infty} G(x, t)^p dt = u^p(x) \int_{-\infty}^{x} v^p(t) dt + v^p(x) \int_{x}^{\infty} u(t)^p dt \\
\leq u^p(x)v^{p-1}(x) \int_{-\infty}^{x} v(t) dt + v(x)^p u(x)^{p-1} \int_{x}^{\infty} u(t) dt \\
= \rho(x)^{p-1} \int_{-\infty}^{\infty} G(x, t) dt \leq c_p d(x)^{p-1} d_0^2 \leq c(p) d_0^{p+1}.
\]

This gives \( H_p \leq c(p) d_0^{p+1} \).

Let us now go to the proof of the theorem. Let \( q_0(a) > 0 \) for some \( a \in (0, \infty) \), and let \( y \in W^{(2)}_1(\mathbb{R}, q) \). Denote

\[
(3.19) \quad f(x) = -y''(x) + q(x)y(x), \quad x \in \mathbb{R}.
\]

Then, clearly, \( f \in L_1(\mathbb{R}) \) since

\[
(3.20) \quad \|f\|_1 \leq \|y''\|_1 + \|qy\|_1 < \infty.
\]

Besides, since \( W^{(2)}_1(\mathbb{R}, q) \hookrightarrow L_1(\mathbb{R}) \) (see Theorems 2.5 and 2.6), in view of the embedding, we have

\[
(3.21) \quad \|y\|_1 \leq c\{\|y''\|_1 + \|qy\|_1\} < \infty.
\]

Let us now consider the equation

\[
(3.22) \quad -\ddot{y}'' + q(x)\dot{y}(x) = f(x), \quad x \in \mathbb{R},
\]

where the function \( f \) is defined according to (3.19). By Theorem 2.5, equation (2.11) is correctly solvable in the space \( L_1(\mathbb{R}) \), and therefore (3.22) has a unique solution \( \bar{y} \in L_1(\mathbb{R}) \), and

\[
(3.23) \quad \|\bar{y}\|_1 \leq c\|f\|_1 \leq c\{\|y''\|_1 + \|qy\|_1\} < \infty.
\]

From (3.22) and (3.19) it follows that the function

\[
z(x) = \bar{y}(x) - y(x), \quad x \in \mathbb{R},
\]

is a solution of (2.2), and \( z \in L_1(\mathbb{R}) \) in view of (3.21) and (3.23). Hence \( z \equiv 0 \) by Lemma 2.1 and \( y \equiv \bar{y} \). Then from (2.18) we get

\[
(3.24) \quad y(x) \equiv \bar{y}(x) = \int_{-\infty}^{\infty} G(x, t) f(t) dt, \quad x \in \mathbb{R},
\]

where the function \( f \) is defined by (3.19). Since \( f \in L_1(\mathbb{R}) \) (see (3.20)) and, by Theorem 2.7, Corollary 3.7 and Theorem 2.5, the operator \( G : L_1(\mathbb{R}) \to L_p(\mathbb{R}) \) is bounded, we obtain from (3.24) that

\[
\|y\|_p = \|Gf\|_p \leq \|G\|_{L_1(\mathbb{R}) \to L_p(\mathbb{R})} \cdot \|f\|_1 \leq c(p) d_0^{1/p+1} \{\|y''\|_1 + \|qy\|_1\},
\]

as required. \( \square \)
References


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