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AN EMBEDDING THEOREM

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Abstract. We consider a weighted space $W_1^{(2)}(\mathbb{R}, q)$ of Sobolev type:

$$W_1^{(2)}(\mathbb{R}, q) = \left\{ y \in AC_{\mathrm{loc}}^{(1)}(\mathbb{R}) : \|y''\|_{L_1(\mathbb{R})} + \|qy\|_{L_1(\mathbb{R})} < \infty \right\},$$

where $0 \le q \in L_1^{\mathrm{loc}}(\mathbb{R})$ and

$$\|y\|_{W_1^{(2)}(\mathbb{R},q)} = \|y''\|_{L_1(\mathbb{R})} + \|qy\|_{L_1(\mathbb{R})}.$$

We obtain a precise condition which guarantees the embedding

$$W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_p(\mathbb{R}),\ p \in [1, \infty).$$

1. Introduction

In the present paper, we consider a weighted function space $W_1^{(2)}(\mathbb{R}, q)$ of Sobolev type:

$$W_1^{(2)}(\mathbb{R}, q) = \left\{ y \in AC_{\mathrm{loc}}^{(1)}(\mathbb{R}) : \|y''\|_{L_1(\mathbb{R})} + \|qy\|_{L_1(\mathbb{R})} < \infty \right\}. \tag{1.1}$$

Here and in the sequel, $\|f\|_p = \|f\|_{L_p(\mathbb{R})}$, $p \in [1, \infty)$, $AC_{\mathrm{loc}}^{(1)}(\mathbb{R})$, is the set of functions absolutely continuous together with the derivative on every finite interval.

In this paper we always require that

$$0 \le q \in L_1^{\mathrm{loc}}(\mathbb{R}) \tag{1.2}$$

and usually do not mention this requirement in the statements.

Our general goal is to find conditions under which there is an embedding

$$W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_p(\mathbb{R}).$$

To be more precise, we give the following definitions.

Definition 1.1 ([6])**.** Let $p \in [1, \infty)$. We say that the space $W_1^{(2)}(\mathbb{R}, q)$ is embedded into the space $L_p(\mathbb{R})$ (and write $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_p(\mathbb{R})$) if $W_1^{(2)}(\mathbb{R}, q) \subseteq L_p(\mathbb{R})$ and

$$\|y\|_p \le c(p)\{\|y''\|_1 + \|qy\|_1\}, \quad \forall y \in W_1^{(2)}(\mathbb{R}, q). \tag{1.3}$$

Our general convention is that the letters $c, c(\cdot)$ stand for absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations.

We now state our main result.

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Theorem 1.2. *Let $p \in [1,\infty)$. Then the embedding $W_1^{(2)}(\mathbb{R},q) \hookrightarrow L_p(\mathbb{R})$ takes place if and only if*

$$\exists a > 0: \quad q_0(a) \stackrel{def}{=} \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} q(t)dt > 0. \tag{1.4}$$

Note that for $p = 1$ this assertion was proved in [4] and will not be reconsidered here (see §2).

Example 1.3. Let

$$q(x) = 1 + \cos(|x|^\theta), \quad x \in \mathbb{R},\ \theta \in (0,\infty). \tag{1.5}$$

In [3] it was shown that in case (1.5) condition (1.4) holds if and only if $\theta \geq 1$. Together with Theorem 1.2, this implies that in case (1.5) the embedding $W_1^{(2)}(\mathbb{R},q) \hookrightarrow L_p(\mathbb{R})$, $p \in [1,\infty)$, takes place if and only if $\theta \geq 1$.

2. Preliminaries

In this section, we present some facts needed for the proofs (see §3).

Lemma 2.1 ([1, 2])**.** *Suppose conditions* (1.2) *and* (2.1) *hold:*

$$\int_{-\infty}^{x} q(t)dt > 0, \quad \int_{x}^{\infty} q(t)dt > 0, \quad \forall x \in \mathbb{R}. \tag{2.1}$$

Then there exists a fundamental system of solutions (FSS) $\{u,v\}$ of the equation

$$z''(x) = q(x)z(x), \quad x \in \mathbb{R}, \tag{2.2}$$

which has the following properties:

$$u(x) > 0, \quad v(x) > 0, \quad u'(x) \leq 0, \quad v'(x) \geq 0, \quad \forall x \in \mathbb{R}, \tag{2.3}$$

$$v'(x)u(x) - u'(x)v(x) = 1, \quad \forall x \in \mathbb{R}, \tag{2.4}$$

$$\lim_{x\to-\infty} \frac{v(x)}{u(x)} = \lim_{x\to\infty} \frac{u(x)}{v(x)} = 0. \tag{2.5}$$

The solutions $\{u,v\}$ and the function ρ,

$$\rho(x) = u(x)v(x), \quad x \in \mathbb{R}, \tag{2.6}$$

satisfy the relations

$$\int_{-\infty}^{0} \frac{dt}{\rho(t)} = \int_{0}^{\infty} \frac{dt}{\rho(t)} = \infty, \tag{2.7}$$

$$|\rho'(x)| < 1, \quad \forall x \in \mathbb{R}, \tag{2.8}$$

$$\frac{v'(x)}{v(x)} = \frac{1+\rho'(x)}{2\rho(x)}, \qquad \frac{u'(x)}{u(x)} = -\frac{1-\rho'(x)}{2\rho(x)}, \quad x \in \mathbb{R}. \tag{2.9}$$

In addition, if $G(x,t)$ is the Green function,

$$G(x,t) = \begin{cases} u(x)v(t), & x \geq t, \\ u(t)v(x), & x \leq t, \end{cases} \tag{2.10}$$

corresponding to the equation

$$-y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{2.11}$$

then we have the inequality

$$\sup_{t\in\mathbb{R}} \Phi(t) \le 1, \qquad \Phi(t) = \int_{-\infty}^{\infty} q(\xi)G(t,\xi)d\xi, \qquad t \in \mathbb{R}. \tag{2.12}$$

Finally, for every $p \in [1,\infty)$, equation (2.2) has no solutions $z \in L_p(\mathbb{R})$ apart from $z \equiv 0$.

Lemma 2.2 ([1]). *Suppose conditions* (1.2) *and* (2.1) *hold. For a given $x \in \mathbb{R}$, consider the equation in $d \ge 0$:*

$$d\int_{x-d}^{x+d} q(t)dt = 2. \tag{2.13}$$

Equation (2.13) *has a unique finite positive solution for every $x \in \mathbb{R}$. Denote this solution by $d(x)$. The function $d(x)$ is continuous for $x \in \mathbb{R}$.*

Note that the function d was introduced by M. Otelbaev (see [6]).

Theorem 2.3 ([1]). *Suppose conditions* (1.2) *and* (2.1) *hold. Then we have the inequalities*

$$\frac{d(x)}{4} \le \rho(x) \le \frac{3}{2}d(x), \qquad x \in \mathbb{R}. \tag{2.14}$$

Estimates of type (2.14) were first obtained (under some additional requirements to the function q) by Otelbaev (see [7]) and are therefore called Otelbaev's inequalities (see also [6]).

Consider equation (2.11) with $f \in L_p(\mathbb{R})$, $p \in [1,\infty)$. Below by a solution of (2.11) we mean any function $y \in AC^{(1)}_{\mathrm{loc}}(\mathbb{R})$ satisfying equality (2.11) almost everywhere on $\mathbb{R}$.

Definition 2.4 ([3]). We say that equation (2.11) is correctly solvable in the space $L_p(\mathbb{R})$, $p \in [1,\infty)$, if assertions I–II hold:

I) for any function $f \in L_p(\mathbb{R})$, there is a unique solution $y \in L_p(\mathbb{R})$ of (2.11);
II) the solution $y \in L_p(\mathbb{R})$ of (2.11) satisfies the estimate

$$\|y\|_p \le c(p)\|f\|_p, \quad \forall f \in L_p(\mathbb{R}). \tag{2.15}$$

Theorem 2.5 ([3]). *Let $p \in [1,\infty)$. Equation* (2.11) *is correctly solvable in $L_p(\mathbb{R})$ if and only if one of the following two equivalent conditions,* A, B, *holds:*

A) *One has inequalities* (2.1) *and $d_0 < \infty$. Here*

$$d_0 = \sup_{x\in\mathbb{R}} d(x). \tag{2.16}$$

B) *There is $a \in (0,\infty)$ such that $q_0(a) > 0$:*

$$q_0(a) = \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} q(t)dt. \tag{2.17}$$

In addition, if equation (2.11) *is correctly solvable in $L_p(\mathbb{R})$, $p \in [1,\infty)$, then its solution $y \in L_p(\mathbb{R})$ is of the form*

$$y(x) = (Gf)(x) \stackrel{def}{=} \int_{-\infty}^{\infty} G(x,t)f(t)dt, \quad x \in \mathbb{R}. \tag{2.18}$$

Theorem 2.6 ([4]). *The embedding $W_1^{(2)}(\mathbb{R},q) \hookrightarrow L_1(\mathbb{R})$ takes place if and only if equation* (2.11) *is correctly solvable in $L_1(\mathbb{R})$.*

Theorem 2.7 ([5, Ch. XI, §6])**.** *Let $K(x,t)$ be a continuous positive function in variables $x,t\in\mathbb{R}$, and let K be the integral operator*

$$(Kf)(x)=\int_{-\infty}^{\infty}K(x,t)f(t)dt,\quad \forall f\in L_1(\mathbb{R}). \tag{2.19}$$

Then for every $p\in(1,\infty)$, we have the equality

$$\|K\|_{L_1(\mathbb{R})\to L_p(\mathbb{R})}=\sup_{t\in\mathbb{R}}\left(\int_{-\infty}^{\infty}K(x,t)^p dx\right)^{1/p}. \tag{2.20}$$

3. Proofs

Proof of Theorem 1.2. *Necessity.*

Lemma 3.1. *Let $W_1^{(2)}(\mathbb{R},q)\hookrightarrow L_p(\mathbb{R})$, $p\in[1,\infty)$. Then relations* (2.1) *hold.*

Proof. Assume the contrary, say,

$$\int_{x_0}^{\infty}q(t)dt=0,$$

for some $x_0\in\mathbb{R}$. Let $\varphi\in C^\infty(\mathbb{R})$ be such that $\operatorname{supp}\varphi=[x_0,\infty)$, $0\le\varphi\le 1$, for $x\in\mathbb{R}$ and $\varphi(x)\equiv 1$ for $x\ge x_0+1$. Clearly, $\varphi\in AC^{(1)}_{\mathrm{loc}}(\mathbb{R})$ and, in addition,

$$\int_{-\infty}^{\infty}|q(t)\varphi(t)|dt=\int_{x_0}^{\infty}q(t)\varphi(t)dt=0,$$
$$0<\int_{-\infty}^{\infty}|\varphi''(t)|dt=\int_{x_0}^{x_0+1}|\varphi''(t)|dt=c<\infty.$$

Hence $\varphi\in W_1^{(2)}(\mathbb{R},q)$, and since $W_1^{(2)}(\mathbb{R},q)\hookrightarrow L_p(\mathbb{R})$, we have

$$\infty>c\left\{\int_{-\infty}^{\infty}|\varphi''(t)|dt+\int_{-\infty}^{\infty}|q(t)\varphi(t)|dt\right\}\ge\left(\int_{-\infty}^{\infty}|\varphi(t)|^p dt\right)\ge\left(\int_{x_0+1}^{\infty}1dt\right)^{1/p}$$
$$=\infty.$$

Contradiction. □

Below, conditions (1.2) and (2.1) are assumed to be valid and do not appear in the statements.

Lemma 3.2. *For a given $x\in\mathbb{R}$, consider the equation in $s\ge 0$:*

$$\int_{x-s}^{x+s}\frac{dt}{d(t)}=2. \tag{3.1}$$

Equation (3.1) *has a unique finite positive solution for every $x\in\mathbb{R}$. Denote it by $s(x)$. Then for every $t\in[x-s(x),x+s(x)]$ and $x\in\mathbb{R}$, we have the inequalities*

$$c^{-1}\le\frac{u(t)}{u(x)},\quad \frac{v(t)}{v(x)}\le c, \tag{3.2}$$

$$c^{-1}d(x)\le s(x)\le cd(x). \tag{3.3}$$

Proof. Consider the function

$$F(s) = \int_{x-s}^{x+s} \frac{d\xi}{d(\xi)}, \qquad s \geq 0. \tag{3.4}$$

Clearly, $F(0) = 0$, the function $F(s)$ is monotone increasing and continuous on $[0, \infty)$, and, in addition, in view of (2.7) and (2.14), we have

$$F(s) = \int_{x-s}^{x+s} \frac{dt}{d(t)} \geq \frac{1}{4} \int_{x-s}^{x+s} \frac{dt}{\rho(t)} \to \infty \quad \text{as} \quad s \to \infty.$$

Hence, for a given $x \in \mathbb{R}$, equation (3.1) has a unique finite positive solution $s(x)$. Further, from (2.8), (2.9) and (2.14), it follows that

$$\frac{v'(\xi)}{v(\xi)} = \frac{1 + \rho'(\xi)}{2\rho(\xi)} \leq \frac{1}{\rho(\xi)} \leq \frac{4}{d(\xi)}, \quad \xi \in \mathbb{R}.$$

For $t \in [x, x + s(x)]$ this implies that

$$\ln \frac{v(t)}{v(x)} \leq 4 \int_x^t \frac{d\xi}{d(\xi)} \leq 4 \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{d(\xi)} = 8.$$

According to (2.3), this gives

$$c^{-1} \leq 1 \leq \frac{v(t)}{v(x)} \leq c \quad \text{for} \quad t \in [x, x + s(x)], \quad x \in \mathbb{R}. \tag{3.5}$$

Similarly, if $t \in [x - s(x), x]$, then

$$\ln \frac{v(x)}{v(t)} \leq 4 \int_t^x \frac{d\xi}{d(\xi)} \leq 4 \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{d(\xi)} = 8$$

$$\Rightarrow c^{-1} \leq 1 \leq \frac{v(x)}{v(t)} \leq c \quad \text{for} \quad t \in [x - s(x), x], \quad x \in \mathbb{R}. \tag{3.6}$$

From (3.5) and (3.6) we obtain (3.2) for v (inequalities (3.2) for u are proved in a similar way). Further, by (3.1), (3.2) and (2.14), we get

$$2 = \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{d(\xi)} = \int_{x-s(x)}^{x+s(x)} \frac{\rho(\xi)}{d(\xi)} \cdot \frac{u(x)}{u(\xi)} \cdot \frac{v(x)}{v(\xi)} \cdot \frac{d(x)}{\rho(x)} \cdot \frac{d\xi}{d(x)} \geq \frac{1}{c} \cdot \frac{s(x)}{d(x)},$$

$$2 = \int_{x-s(x)}^{x+s(x)} \frac{d\xi}{d(\xi)} = \int_{x-s(x)}^{x+s(x)} \frac{\rho(\xi)}{d(\xi)} \cdot \frac{u(x)}{u(\xi)} \cdot \frac{v(x)}{v(\xi)} \cdot \frac{d(x)}{\rho(x)} \cdot \frac{d\xi}{d(x)} \leq c \frac{s(x)}{d(x)}.$$

□

Lemma 3.3. *Suppose conditions* (1.2) *and* (2.1) *hold. Set*

$$f_x(t) = \begin{cases} 1, & \text{if} \quad t \in [x - s(x), x + s(x)], \\ 0, & \text{if} \quad t \notin [x - s(x), x + s(x)], \end{cases} \tag{3.7}$$

$$y_x(t) = \int_{-\infty}^{\infty} G(t, \xi) f_x(\xi) d\xi, \quad t \in \mathbb{R} \quad (\textit{see } (2.10)). \tag{3.8}$$

Then we have the relations

$$-(y_x(t))'' + q(t) y_x(t) = f(t), \quad t \in \mathbb{R}, \tag{3.9}$$

$$\|(y_x)''\|_1 + \|q y_x\|_1 \leq c s(x), \quad x \in \mathbb{R}. \tag{3.10}$$

Proof. From (2.10) and (3.8) it follows that

$$y_x(t) = u(t)\int_{-\infty}^{t} v(\xi)f_x(\xi)d\xi + v(t)\int_{t}^{\infty} u(\xi)f_x(\xi)d\xi. \tag{3.11}$$

According to (3.7), integration in (3.11) is along segments which are contained in the finite segment $[x-s(x), x+s(x)]$, and therefore the function y_x is well-defined. Equality (3.9) can be checked directly using Lemma 2.1. Further, from Fubini's Theorem and (2.12), it follows that

$$\begin{aligned}\|qy\|_1 &= \int_{-\infty}^{\infty} q(t)\left|\int_{-\infty}^{\infty} G(t,\xi)f_x(\xi)d\xi\right| dt = \int_{-\infty}^{\infty} q(t)\int_{-\infty}^{\infty} G(t,\xi)f_x(\xi)d\xi dt \\ &= \int_{-\infty}^{\infty} f_x(\xi)\left[\int_{-\infty}^{\infty} q(t)G(t,\xi)dt\right]d\xi \le \int_{-\infty}^{\infty} f_x(\xi)d\xi = 2s(x). \end{aligned} \tag{3.12}$$

Estimate (3.10) follows from (3.12), (3.7), (3.9) and the triangle inequality for norms. □

Corollary 3.4. *Suppose conditions* (1.2) *and* (2.1) *hold. Let* $x \in \mathbb{R}$, *and let* $y_x(t)$, $t \in \mathbb{R}$, *be the function defined in* (3.8). *Then we have the inequalities*

$$y_x(t) \ge c^{-1}s^2(x) \qquad \text{if} \qquad |t-x| \le s(x), \tag{3.13}$$

$$\|y_x\|_p \ge c^{-1}s(x)^{2+1/p}, \quad x \in \mathbb{R}, \quad p \in [1,\infty). \tag{3.14}$$

Proof. Now we use (3.11), (3.2), (3.3) and (2.14):

$$\begin{aligned} y_x(t) &= u(t)\int_{x-s(x)}^{t} v(\xi)d\xi + v(t)\int_{t}^{x+s(x)} u(\xi)d\xi \\ &= \rho(x)\left\{\frac{u(t)}{u(x)}\int_{x-s(x)}^{t}\frac{v(\xi)}{v(x)}d\xi + \frac{v(t)}{v(x)}\int_{t}^{x+s(x)}\frac{u(\xi)}{u(x)}d\xi\right\} \\ &\ge \rho(x)s(x)c^{-1} \ge c^{-1}d(x)s(x) \ge c^{-1}s^2(x) \;\Rightarrow\; (3.13). \end{aligned}$$

Estimate (3.14) follows from (3.13):

$$\|y_x\|_p \ge \left[\int_{x-s(x)}^{x+s(x)} |y_x(t)|^p dt\right]^{1/p} \ge c^{-1}s(x)^{2+1/p}.$$

□

Let us now go to the proof of the theorem. Let $p \in (1,\infty)$, and let $W_1^{(2)}(\mathbb{R},q) \hookrightarrow L_p(\mathbb{R})$. According to the facts proven above, we have (2.1), the functions d, s and y_x (see (3.8)) are defined, and $y_x \in W_1^{(2)}(\mathbb{R},q)$ in view of (3.10). This implies that

$$cs(x) \ge \|(y_x)''\|_1 + \|qy_x\|_1 \ge c^{-1}\|y_x\|_p \ge c^{-1}s(x)^{2+1/p}.$$

Hence $s(x) \le c < \infty$, $x \in \mathbb{R}$, and therefore $d_0 < \infty$ (see (3.3), (2.16)). Then $q_0(a) > 0$ for some $a \in (0,\infty)$ (see Theorem 2.5).

Proof of Theorem 1.2. *Sufficiency.*

Lemma 3.5. *Suppose conditions* (1.2) *and* (2.1) *hold. Then for every* $x \in \mathbb{R}$, *we have the equality*

$$4\sqrt{\rho(x)} = \int_{-\infty}^{\infty}\frac{G(x,\xi)d\xi}{\rho(\xi)^{3/2}}. \tag{3.15}$$

Proof. The following elementary relations follow from Lemma 2.1:

$$\begin{aligned}
\left(\sqrt{\frac{v(\xi)}{u(\xi)}}\right)' &= \frac{1}{2}\sqrt{\frac{u(\xi)}{v(\xi)}}\frac{v'(\xi)u(\xi)-u'(\xi)v(\xi)}{u^2(\xi)} \\
&= \frac{1}{2\sqrt{v(\xi)}}\frac{1}{u(\xi)^{3/2}} = \frac{1}{2}\frac{v(\xi)}{\rho(\xi)^{3/2}}, \quad \xi\in\mathbb{R} \\
&\Rightarrow \sqrt{\frac{v(x)}{u(x)}} - \sqrt{\frac{v(t)}{u(t)}} = \frac{1}{2}\int_t^x \frac{v(\xi)d\xi}{\rho(\xi)^{3/2}}, \quad t\le x \\
&\Rightarrow \sqrt{\frac{v(x)}{u(x)}} = \sqrt{\frac{v(x)}{u(x)}} - \lim_{t\to-\infty}\sqrt{\frac{v(t)}{u(t)}} = \frac{1}{2}\int_{-\infty}^x \frac{v(\xi)d\xi}{\rho(\xi)^{3/2}} \\
&\Rightarrow \sqrt{\rho(x)} = \frac{1}{2}u(x)\int_{-\infty}^x \frac{v(\xi)d\xi}{\rho(\xi)^{3/2}} = \frac{1}{2}\int_{-\infty}^x \frac{G(x,\xi)d\xi}{\rho(\xi)^{3/2}}, \quad x\in\mathbb{R}.
\end{aligned}$$

Similarly, one can check the equality

$$\sqrt{\rho(x)} = \frac{1}{2}\int_x^\infty \frac{G(x,\xi)d\xi}{\rho(\xi)^{3/2}}, \quad x\in\mathbb{R}.$$

The latter two relations imply (3.15). □

Let us introduce the notation

$$H_p = \sup_{x\in\mathbb{R}}\int_{-\infty}^\infty G(x,t)^p dt, \quad p\in[1,\infty). \tag{3.16}$$

Lemma 3.6. *Suppose conditions* (1.2) *and* (2.1) *hold. Then we have the inequalities (see* (2.16)*)*

$$c^{-1}d_0^2 \le H_1 \le cd_0^2. \tag{3.17}$$

Proof. Let $x\in\mathbb{R}$. Now we use Lemmas 2.2, 3.2, and the estimate (3.13):

$$\begin{aligned}
\int_{-\infty}^\infty G(x,t)dt &\ge \int_{x-s(x)}^{x+s(x)} G(x,t)dt = u(x)\int_{x-s(x)}^x v(t)dt + v(x)\int_x^{x+s(x)} u(t)dt \\
&\ge c^{-1}s^2(x) \ge c^{-1}d^2(x) \quad \Rightarrow \quad H_1 \ge c^{-1}d_0^2.
\end{aligned}$$

In particular, if $d_0=\infty$, then from the facts proven above it follows that $H_1=\infty$, and therefore in this case the upper estimate in (3.17) also holds. Now let $d_0<\infty$. In the following relations, we use (3.15) and (2.14):

$$\begin{aligned}
4\sqrt{d_0} \ge 4\sqrt{d(x)} \ge c^{-1}\sqrt{\rho(x)} &= \frac{c^{-1}}{4}\int_{-\infty}^\infty \frac{G(x,\xi)d\xi}{\rho(\xi)^{3/2}} \\
&\ge c^{-1}\int_{-\infty}^\infty \frac{G(x,\xi)d\xi}{d(\xi)^{3/2}} \ge \frac{c^{-1}}{d_0^{3/2}}\int_{-\infty}^\infty G(x,\xi)d\xi, \quad x\in\mathbb{R}.
\end{aligned}$$

Hence $H_1\le cd_0^2$, as required. □

Corollary 3.7. *Suppose conditions* (1.2) *and* (2.1) *hold, and let* $p\in(1,\infty)$. *Then we have the inequalities*

$$c^{-1}(p)d_0^{p+1} \le H_p \le c(p)d_0^{p+1}. \tag{3.18}$$

Proof. The lower estimate in (3.18) is proved exactly as the lower estimate in (3.17). Here, if $d_0 = \infty$, then also $H_p = \infty$, and then the upper estimate in (3.18) also holds. Now let $d_0 < \infty$. The following relations are based on Lemma 2.1, (2.14) and (3.17):

$$\begin{aligned}\int_{-\infty}^{\infty} G(x,t)^p dt &= u^p(x)\int_{-\infty}^{x} v^p(t)dt + v^p(x)\int_{x}^{\infty} u(t)^p dt \\ &\le u^p(x)v^{p-1}(x)\int_{-\infty}^{x} v(t)dt + v(x)^p u(x)^{p-1}\int_{x}^{\infty} u(t)dt \\ &= \rho(x)^{p-1}\int_{-\infty}^{\infty} G(x,t)dt \le c_p d(x)^{p-1} d_0^2 \le c(p) d_0^{p+1}.\end{aligned}$$

This gives $H_p \le c(p) d_0^{p+1}$. □

Let us now go to the proof of the theorem. Let $q_0(a) > 0$ for some $a \in (0,\infty)$, and let $y \in W_1^{(2)}(\mathbb{R}, q)$. Denote

$$f(x) = -y''(x) + q(x)y(x), \qquad x \in \mathbb{R}. \tag{3.19}$$

Then, clearly, $f \in L_1(\mathbb{R})$ since

$$\|f\|_1 \le \|y''\|_1 + \|qy\|_1 < \infty. \tag{3.20}$$

Besides, since $W_1^{(2)}(\mathbb{R}, q) \hookrightarrow L_1(\mathbb{R})$ (see Theorems 2.5 and 2.6), in view of the embedding, we have

$$\|y\|_1 \le c\{\|y''\|_1 + \|qy\|_1\} < \infty. \tag{3.21}$$

Let us now consider the equation

$$-\tilde{y}'' + q(x)\tilde{y}(x) = f(x), \qquad x \in \mathbb{R}, \tag{3.22}$$

where the function f is defined according to (3.19). By Theorem 2.5, equation (2.11) is correctly solvable in the space $L_1(\mathbb{R})$, and therefore (3.22) has a unique solution $\tilde{y} \in L_1(\mathbb{R})$, and

$$\|\tilde{y}\|_1 \le c\|f\|_1 \le c\{\|y''\|_1 + \|qy\|_1\} < \infty. \tag{3.23}$$

From (3.22) and (3.19) it follows that the function

$$z(x) = \tilde{y}(x) - y(x), \qquad x \in \mathbb{R},$$

is a solution of (2.2), and $z \in L_1(\mathbb{R})$ in view of (3.21) and (3.23). Hence $z \equiv 0$ by Lemma 2.1 and $y \equiv \tilde{y}$. Then from (2.18) we get

$$y(x) \equiv \tilde{y}(x) = \int_{-\infty}^{\infty} G(x,t)f(t)dt, \quad x \in \mathbb{R}, \tag{3.24}$$

where the function f is defined by (3.19). Since $f \in L_1(\mathbb{R})$ (see (3.20)) and, by Theorem 2.7, Corollary 3.7 and Theorem 2.5, the operator $G : L_1(\mathbb{R}) \to L_p(\mathbb{R})$ is bounded, we obtain from (3.24) that

$$\|y\|_p = \|Gf\|_p \le \|G\|_{L_1(\mathbb{R})\to L_p(\mathbb{R}} \cdot \|f\|_1 \le c(p) d_0^{1+1/p}\{\|y''\|_1 + \|qy\|_1\},$$

as required. □

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