PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 10, October 2013, Pages 3447–3451
S 0002-9939(2013)11599-6
Article electronically published on June 5, 2013

THE KALTON CENTRALIZER ON $L_p[0,1]$ IS NOT STRICTLY SINGULAR

JESÚS SUÁREZ DE LA FUENTE

(Communicated by Thomas Schlumprecht)

ABSTRACT. We prove that the Kalton centralizer on $L_p[0,1]$, for $0 < p < \infty$, is not strictly singular: in all cases there is a Hilbert subspace on which it is trivial. Moreover, for $0 < p < 2$ there are copies of ℓ_q, with $p < q < 2$, on which it becomes trivial. This is in contrast to the situation for ℓ_p spaces, in which the Kalton-Peck centralizer is strictly singular.

1. INTRODUCTION

In [7, Cor. 4.8, Thm. 6.4] Kalton and Peck obtained an extremal solution to the 3-space problem for Hilbert spaces by showing the existence of a non-trivial exact sequence

$$0 \longrightarrow \ell_2 \xrightarrow{\ j\ } Z_2 \xrightarrow{\ q\ } \ell_2 \longrightarrow 0$$

in which the quotient map q is strictly singular. Recall that an operator $q : Z \to Y$ acting between quasi-Banach spaces is said to be strictly singular if its restriction to any infinite dimensional subspace of Z is not an isomorphism. The correspondence between exact sequences $0 \to Y \to X \to Z \to 0$ and quasi-linear maps Ω from Z to Y (see the Preliminaries section below) allows one to naturally define [4, Prop. 3] a quasi-linear map Ω (from Z to Y) to be strictly singular if its restrictions to infinite dimensional subspaces of Z are never trivial. Of course, strictly singular quasi-linear maps correspond to exact sequences with a strictly singular quotient map. The Kalton quasi-linear map is given on $L_p(\mu)$ spaces, roughly speaking, by the rule

$$\Omega_p(f) = f \log \frac{|f|}{\|f\|_p},$$

for $f \in L_p(\mu)$. If μ is the counting measure on $\mathbb{N}$, then $L_p(\mu) = \ell_p$ and Ω_p is known as the Kalton-Peck centralizer. For this choice of μ, Kalton and Peck proved that Ω_p is strictly singular if $1 < p < \infty$ [7, Thm. 6.4]. Thus, we focus on when μ is the Lebesgue measure on the unit interval. In this case, Ω_p fails to be strictly singular for $0 < p < \infty$, in contrast to the previous situation. In all cases the centralizer is trivial when restricted to the copy of ℓ_2 spanned by the Rademacher (or Gaussian) functions. Moreover, for $0 < p < q < 2$, the centralizer is also trivial when restricted to the copy of ℓ_q spanned in L_p by a sequence of independent q-stable random variables.

Received by the editors March 31, 2011 and, in revised form, October 20, 2011, October 24, 2011, November 16, 2011 and December 8, 2011.

2010 *Mathematics Subject Classification.* Primary 46B20, 46B07, 46A16.

The author was partially supported by MTM2010-20190-C02-01 and Junta de Extremadura CR10113 "IV Plan Regional I+D+i", Ayudas a Grupos de Investigación.

2. Preliminaries

The basic theory of quasi-linear maps and twisted sums of quasi-Banach spaces can be seen in [3, 7]. Let us recall a few basic facts. An exact sequence of quasi-Banach spaces is a diagram

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0$$

formed by quasi-Banach spaces and linear continuous operators such that the image of each arrow coincides with the kernel of the next one. The middle space X is also called a *twisted sum* of Y and Z. It follows from the definition that Y is a subspace of X and, thanks to the open mapping theorem, Z is isomorphic to X/Y. The sequence is said to be trivial if $j(Y)$ is complemented in X, in which case X is isomorphic to $Y \oplus Z$. The theory developed by Kalton [5] and Kalton and Peck [7] establishes that there is a correspondence between exact sequences (or twisted sums) and *quasi-linear* maps. By a quasi-linear map Ω from Z to Y, we mean a map satisfying

$$\|\Omega(z+z') - \Omega(z) - \Omega(z')\|_Y \leq K(\|z\| + \|z'\|),$$

for some constant K and all $z, z' \in Z$. The correspondence between X (algebraically $Y \oplus Z$) and Ω is clearly expressed by the formula

$$\|(y,z)\|_X = \|y - \Omega z\|_Y + \|z\|_Z.$$

We usually write $X = Y \oplus_\Omega Z$. With this correspondence, Ω is *trivial* ([3, Corollary 1.5.e]) if and only if there is a linear map (not necessarily bounded) $L : Z \to Y$ such that

$$\sup_{z \in B(Z)} \|\Omega(z) - L(z)\| < \infty.$$

In other words, Ω is trivial if and only if it is at a finite distance from a linear map (see [3, page 15] for the precise definition of distance).

Although Kalton-Peck construction can be developed for certain class of spaces, we are only interested in $L_p(\mu)$ spaces. This is enough for our purposes. The Kalton quasi-linear map is given for $f \in L_p(\mu)$ by the formula

$$\Omega_p(f) = f \log \frac{|f|}{\|f\|_p}, \tag{2.1}$$

with the convention $0 \cdot \log 0 = 0$. If μ is the counting measure on $\mathbb{N}$, then $L_p(\mu) = \ell_p$ and Ω_p is known as the Kalton-Peck centralizer. In what follows, Ω_p will denote the map given by (2.1) and L_p will denote $L_p[0,1]$ endowed with the Lebesgue measure. The map Ω_p has an extra property on L_p: it is an L_∞-centralizer (see [6] for an excellent exposition about centralizers). This means basically that the expression

$$\|\Omega_p(af) - a\Omega_p(f)\|_p \leq K\|a\|_\infty \|f\|_p$$

holds for some constant K, every $a \in L_\infty$ and $f \in L_p$. In practice, this property reveals that the corresponding twisted sum $L_p \oplus_{\Omega_p} L_p$ is an L_∞-module.

3. The Hilbert copy

Kalton and Peck proved that their centralizer is strictly singular on ℓ_p for $1 < p < +\infty$. Furthermore, it has been proved in [2] that the same holds true for ℓ_p with $0 < p < \infty$. The situation for L_p is the opposite:

Proposition 3.1. *Let Ω_p be the Kalton centralizer (2.1) on L_p for $0 < p < \infty$. Then Ω_p is trivial when restricted to the copy of ℓ_2 spanned in L_p by the Rademacher (or Gaussian) functions.*

Proof. We perform the Rademacher case. The embedding $j : \ell_2 \to L_p$ is given for a point $a = (a_i)_{i=1}^{\infty}$ by the rule

$$j(a) = \sum_{i=1}^{\infty} r_i a_i.$$

That j is an into isomorphism follows from Khintchine's inequality. Fix $0 < p < \infty$ and a point such that $\|j(a)\|_p = 1$. Let us consider the random variable $f : [0,1] \to \mathbb{R}$ given by

$$f(t) = \left|\sum_{i=1}^{\infty} r_i(t)a_i\right|^p \left|\log\left|\sum_{i=1}^{\infty} r_i(t)a_i\right|\right|^p$$

if $\sum_{i=1}^{\infty} r_i(t)a_i$ converges and $f(t) = 0$ otherwise. It is a routine calculation to check that

$$\|\Omega_p(ja)\|_p^p = \mathbb{E}_\mu f,$$

where μ denotes the Lebesgue measure. So let us write

$$\mathbb{E}_\mu f = \int_0^1 |f(t)| d\mu(t) = \int_0^{\infty} \mu\{t \in [0,1] : |f(t)| \geq s\} ds. \tag{3.1}$$

For the last equality, see [10, (13.8)]. Now observe that if $|f(t)| > s$ for $s \geq 1$, then $|\sum_{i=1}^{\infty} r_i(t)a_i| > 1$ and thus $\log|\sum_{i=1}^{\infty} r_i(t)a_i| > 0$. Consequently, from $|f(t)| > s$ for $s \geq 1$ it must be $|\sum_{i=1}^{\infty} r_i(t)a_i| \geq s^{\frac{1}{2p}}$. Therefore, we deduce that

$$\mathbb{E}_\mu f \leq 1 + \int_1^{\infty} \mu\left\{t \in [0,1] : \left|\sum_{i=1}^{\infty} r_i(t)a_i\right| \geq s^{\frac{1}{2p}}\right\} ds. \tag{3.2}$$

Since $0 < c \leq \|a\|_2 \leq C < \infty$, the bound (3.2) and the exponential decay of the tails of the distribution of the sums of Rademacher functions [8, (4.1)] yield the upper bound:

$$1 + 2\int_1^{\infty} \exp\left\{-\frac{s^{\frac{1}{p}}}{2C^2}\right\} ds. \tag{3.3}$$

The change of variable $u = s^{\frac{1}{p}}$ shows that the expression (3.3) is a constant that can be estimated in terms of the Γ function. All together this yields, by homogenity, that we have for any $a \in \ell_2$:

$$\|\Omega_p(ja)\|_p \leq K_p\|a\|_{\ell_2}. \tag{3.4}$$

This means, in particular, that $\Omega_p \circ j$ is a finite distance (see [3, page 15]) from the linear map identically 0 and thus $\Omega_p \circ j$ is trivial by using [3, Corollary 1.5.e]. The Gaussian case is analogous since the proof is based on the exponential behaviour of the tails of the distribution [8]. $\square$

4. The ℓ_q copies for $0 < p < q < 2$

There is an analogue of Proposition 3.1 replacing ℓ_2 copies by ℓ_q copies whenever $0 < p < q < 2$.

Proposition 4.1. *Let Ω_p be the Kalton centralizer on L_p for $0 < p < 2$. For every $p < q < 2$, Ω_p is trivial when restricted to the copy of ℓ_q spanned in L_p by a sequence of independent and identically distributed q-stable random variables.*

Proof. Take θ to be a q-stable random variable and denote by $(\theta_i)_{i=1}^{\infty}$ a sequence of independent copies. It is well known (see e.g. [1, Chapter 6] or the corollary appearing in [11, p. 94]) that $\|\sum_{i=1}^{\infty} a_i\theta_i\|_p = \|\theta_1\|_p\,(\sum_{i=1}^{\infty}|a_i|^q)^{\frac{1}{q}}$, so the map $j : \ell_q \to L_p[0,1]$ given by $j(a) = \sum_{i=1}^{\infty} a_i\theta_i$ is an into isomorphism. Assume $\|\sum_{i=1}^{\infty} a_i\theta_i\|_p = 1$; then observe that

$$\left\|\Omega\left(\sum_{i=1}^{\infty} a_i\theta_i\right)\right\|_p^p = \int_0^{\infty} \mu\left\{\left|\sum_{i=1}^{\infty} a_i\theta_i\right|^p \left|\log\left|\sum_{i=1}^{\infty} a_i\theta_i\right|\right|^p > t\right\} dt. \tag{4.1}$$

For $t > 1$ it must be $|\log|\sum_{i=1}^{\infty} a_i\theta_i|| = \log|\sum_{i=1}^{\infty} a_i\theta_i|$. On the other hand, notice that for any $\alpha > 1$ there exists t_α depending only on α such that $\log t \leq \sqrt[\alpha]{t}$ for all $t \geq t_\alpha$. Given q, since $\frac{q}{p} > 1$, pick $\alpha > 1$ such that $\frac{q}{p} > \alpha$.

Claim. $\forall \alpha > 1 \exists t_\alpha > 1 : \forall t \geq t_\alpha, s\log s > t \Longrightarrow s > \sqrt[\alpha]{t}$.

To prove the claim, assume to the contrary that

$$\exists \alpha > 1 : \forall T > 1, \exists t \geq T, \exists s > 0 : s\log s > t \text{ but } s \leq \sqrt[\alpha]{t}.$$

Using the expressions $s\log s > t$ and $s \leq \sqrt[\alpha]{t}$ we find that

$$t^\alpha < s^\alpha (\log s)^\alpha \leq \left(\frac{1}{\alpha}\right)^\alpha t(\log t)^\alpha,$$

and thus

$$t^{\frac{\alpha-1}{\alpha}} \leq \frac{1}{\alpha}\log t. \tag{4.2}$$

Making T range over, for example, the natural numbers, one may construct sequences $\{t_n\}_{n=1}^{\infty}$ (with $t_n \to \infty$) and $\{s_n\}_{n=1}^{\infty}$, verifying that $s_n \log s_n > t_n$ but $s_n \leq \sqrt[\alpha]{t_n}$. Consequently, using (4.2) we find that

$$t_n^{\frac{\alpha-1}{\alpha}} \leq \frac{1}{\alpha}\log t_n,$$

which is impossible because $t_n \to \infty$. So the claim is proved. According to the claim, we may bound the integral (4.1) as

$$t_\alpha + \int_{t_\alpha}^{\infty} \mu\left\{\left|\sum_{i=1}^{\infty} a_i\theta_i\right| > t^{\frac{1}{\alpha p}}\right\} dt. \tag{4.3}$$

To conclude we recall that the distribution of $\sum_{i=1}^{\infty} a_i\theta_i$ is the same as the distribution of $(\sum_{i=1}^{\infty}|a_i|^q)^{\frac{1}{q}}\,\theta_1$ (see [8, Chapter 5] or [9, 8.1]). Furthermore, we have that $\mu\{|\theta_1| > t\} \leq ct^{-q}$, where $t > 0$ and c depends only on q ([9, 8.1.2]). All together this yields the following upper bound for (4.3):

$$t_\alpha + c' \int_{t_\alpha}^{\infty} t^{-\frac{q}{\alpha p}}\,dt.$$

Since by our choice $\frac{q}{\alpha p} > 1$, the integral above converges. By homogenity we are done. Notice that α (and thus t_α) depends only on the choice of q. $\square$

Acknowledgements

The author thanks Félix Cabello for some very helpful conversations on the paper. He also informed the author that Nigel J. Kalton has already proved (unpublished) Proposition 3.1. He also thanks the referee for good comments.

References

1. F. Albiac, N. J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, 233. Springer-Verlag. MR2192298 (2006h:46005)
2. F. Cabello, J. M. F. Castillo, J. Suárez, *On strictly singular non-linear centralizers*, Nonlinear Anal. 75 (2012), no. 7, 3313-3321.
3. J. M. F. Castillo, M. González, *Three-space problems in Banach space theory*, Lecture Notes in Mathematics, 1667. Springer-Verlag, Berlin, 1997. MR1482801 (99a:46034)
4. J. M. F. Castillo, Y. Moreno, *Singular and cosingular exact sequences of quasi-Banach spaces*, Arch. Math. 88 (2007), No. 2, 123-132. MR2299035 (2008b:46004)
5. N. J. Kalton, *Differentials of complex interpolation processes for Köthe function spaces*, Trans. Amer. Math. Soc. 333 (1992), No. 2, 479-529. MR1081938 (92m:46111)
6. N. J. Kalton, *Nonlinear commutators in interpolation theory*, Mem. Amer. Math. Soc. 73 (1988), No. 385. MR938889 (89h:47104)
7. N. J. Kalton, N. T. Peck, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. 255 (1979), 1–30. MR542869 (82g:46021)
8. M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 23. Springer-Verlag. MR2814399
9. V. D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Mathematics, 1200. Springer-Verlag, Berlin, 1986. MR856576 (87m:46038)
10. R. L. Schilling, *Measures, integrals and martingales*. Cambridge University Press, New York, 2005. MR2200059 (2007g:28001)
11. P. Wojtaszczyk, *Banach spaces for analysts*. Cambridge Studies in Advances Mathematics, 25. Cambridge University Press, Cambridge, 1991. MR1144277 (93d:46001)

Escuela Politécnica, Universidad de Extremadura, Avenida Universidad s/n, 10071 Cáceres, Spain

E-mail address: jesus@unex.es