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ON THE JULIA SET OF KÖNIG'S ROOT–FINDING ALGORITHMS

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Dedicated to the memory of Sergio Plaza S.

ABSTRACT. As is well known, the Julia set of Newton's method applied to complex polynomials is connected. The family of König's root–finding algorithms is a natural generalization of Newton's method. We show that the Julia set of König's root–finding algorithms of order $\sigma \geq 3$ applied to complex polynomials is not always connected.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let N_f be the classical Newton iterative root–finding algorithm or, in other words, $N_f = Id - f/f'$, where f is a degree d complex polynomial. In [3], J. Hubbard, D. Schleicher and S. Sutherland show that if f is a complex polynomial of degree d, then there is a finite set S_d depending only on d such that, given any root α of f, there exists at least one point in S_d converging under iterations of N_f to α. This solves the ancient problem: Find all of the roots of a complex polynomial.

By a *root–finding algorithm* or *root–finding method* we mean a rational map $T_f : Poly_d \to Rat_k$ such that the roots of the polynomial map f are attracting fixed points of T_f. We say that a root–finding algorithm T_f has *order* σ if the local degree of T_f at each simple root of f is σ.

In order to construct the set S_d, it is essential that the immediate basins of attraction of Newton's method be simply connected. This fact is shown by F. Przytycki in [6]. He shows that if α is an attracting fixed point for Newton's method N_f, and if U is the immediate basin of attraction of α, then U is simply connected. A stronger result due to M. Shishikura establishes that the Julia set of Newton's method of a complex polynomial is connected. (See [10].) The case of polynomials of degree three has been studied by H. Meier in [5] and by L. Tan in [11]. Przytycki's result has been extended to entire holomorphic maps by S. Mayer and D. Schleicher in [4]. Combining the previous results with the Riemann Mapping Theorem we obtain a conformal isomorphism between the unit disk and the immediate basin of attraction of Newton's method, which gives a source of geometric information.

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A natural question is whether Hubbard, Schleicher and Sutherland's results of [3] also hold for root–finding algorithms of higher order than Newton's method.

This paper deals with the preceding question. In particular, we focus on König's root–finding algorithm, which may be thought of as an arbitrarily high order generalization of Newton's method. Indeed, when the order $\sigma = 2$, König's method is the classical Newton's method. The well known Halley's method is obtained when $\sigma = 3$ (see [8]). A geometric interpretation of Halley's method is as follows. We may naturally approximate f at z_0 by the unique automorphism of the Riemann sphere whose 2-jet agrees with that of f, and define z_1 as the unique zero of this automorphism.

Our main result is the following.

Theorem 1.1. *For all order $\sigma \geq 3$, there exists a complex polynomial p such that the Julia set of König's method applied to this polynomial is not connected.*

Thus, Przytycki's and Shishikura's results are, in general, no longer true for König's method when $\sigma \geq 3$. Consequently, our result establishes restrictions for extending the main result of [3] to higher order root–finding algorithms.

We remark that the proof of Theorem 1.1 hardly involves iteration. It is actually a constructive example based mainly on the geometry of the map $z+1/z$. The proof outline proceeds as follows:

Outline of Proof of Theorem 1.1. We denote by $K_{h,\sigma}$ König's method (2.1) applied to an arbitrary function h, and we consider a disk $C_1 \subset \mathbb{C}$ containing the unit circle S^1. For each $\sigma \geq 3$ we can construct an analytic function f_σ on C_1 so that $K_{f_\sigma,\sigma}$ has a pole at the origin and maps S^1 onto a segment I_σ. This situation is illustrated in Figure 1. Now we consider a quadratic polynomial g_σ and we show that there exists a domain $C_2 \subset \mathbb{C}$ which contains I_σ, is disjoint from C_1, and is contained in the basin of attraction of a fixed point of the map $K_{g_\sigma,\sigma}$.

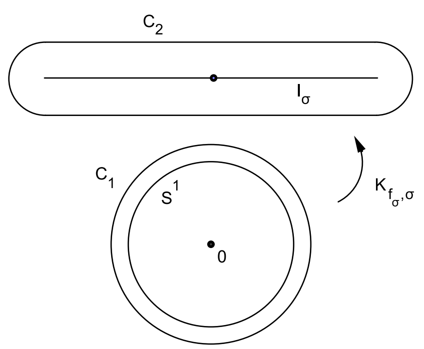

FIGURE 1. The geometry of $K_{f_\sigma,\sigma}$.

Now according to Mergelyan's theorem, there exists a polynomial p close to both f_σ on C_1 and g_σ on C_2. The most technical part of the proof is to check that the assertions of the previous paragraph remain true for this polynomial. Once this is done, we see that S^1 is contained in the Fatou set of $K_{p,\sigma}$ and also that there exists a preimage of infinity b under the map $K_{p,\sigma}$ which is in the interior of the unit disk. Since the Julia set of $K_{p,\sigma}$ contains infinity, we conclude that b is also in this Julia set. It follows that the Julia set of $K_{p,\sigma}$ is disconnected.

In the next section we introduce the König root–finding algorithm and present some preliminary results. The detailed proof of Theorem 1.1 is in section 3.

2. König's root–finding algorithms

We first establish definitions and basic properties.

Definition 2.1. Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic map and let $\sigma \geq 2$ be an integer. König's method of order σ associated to f is the rational map $K_{f,\sigma} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by

$$K_{f,\sigma} = Id + (\sigma - 1)\frac{(1/f)^{[\sigma-2]}}{(1/f)^{[\sigma-1]}}, \tag{2.1}$$

where $(1/f)^{[k]}$ is the k^{th} derivative of $1/f$.

Proposition 2.2 ([2]). *Let $f : U \to \mathbb{C}$ be a meromorphic map on a domain $U \subset \mathbb{C}$. We denote its zeros by α_i and their multiplicities by $n_i \geq 1$. Then for any integer $\sigma \geq 2$, the fixed points of König's method $K_{f,\sigma} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are either (super)attracting or repelling. We have:*

1. *The (super)attracting fixed points are exactly the zeros α_i, and their corresponding multipliers are $1 - (\sigma - 1)/(n_i + \sigma - 2)$. When $n_i = 1$, the local degree of $K_{f,\sigma}$ at α_i is exactly σ.*
2. *The extraneous fixed points of $K_{f,\sigma}$ are exactly the zeros of $(1/f)^{[\sigma-2]}$. If β_j is a zero of $(1/f)^{[\sigma-2]}$ with multiplicity m_j, then it is a repelling fixed point of $K_{f,\sigma}$ with multiplier $1 + (\sigma - 1)/m_j$.*
3. *König's method has a repelling fixed point at ∞ with multiplier $1 + [(\sigma - 1)/(d - 1)]$.*

We next record a useful result about spherical convergence which we will use to prove Lemma 2.4 below. For a proof the reader can consult, for example, [9].

Proposition 2.3. *Let $\{h_n\}$ be a sequence of meromorphic functions on a domain $U \subset \overline{\mathbb{C}}$. Then h_n converges to h for the spherical metric on compact subsets of U if and only if about each point $z_0 \in U$ there is a closed disk $D(z_0, r)$ with center z_0 and radius $r > 0$ such that either:*

1. $|h_n - h| \to 0$ *or*
2. $|\frac{1}{h_n} - \frac{1}{h}| \to 0$

on $D(z_0, r)$, uniformly as $n \to \infty$.

The following result has been proved in [2] under a more stringent hypothesis.

Lemma 2.4. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials converging uniformly to an analytic function f on an open set $U \subset \mathbb{C}$. Suppose that f has only simple zeros and that $K_{f,\sigma}$ is meromorphic in U. Then the sequence of rational maps $K_{f_n,\sigma}$ converges uniformly to $K_{f,\sigma}$ on every compact subset of U for the spherical metric on $\overline{\mathbb{C}}$.*

Proof. We first recall the construction of König's method given by N. Argiropoulos, V. Drakopoulos and A. Böhm in [1]. Define $h_1(z) = 1$ and, inductively,

$$\left(\frac{h_1(z)}{f(z)}\right)' = \frac{h_1'(z)f(z) - h_1 f'(z)}{[f(z)]^2} = \frac{h_2(z)}{[f(z)]^2},$$

$$\left(\frac{h_2(z)}{[f(z)]^2}\right)' = \frac{h_2'(z)f(z) - 2h_2 f'(z)}{[f(z)]^3} = \frac{h_3(z)}{[f(z)]^3}.$$

In general, we have

$$\left(\frac{1}{f(z)}\right)^{[k]} = \frac{h_{k+1}(z)}{[f(z)]^{k+1}},$$

where $h_{k+1}(z) = h_k'(z)f(z) - kh_k(z)f'(z)$, with $k = 1, 2, \ldots, \sigma - 1$. Now we rewrite

$$K_{f,\sigma}(z) = z + (\sigma - 1)\frac{h_{\sigma-1}(z)f(z)}{h_\sigma(z)}.$$

Let U be an open set of $\mathbb{C}$. We write $h_{f,k}$ instead of h_k to emphasize the dependence on f. We show by induction on k that $h_{f_n,k}$ converges to $h_{f,k}$ uniformly on every compact subset of U. Since f_n converges uniformly to f on every compact subset of U, we have that $h_{f_n,2} = h'_{f_n,1}f_n - h_{f_n,1}f_n' = -f_n'$ converges to $h_{f,2}$ uniformly on every compact subset of U. Thus, the assertion is true for $k = 1$. Assume that $h_{f_n,k}$ converges to $h_{f,k}$ uniformly on every compact subset of U. We have that $h_{f_n,k+1} = h'_{f_n,k}f_n - kh_{f_n,k}f_n'$ converges to $h_{f,k+1} = h'_{f,k}f - kh_{f,k}f'$ uniformly on every compact subset of U. This completes the induction. Define

$$P_{f,\sigma}(z) = zh_{f,\sigma}(z) + (\sigma - 1)h_{f,\sigma-1}(z)f(z)$$

and $Q_{f,\sigma}(z) = h_{f,\sigma}(z)$. Rewrite $K_{f,\sigma} = P_{f,\sigma}/Q_{f,\sigma}$.

Next we show that if $\sigma \geq 2$, König's method $K_{f_n,\sigma}$ converges uniformly to $K_{f,\sigma}$ on every compact subset of U for the spherical metric on $\overline{\mathbb{C}}$. To prove this, we need only show that the sequence $K_{f_n,\sigma}$ satisfies condition (1) or condition (2) of Proposition 2.3. Thus, for an arbitrary $z_0 \in U$ there are two cases. Suppose $Q_{f,\sigma}(z) \neq 0$ in a closed disk $D(z_0, r) \subset U$. By Proposition 2.2(1), the local degree of $K_{f,\sigma}$ at the zeros of f is σ. Hence $P_{f,\sigma}$ and $Q_{f,\sigma}$ have no common factors. It follows from the induction above, and Hurwitz's theorem, that $K_{f_n,\sigma} = P_{f_n,\sigma}/Q_{f_n,\sigma}$ converges uniformly to $K_{f,\sigma} = P_{f,\sigma}/Q_{f,\sigma}$ on $D(z_0, r)$ for the Euclidean metric. Hence condition (1) of Proposition 2.3 is satisfied. On the other hand, if $Q_{f,\sigma}(z_0) = 0$, we choose $r > 0$ sufficiently small so that $P_{f,\sigma}(z) \neq 0$ in $D(z_0, r)$, and again $P_{f,\sigma}$ and $Q_{f,\sigma}$ have no common factors. Combining the induction above with Hurwitz's theorem, we have that $1/K_{f_n,\sigma} = Q_{f_n,\sigma}/P_{f_n,\sigma}$ converges uniformly to $1/K_{f,\sigma} = Q_{f,\sigma}/P_{f,\sigma}$ on $D(z_0, r)$ for the Euclidean metric, so condition (2) of Proposition 2.3 is satisfied and the lemma now follows. □

Remark 2.5. Here the fact that f has no multiple zeros is essential. If we consider the sequence $f_n(z) = z^2 + 1/n^2$, the corresponding Newton's method

$$N_{f_n}(z) = \frac{(z^2n^2 - 1)}{2n^2 z}$$

fails to be normal in any domain containing the origin. In fact, N_{f_n} is ∞ at $z = 0$, whereas Newton's method applied to the limit function f, that is, $N_f(z) = z/2$, satisfies $N_f(0) = 0$.

The proof of our main result depends on the following classical theorem, whose importance lies in its interplay between interpolation and approximation. We state it without proof. (See [7].)

Theorem 2.6 (Mergelyan's approximation). *Let C be a compact subset of the complex plane whose complement is connected. Then every function continuous on C and analytic on the interior of C can be approximated uniformly on C by polynomials.*

3. Proof of the main theorem

Proof. Let $r \in (0, \frac{1}{8})$. For each $\sigma \geq 3$, define the function

$$f_\sigma(z) = \frac{(-i\sqrt{\sigma-1})^{\sigma-1}\exp(i\sqrt{\sigma-1}z)}{[-i\sqrt{\sigma-1}z-(\sigma-1)]}.$$

Note that f_σ is a meromorphic function and that for every $\sigma \geq 3$, we have that $i\sqrt{\sigma-1}$ is a simple pole of f_σ. In particular, f_σ is holomorphic on $C_1 = \{z \in \mathbb{C} : |z| \leq 1+r\}$.

König's method of order σ applied to f_σ is given by

$$K_{f_\sigma,\sigma}(z) = z + \frac{1}{z} + i\sqrt{\sigma-1}.$$

Let $I_\sigma = i\sqrt{\sigma-1} + [-2, 2]$. For $z_0 \notin I_\sigma$, the equation $K_{f_\sigma,\sigma}(w) = z_0$ has two solutions, one of which lies inside the unit circle and one of which lies outside. Hence K_{f_σ} maps the exterior of the closed unit disk isomorphically onto the complement $\mathbb{C} \setminus I_\sigma$. In particular, the map $K_{f_\sigma,\sigma}$ carries the unit circle S^1 in a two–to–one manner onto I_σ.

Now consider the quadratic polynomial

$$g_\sigma(z) = (z - i\sqrt{\sigma-1})(z-(i\sqrt{\sigma-1}+N_0))$$

where N_0 is a positive integer. It is well known that König's method for quadratic polynomials is conjugated to $z \to z^\sigma$. For further details, see [2]. We first prove the following.

Lemma 3.1. *There exists $\delta_0 > 0$ such that, for all $\delta < \delta_0$, the set $C_2 = \{z \in \mathbb{C} : d(z, I_\sigma) \leq \delta\}$ is contained completely in the basin of attraction of the fixed point $i\sqrt{\sigma-1}$ associated to $K_{g_\sigma,\sigma}$.*

Proof. König's method of order σ applied to g_σ has two superattracting fixed points, namely $i\sqrt{\sigma-1}$ and $i\sqrt{\sigma-1}+N_0$. After composing $K_{g_\sigma,\sigma}$ with a Möbius transformation if necessary, we may choose $\delta_0 > 0$ and an N_0 sufficiently large so that the open ball of radius δ_0 centered at $z_1 = 2 + i\sqrt{\sigma-1}$ is contained in the basin of attraction of $i\sqrt{\sigma-1}$. It follows from compactness that there exists a finite subset $\{z_2, \ldots, z_l\}$ of I_σ such that $\bigcup_{i=1}^{l} B_{\delta_0}(z_i)$ is contained in the basin of attraction of $i\sqrt{\sigma-1}$. The proof finishes by taking $\delta < \delta_0$ so that C_2 is contained in the preceding union. $\square$

We continue the main proof. Choose δ as in Lemma 3.1 so that $\delta < r$. We let C denote the union of C_1 and C_2. Let

$$F(z) = \begin{cases} f_\sigma(z) & \text{if } z \in C_1, \\ g_\sigma(z) & \text{if } z \in C_2. \end{cases}$$

By Mergelyan's theorem, there exists a sequence of polynomials $\{p_n\}$ which converges uniformly to the map F on the compact subset C. Since F has only one simple zero at $i\sqrt{\sigma-1}$, it follows from Lemma 2.4 that $K_{p_n,\sigma}$ converges uniformly to $K_{F,\sigma}$ on every compact subset of C. Now we need two auxiliary lemmas.

Lemma 3.2. *For some positive integer k, the unit circle S^1 lies in a Fatou component of $K_{p_n,\sigma}$, for all $n \geq k$.*

Proof. By Hurwitz's theorem, there exist a sequence $\{a_n\}$ in C converging to $i\sqrt{\sigma-1}$ and a positive integer n_0 such that $p_n(a_n) = 0$, for all $n \geq n_0$. By part (1) of Proposition 2.2, $K_{p_n,\sigma}$ has an attracting fixed point at a_n, for every $n \geq n_0$. Since $i\sqrt{\sigma-1}$ is a superattracting fixed point of the rational map $K_{g_\sigma,\sigma}$, there is a real number α satisfying $|K'_{g_\sigma,\sigma}(i\sqrt{\sigma-1})| < \alpha < 1$. From the preceding inequality we conclude that there exists a positive integer $n_1 > n_0$ such that, for all $z \in C_2$, we have

$$|K_{p_n,\sigma}(z) - a_n| < \alpha|z - a_n|$$

for $n \geq n_1$. This shows that for each $n \geq n_1$, $K_{p_n,\sigma}$ maps C_2 into itself. Consequently, C_2 lies in the Fatou set of the map $K_{p_n,\sigma}$, for all $n \geq n_1$.

Note that there exists a positive integer n_2 such that $K_{p_n,\sigma}(S^1) \subset C_2$, for all $n \geq n_2$. Indeed, if $z \in S^1$, then $K_{F,\sigma}(z) \in S^1$. Since $K_{p_n,\sigma}(z)$ converges to $K_{F,\sigma}(z)$ as $n \to \infty$, we may choose a positive integer n_2 so that, for all $n \geq n_2$, we have $|K_{F,\sigma}(z) - K_{p_n,\sigma}(z)| < \delta$. Thus $K_{p_n,\sigma}(z) \in C_2$ whenever $n \geq n_2$. Taking $k > \max\{n_1, n_2\}$, we obtain the lemma. □

Lemma 3.3. *There exists a positive integer k' such that infinity has preimages $\{b_n\}$ under $K_{p_n,\sigma}$ which lie in the interior of the unit disk whenever $n \geq k'$.*

Proof. Note that König's method applied to F has a pole of first order in 0. Now arguing as in Lemma 3.2, we can use Hurwitz's theorem to obtain a sequence $\{b_n\}$ in C converging to 0 and a positive integer k' such that for $n \geq k'$, each element of $\{b_n\}$ is a pole of the corresponding rational map $K_{p_n,\sigma}$. □

These two lemmas allow us to finish the proof of Theorem 1.1. Let k and k' be two positive integers as in Lemmas 3.2 and 3.3. Let m be a positive integer such that $m \geq \max\{k, k'\}$. There is a preimage of infinity b_m under the rational map $K_{p_m,\sigma}$ which lies in the interior of the unit disk. Moreover, by part (3) of Proposition 2.2, b_m belongs to the Julia set of $K_{p_m,\sigma}$. Since S^1 lies in the Fatou set of $K_{p_m,\sigma}$, it follows that S^1 disconnects the Julia set of $K_{p_m,\sigma}$, which completes the proof. □

Remark 3.4. It is noteworthy that the sets I_σ and S^1 are separated only if $\sigma > 2$. In the case $\sigma = 2$, we cannot disconnect these sets. This is consistent with the fact that the Julia set of Newton's method applied to complex polynomials is connected.

Remark 3.5. Up to now we have been unable to show explicitly a disconnected Julia set for König's method applied to complex polynomials.

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