A NOTE ON A BRUNN-MINKOWSKI INEQUALITY FOR THE GAUSSIAN MEASURE

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Abstract. We give counterexamples related to a Gaussian Brunn-Minkowski inequality and the (B) conjecture.

1. INTRODUCTION AND NOTATION

Let $\gamma_n$ be the standard Gaussian distribution on $\mathbb{R}^n$, i.e. the measure with the density

$$g_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2},$$

where $|\cdot|$ stands for the standard Euclidean norm. A powerful tool in convex geometry is the Brunn-Minkowski inequality for Lebesgue measure (see [Sch] for more information). Concerning the Gaussian measure, the following question has recently been posed.

Question (R. Gardner and A. Zvavitch, [GZ]). Let $0 < \lambda < 1$ and let $A$ and $B$ be closed convex sets in $\mathbb{R}^n$ such that $o \in A \cap B$. Is it true that

$$(GBM) \quad \gamma_n(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \gamma_n(A)^{1/n} + (1 - \lambda)\gamma_n(B)^{1/n}?$$

A counterexample is given in this note. However, we believe that this question has an affirmative answer in the case of $o$-symmetric convex sets, i.e. the sets satisfying $K = -K$.

In [CFM] it is proved that for an $o$-symmetric convex set $K$ in $\mathbb{R}^n$ the function

$$(1.1) \quad \mathbb{R} \ni t \mapsto \gamma_n(e^t K)$$

is log-concave. This was conjectured by W. Banaszczyk and was popularized by R. Latała [Lat]. It turns out that the (B) conjecture cannot be extended to the class of sets which are not necessarily $o$-symmetric yet contain the origin, as one of the sets provided in our counterexample shows.

As for the notation, we frequently use the function

$$T(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt.$$
2. Counterexamples

Now we construct the convex sets $A, B \subset \mathbb{R}^2$ containing the origin such that inequality \textbf{GBM} does not hold. Later on we show that for the set $B$ the (B) conjecture is not true.

Fix $\alpha \in (0, \pi/2)$ and $\varepsilon > 0$. Take

$$A = \{(x, y) \in \mathbb{R}^2 \mid y \geq |x| \tan \alpha\},$$

$$B = B_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid y \geq |x| \tan \alpha - \varepsilon\} = A - (0, \varepsilon).$$

Clearly, $A, B$ are convex and $0 \in A \cap B$. Moreover, from the convexity of $A$ we have $\lambda A + (1 - \lambda)A = A$, and therefore

$$\lambda A + (1 - \lambda)B = \lambda A + (1 - \lambda)(A - (0, \varepsilon)) = A - (1 - \lambda)(0, \varepsilon).$$

Observe that

$$\gamma_2(A) = \frac{1}{2} - \frac{\alpha}{\pi},$$

$$\gamma_2(B) = 2 \int_0^{+\infty} T(x \tan \alpha - \varepsilon) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx,$$

$$\gamma_2(\lambda A + (1 - \lambda)B) = 2 \int_0^{+\infty} T(x \tan \alpha - \varepsilon(1 - \lambda)) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

and that these expressions are analytic functions of $\varepsilon$. We will expand these functions in $\varepsilon$ up to the order 2. Let

$$a_k = \int_0^{+\infty} T^{(k)}(x \tan \alpha) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx,$$

for $k = 0, 1, 2$, where $T^{(k)}$ is the $k$-th derivative of $T$ (we adopt the standard notation $T^{(0)} = T$). We get

$$\gamma_2(A) = 2a_0,$$

$$\gamma_2(B) = 2a_0 - 2\varepsilon a_1 + \varepsilon^2 a_2 + o(\varepsilon^2),$$

$$\gamma_2(\lambda A + (1 - \lambda)B) = 2a_0 - 2\varepsilon(1 - \lambda) a_1 + \varepsilon^2(1 - \lambda)^2 a_2 + o(\varepsilon^2).$$

Thus

$$\sqrt{\gamma_2(B)} = \sqrt{2a_0} - \frac{a_1}{\sqrt{2a_0}} \varepsilon + \left( \frac{a_2}{2\sqrt{2a_0}} - \frac{a_1^2}{2(2a_0)^{3/2}} \right) \varepsilon^2 + o(\varepsilon^2).$$

Taking $\varepsilon(1 - \lambda)$ instead of $\varepsilon$ we obtain

$$\sqrt{\gamma_2(\lambda A + (1 - \lambda)B)} = \sqrt{2a_0} - \frac{a_1}{\sqrt{2a_0}} (1 - \lambda) \varepsilon$$

$$+ \left( \frac{a_2}{2\sqrt{2a_0}} - \frac{a_1^2}{2(2a_0)^{3/2}} \right) (1 - \lambda)^2 \varepsilon^2 + o(\varepsilon^2).$$

Since

$$\sqrt{\gamma_2(\lambda A + (1 - \lambda)B)} - \lambda \sqrt{\gamma_2(A)} - (1 - \lambda) \sqrt{\gamma_2(B)}$$

$$= -\lambda(1 - \lambda) \frac{1}{2(2a_0)^{3/2}} (2a_0 a_2 - a_1^2) \varepsilon^2 + o(\varepsilon^2),$$

we will have a counterexample if we find $\alpha \in (0, \pi/2)$ such that

$$2a_0 a_2 - a_1^2 > 0.$$
Recall that \( a_0 = \frac{1}{2} \gamma_2(A) = \frac{1}{2} \left( \frac{1}{2} - \frac{\alpha}{\pi} \right) \). The integrals that define the \( a_k \)'s can be calculated. Namely,

\[
a_1 = \int_0^\infty T'(x \tan \alpha) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = -\frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{\mathbb{R}} e^{-(1+\tan^2 \alpha)x^2/2} \, dx = -\frac{1}{\sqrt{2\pi}} \frac{1}{2 \sqrt{1 + \tan^2 \alpha}},
\]

\[
a_2 = \int_0^\infty T''(x \tan \alpha) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (x \tan \alpha) e^{-(1+\tan^2 \alpha)x^2/2} \, dx = \frac{1}{2\pi} \frac{\tan \alpha}{1 + \tan^2 \alpha}.
\]

Therefore,

\[
2a_0a_2 - a_1^2 = 2 \left( \frac{1}{2} \left( \frac{1}{2} - \frac{\alpha}{\pi} \right) \cdot \frac{1}{2\pi} \frac{\tan \alpha}{1 + \tan^2 \alpha} \right) - \frac{1}{2\pi} \cdot \frac{1}{4(1 + \tan^2 \alpha)} = -\frac{1}{8\pi} \frac{1}{1 + \tan^2 \alpha} \left( \tan \alpha \left( 2 - \frac{4\alpha}{\pi} \right) - 1 \right),
\]

which is positive for \( \alpha \) close to \( \pi/2 \).

Now we turn our attention to the (B) conjecture. We are going to check that for the set \( B = B_\varepsilon \) the function \( t \mapsto \gamma_n(e^t \gamma_1) \) is not log-concave, provided that \( \varepsilon \) is sufficiently small. Since \( e^t B = \{(x, y) \in \mathbb{R}^2 \mid y \geq \tan \alpha |x| - \varepsilon e^t\} \), we get

\[
\ln \gamma_2(e^t B) = \ln \left( 2 \int_0^\infty T(x \tan \alpha - e^t \varepsilon) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx \right) = \ln \left( 2 \int_0^\infty T(x \tan \alpha) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx \right) - e^t \varepsilon \frac{\int_0^\infty T'(x \tan \alpha) e^{-x^2/2} \, dx}{\int_0^\infty T(x \tan \alpha) e^{-x^2/2} \, dx} + o(\varepsilon).
\]

This produces the desired counterexample for sufficiently small \( \varepsilon \) as the function \( t \mapsto \beta e^t \), where

\[
\beta = -\frac{\int_0^\infty T'(x \tan \alpha) e^{-x^2/2} \, dx}{\int_0^\infty T(x \tan \alpha) e^{-x^2/2} \, dx} > 0,
\]

is convex. \( \square \)

**Remark.** The set \( B_\varepsilon \) which serves as a counterexample to the (B) conjecture in the nonsymmetric case works when the parameter \( \alpha = 0 \) as well (and \( \varepsilon \) is sufficiently small). Since \( B_\varepsilon \) is simply a halfspace in this case, it shows that the symmetry of \( K \) is required for log-concavity of \( \gamma_1 \), even in the one-dimensional case.

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References


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