PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 11, November 2013, Pages 3739–3753
S 0002-9939(2013)11659-X
Article electronically published on July 16, 2013

COHOMOLOGY RINGS FOR QUANTIZED ENVELOPING ALGEBRAS

CHRISTOPHER M. DRUPIESKI

(Communicated by Gail R. Letzter)

Abstract. We compute the structure of the cohomology ring for the quantized enveloping algebra (quantum group) U_q associated to a finite-dimensional simple complex Lie algebra $\mathfrak{g}$. We show that the cohomology ring is generated as an exterior algebra by homogeneous elements in the same odd degrees as those that generate the cohomology ring for the Lie algebra $\mathfrak{g}$. Partial results are also obtained for the cohomology rings of the non-restricted quantum groups obtained from U_q by specializing the parameter q to a non-zero value $\varepsilon \in \mathbb{C}$.

1. Introduction

1.1. Let G be a simple compact connected Lie group of dimension d. It is a famous theorem from algebraic topology that the homology and cohomology algebras for G (as a topological space) are exterior algebras over graded subspaces concentrated in odd degrees [22]. By a result of Cartan, the homology and cohomology algebras for G identify with those for its Lie algebra $\mathfrak{g}$, so we also get the ring structure of the Lie algebra cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C}) = \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$. Here $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g} = \mathrm{Lie}(G)$. In recent years there has been much interest in homological and cohomological properties for various classes of noetherian Hopf algebras [6, 8, 7], important examples of which are the universal enveloping algebras and quantized enveloping algebras associated to a finite-dimensional simple complex Lie algebra. A common theme to some of the recent work has been the desire to generalize Poincaré duality to these classes of noetherian Hopf algebras [7, 15].

Let q be an indeterminate, and set $k = \mathbb{C}(q)$. Let U_q be the quantized enveloping algebra over k associated to the finite-dimensional simple complex Lie algebra $\mathfrak{g}$. Though the above-cited works provide general results relating the dimensions of the homology and cohomology groups

$$\mathrm{H}_n(U_q, k) = \mathrm{Tor}_n^{U_q}(k, k) \qquad \text{and} \qquad \mathrm{H}^n(U_q, k) = \mathrm{Ext}^n_{U_q}(k, k),$$

namely, $\dim_k \mathrm{H}^n(U_q, k) = \dim_k \mathrm{H}_{d-n}(U_q, k)$, there have been no explicit calculations of the dimensions of these groups nor of the ring structure for the cohomology ring $\mathrm{H}^\bullet(U_q, k)$. Similarly, one would like to know the dimension and ring structure of the cohomology ring $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ associated to the quantized enveloping algebra U_ε with parameter q specialized to a value $\varepsilon \in \mathbb{C}^\times := \mathbb{C} - \{0\}$.

Received by the editors October 21, 2010 and, in revised form, December 23, 2011 and January 18, 2012.

2010 *Mathematics Subject Classification.* Primary 17B37, 17B56.

The author was supported in part by NSF VIGRE grant DMS-0738586.

In this paper we show that the cohomology ring $\mathrm{H}^\bullet(U_q, k)$ is an exterior algebra generated by homogeneous elements in the same odd degrees as those of $\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$, and thus for each $n \in \mathbb{N}$ that the cohomology group $\mathrm{H}^n(U_q, k)$ for U_q is of the same dimension as the corresponding group for $U(\mathfrak{g})$. Our proof relies on an integral form U_{A} for U_q, which enables us to relate, via the universal coefficient theorem, cohomology for U_q to that for $U(\mathfrak{g})$. The main steps of this argument are carried out in Sections 3.4 and 4.1. A key step in the proof is the calculation of the restriction maps in Lie algebra cohomology associated to an inclusion $F \subset E$ of simple Lie algebras; see Section 2.3. Finally, assuming $\varepsilon \in \mathbb{C}$ is a root of unity of sufficiently large prime order p, we obtain the structure of the cohomology ring $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ for the quantized enveloping algebra U_ε. This last computation exploits a connection between U_ε and the characteristic p universal enveloping algebra of $\mathfrak{g}$.

1.2. **Notation.** Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denote the set of non-negative integers. Let (a_{ij}) be the $r \times r$ Cartan matrix associated to the finite-dimensional simple complex Lie algebra $\mathfrak{g}$, and let $(d_1, \dots, d_r) \in \mathbb{N}^r$ be the unique vector such that $\gcd(d_1, \dots, d_r) = 1$ and the matrix $(d_i a_{ij})$ is symmetric. The ordering of the rows and columns for the matrix (a_{ij}) corresponds to a labeling of the Dynkin diagram associated to $\mathfrak{g}$; we assume this is done as in Bourbaki [5, Plates I–IX].

Let $\mathbb{C}[q, q^{-1}]$ be the Laurent polynomial ring over $\mathbb{C}$ in the indeterminate q, and let $k = \mathbb{C}(q)$ be its quotient field. Then the quantized enveloping algebra (or quantum group) associated to $\mathfrak{g}$ is the k-algebra $U_q = U_q(\mathfrak{g})$ defined by the generators $\{E_i, F_i, K_i^{\pm 1} : i \in [1, r]\}$ and the relations given in [11, (1.2.1–1.2.5)]. The algebra U_q is also a Hopf algebra via the maps in [11, (1.2.6–1.2.8)].

For $n \in \mathbb{Z}$ and $d \in \mathbb{N}$, put $[n]_d = (q^{nd} - q^{-nd})/(q^d - q^{-d}) \in \mathbb{Z}[q, q^{-1}]$. Given $\ell \in \mathbb{N}$, let $\phi_\ell \in \mathbb{Z}[q]$ be the ℓ-th cyclotomic polynomial. Now define $S \subset \mathbb{Z}[q, q^{-1}]$ to be the multiplicatively closed set generated by

$$
\begin{array}{ll}
\{1\} & \text{if } \mathfrak{g} \text{ has Lie type } ADE, \\
\{\phi_4, \phi_8\} & \text{if } \mathfrak{g} \text{ has Lie type } BCF, \\
\{\phi_3, \phi_4, \phi_6, \phi_9, \phi_{12}, \phi_{18}\} & \text{if } \mathfrak{g} \text{ has Lie type } G_2.
\end{array}
\tag{1.1}
$$

Then the generators for S are precisely the irreducible factors of $[n]_{d_i}$ in $\mathbb{Z}[q, q^{-1}]$ when $1 \le n \le |a_{ij}|$ and $i \neq j$. Set $\mathcal{Z} = S^{-1}\mathbb{Z}[q, q^{-1}]$, the localization at S.

Let $U_{\mathcal{Z}}$ be the $\mathcal{Z}$-subalgebra of U_q generated by the set $\{E_i, F_i, K_i^{\pm 1} : i \in [1, r]\}$. This algebra is the De Concini–Kac integral form of U_q over $\mathcal{Z}$. Given a $\mathcal{Z}$-algebra B, set $U_B = U_{\mathcal{Z}} \otimes_{\mathcal{Z}} B$. In particular, given $\varepsilon \in \mathbb{C}^\times$ with $f(\varepsilon) \neq 0$ for all $f \in S$, write $\mathbb{C}_\varepsilon$ for the field $\mathbb{C}$ considered as a $\mathcal{Z}$-algebra via the map $q \mapsto \varepsilon$, and set $U_\varepsilon = U_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{C}_\varepsilon$. If $\varepsilon^{2d_i} \neq 1$ for all $i \in [1, r]$, then U_ε is the $\mathbb{C}$-algebra with the same generators and relations as U_q, but with q replaced by ε. For this reason, we call U_ε a specialization of U_q.

For each $i \in [1, r]$, let T_i be the braid group operator on U_q as defined in [11, §1.6], and let Φ be the root system associated to $\mathfrak{g}$. Then for each positive root $\beta \in \Phi^+$, there exist root vectors $E_\beta, F_\beta \in U_q$, defined in terms of the T_i [11, §1.7]. By the definition of the denominator set S, the T_i restrict to automorphisms of the algebra $U_{\mathcal{Z}}$, so also $E_\beta, F_\beta \in U_{\mathcal{Z}}$. This is why we work with $\mathcal{Z}$ instead of with $\mathbb{Z}[q, q^{-1}]$.

2. Lie algebra cohomology

2.1. **An isomorphism with U_1.** We begin with an observation on the relationship between the cohomology spaces for the universal enveloping algebra $U(\mathfrak{g})$ and for the specialization U_1. Recall from [11, Proposition 1.5] that U_1 is a central extension of $U(\mathfrak{g})$ by the group algebra over $\mathbb{C}$ for the finite group $(\mathbb{Z}/2\mathbb{Z})^r$, so there exists a surjective Hopf algebra homomorphism $U_1 \to U(\mathfrak{g})$.

Lemma 2.1. *The homomorphism $U_1 \to U(\mathfrak{g})$ induces an algebra isomorphism*

$$\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C}) \xrightarrow{\sim} \mathrm{H}^\bullet(U_1, \mathbb{C}).$$

Proof. Set $G = (\mathbb{Z}/2\mathbb{Z})^r$, and consider the Lyndon–Hochschild–Serre (LHS) spectral sequence for the algebra U_1 and its normal Hopf-subalgebra isomorphic to $\mathbb{C}G$:

$$E_2^{i,j} = \mathrm{H}^i(U(\mathfrak{g}), \mathrm{H}^j(\mathbb{C}G, \mathbb{C})) \Rightarrow \mathrm{H}^{i+j}(U_1, \mathbb{C});$$

for details on the LHS spectral sequence, see [3]. The algebra $\mathbb{C}G$ is semisimple, so $E_2^{i,j} = 0$ for all $j > 0$. Since the spectral sequence respects cup products, it follows that the edge map $E_2^{\bullet,0} = \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C}) \to \mathrm{H}^\bullet(U_1, \mathbb{C})$ is an algebra isomorphism. □

2.2. **The structure of Lie algebra cohomology.** Lemma 2.1 reduces the problem of studying the cohomology ring $\mathrm{H}^\bullet(U_1, \mathbb{C})$ to the classical problem of studying the cohomology ring $\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$. We summarize some details on the computation of $\mathrm{H}^\bullet(E, \mathbb{C}) = \mathrm{H}^\bullet(U(E), \mathbb{C})$ for E an arbitrary finite-dimensional reductive Lie algebra over $\mathbb{C}$. Our main reference is [14, Chapters V–VI].

Let $\Lambda^\bullet(E^*)$ denote the exterior algebra on the dual space $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$, considered as a graded complex with E^* concentrated in degree 1. The map $\Lambda^2(E) \to E$ defined by $x \wedge y \mapsto [x, y]$ induces a map $d : E^* \to \Lambda^2(E^*)$, which extends by derivations to a differential on $\Lambda^\bullet(E^*)$, also denoted d. Then $\mathrm{H}^\bullet(E, \mathbb{C})$ is the cohomology of the complex $\Lambda^\bullet(E^*)$ with respect to the differential d. The space $\Lambda^\bullet(E^*)$ is also naturally an E-module, with E-action induced by the coadjoint action of E on E^*. Let $\Lambda^\bullet(E^*)^E$ denote the space of E-invariants in $\Lambda^\bullet(E^*)$. Then the inclusion $\Lambda^\bullet(E^*)^E \hookrightarrow \Lambda^\bullet(E^*)$ induces an algebra isomorphism $\Lambda^\bullet(E^*)^E \cong \mathrm{H}^\bullet(E, \mathbb{C})$.

The addition map $E \times E \to E$, $(x, y) \mapsto x + y$, induces on $\Lambda^\bullet(E^*)$ the structure of a bialgebra, and the bialgebra structure restricts to one on $\Lambda^\bullet(E^*)^E \cong \mathrm{H}^\bullet(E, \mathbb{C})$. Then $\mathrm{H}^\bullet(E, \mathbb{C})$ is generated as an algebra by its subspace of primitive elements, which we denote by P_E. The subspace P_E is concentrated in odd degrees, and the induced map $\Lambda(P_E) \to \Lambda^\bullet(E^*)^E \to \mathrm{H}^\bullet(E, \mathbb{C})$ is an algebra isomorphism.

Theorem 2.2 ([22, 14])**.** *Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra. Then $\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$ is an exterior algebra generated by homogeneous elements in the odd degrees listed in Table* 1*.*

2.3. **Restriction maps.** Let E and F be finite-dimensional reductive Lie algebras over $\mathbb{C}$ with $F \subseteq E$. Write $j : F \to E$ for the inclusion map. Let $W(E)$ and $W(F)$ be the Weyl groups associated to E and F, respectively. The cohomological restriction map $\mathrm{H}^\bullet(E, \mathbb{C}) \to \mathrm{H}^\bullet(F, \mathbb{C})$ is completely determined by the induced map $j^* : P_E \to P_F$ on the spaces of primitive elements.

Let $H \subset F$ be a Cartan subalgebra of F, and let $H' \subset E$ be a Cartan subalgebra of E with $j(H) \subset H'$. Let $S(E^*)$ be the ring of polynomial functions on E, but with the subspace E^* concentrated in degree 2. Similarly, define $S(F^*)$, $S(H^*)$, and $S(H'^*)$ to be the evenly graded rings of polynomial functions on F, H, and H'. The

TABLE 1. Degrees of homogeneous generators for $\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$.

Type	Degrees
A_r	$3,5,7,\ldots,2r+1$
B_r	$3,7,11,\ldots,4r-1$
C_r	$3,7,11,\ldots,4r-1$
D_r $(r\geq 4)$	$3,7,11,\ldots,4r-5,2r-1$
E_6	$3,9,11,15,17,23$
E_7	$3,11,15,19,23,27,35$
E_8	$3,15,23,27,35,39,47,59$
F_4	$3,11,15,23$
G_2	$3,11$

coadjoint action of E on E^* extends to an action of E on $S(E^*)$, and similarly for F on $S(F^*)$. Then the restriction map $S(E^*)\to S(F^*)$ induces a map $S(E^*)^E\to S(F^*)^F$. By [14, §11.9], the restriction maps $S(E^*)\to S(H'^*)$ and $S(F^*)\to S(H^*)$ induce isomorphisms $S(E^*)^E\cong S(H'^*)^{W(E)}$ and $S(F^*)^F\cong S(H^*)^{W(F)}$. Since $W(E)$ and $W(F)$ are finite reflection groups, the rings $S(H'^*)^{W(E)}$ and $S(H^*)^{W(F)}$ are generated by algebraically independent homogeneous elements.

By [14, §6.7], there exists a canonical linear map $\rho_E : S(E^*)^E\to\Lambda^\bullet(E^*)^E$, homogeneous of degree -1 and natural with respect to the inclusion $F\subseteq E$. By [14, §6.14], $\operatorname{im}\rho_E=P_E$ and $\ker\rho_E=(S(E^*)^E)^2$. Then

$$j^*(P_E)\cong j^*(S(E^*)^E)/(S(F^*)^F)^2\cong j^*(S(H'^*)^{W(E)})/(S(H^*)^{W(F)})^2,$$

so to compute the map $j^*:P_E\to P_F$, and hence the map $j^*:\mathrm{H}^\bullet(E,\mathbb{C})\to\mathrm{H}^\bullet(F,\mathbb{C})$, it suffices to determine which polynomial generators for $S(H'^*)^{W(E)}$ restrict to a sum of decomposable elements in $S(H^*)^{W(F)}$.

In the following theorem we explicitly describe the cohomological restriction map $\mathrm{H}^\bullet(E,\mathbb{C})\to\mathrm{H}^\bullet(F,\mathbb{C})$ for certain simple pairs (E,F). Specifically, let E be a simple complex Lie algebra with associated root system Φ, and let $\Delta=\{\alpha_1,\ldots,\alpha_r\}$ be a set of simple roots for Φ, ordered as in [5, Plates I-IX]. Then we will assume that F is a simple subalgebra of E of rank $r-1$ corresponding to removing some simple root α_F from Δ. For ease in stating the theorem, we identify the Lie algebras E and F with their respective Lie types.

Theorem 2.3. *Let E, F, α_F be as in the previous paragraph. Write $\mathrm{H}^\bullet(E,\mathbb{C})=\Lambda(x_{i_1},\ldots,x_{i_r})$ and $\mathrm{H}^\bullet(F,\mathbb{C})=\Lambda(y_{j_1},\ldots,y_{j_{r-1}})$ as in Theorem 2.2, with the x_i and y_j homogeneous of degrees i and j, respectively. If one of E or F is of type D, write $\widetilde{x}_i$ or $\widetilde{y}_j$ for the generator of the last degree listed in Table 1. Then the homogeneous generators can be chosen so that the cohomological restriction map $\mathrm{H}^\bullet(E,\mathbb{C})\to\mathrm{H}^\bullet(F,\mathbb{C})$ admits the following description:*

(1) *If $(E,F)=(A_r,A_{r-1})$ and $\alpha_F=\alpha_1$, then*

$$x_3\mapsto y_3,\ x_5\mapsto y_5,\ \ldots,\ x_{2r-1}\mapsto y_{2r-1},\ x_{2r+1}\mapsto 0.$$

(2) *If $(E,F)=(B_r,B_{r-1})$ or (C_r,C_{r-1}) and $\alpha_F=\alpha_1$, then*

$$x_3\mapsto y_3,\ x_7\mapsto y_7,\ \ldots,\ x_{4r-5}\mapsto y_{4r-5},\ x_{4r-1}\mapsto 0.$$

(3) *If $(E,F)=(D_r,D_{r-1})$, $r\geq 5$, and $\alpha_F=\alpha_1$, then*

$$x_3\mapsto y_3,\ x_7\mapsto y_7,\ \ldots,\ x_{4r-9}\mapsto y_{4r-9},\ x_{4r-5}\mapsto 0,\ \widetilde{x}_{2r-1}\mapsto 0.$$

(4) *If* $(E,F) = (D_r, A_{r-1})$ *and* $\alpha_F = \alpha_r$, *then* $\widetilde{x}_{2r-1} \mapsto y_{2r-1}$, *and*

$$x_i \mapsto \begin{cases} 2y_i & \textit{if } i = 4j-1 \textit{ for some } j \geq 1 \textit{ with } 2j \leq r, \\ 0 & \textit{otherwise.} \end{cases}$$

(5) *If* $(E,F) = (E_6, D_5)$ *and* $\alpha_F = \alpha_6$, *then*

$$x_3 \mapsto y_3,\ x_9 \mapsto \widetilde{y}_9,\ x_{11} \mapsto y_{11},\ x_{15} \mapsto y_{15},\ x_{17} \mapsto 0,\ x_{23} \mapsto 0.$$

(6) *If* $(E,F) = (E_7, E_6)$ *and* $\alpha_F = \alpha_7$, *then*

$$x_3 \mapsto 2y_3,\ x_{11} \mapsto 2y_{11},\ x_{15} \mapsto 2y_{15},\ x_{19} \mapsto 0,\ x_{23} \mapsto 2y_{23},\ x_{27} \mapsto 0,\ x_{35} \mapsto 0.$$

Proof. The various cases of the theorem are established through direct computation of either $P_E \to P_F$, $S(E^*)^E \to S(F^*)^F$ or $S(H'^*)^{W(E)} \to S(H^*)^{W(F)}$. The case (E_6, D_5) is computed in [23, (5.6)], the case (E_7, E_6) is computed in [24, (2.3)], and the remaining cases (and many others) are computed in [14, Chapter XI.4]. □

3. Cohomology for the integral form U_{A}

Next we study the cohomological properties of a certain integral form U_{A} of U_q, to be defined in Section 3.3, which will enable us to relate cohomology for U_q to that for the Lie algebra $\mathfrak{g}$. First we collect some results on the algebra $U_{\mathcal{Z}}$.

3.1. **A resolution of the trivial module.** We begin with the following lemma, which is well-known for the Lusztig integral form of U_q, though we could find no analogous statement in the literature for the De Concini–Kac integral form $U_{\mathcal{Z}}$ as we have defined it here. We thus record the result now.

Lemma 3.1. *The algebra* $U_{\mathcal{Z}}$ *is a free* $\mathcal{Z}$*-module. Consequently, for any* $\mathcal{Z}$*-algebra* B, *the algebra* $U_B = U_{\mathcal{Z}} \otimes_{\mathcal{Z}} B$ *is free over* B.

Proof sketch. The algebra $U_{\mathcal{Z}}$ inherits from U_q the triangular decomposition $U_{\mathcal{Z}} \cong U_{\mathcal{Z}}^+ \otimes_{\mathcal{Z}} U_{\mathcal{Z}}^0 \otimes_{\mathcal{Z}} U_{\mathcal{Z}}^-$, where $U_{\mathcal{Z}}^+$ (resp. $U_{\mathcal{Z}}^-$) is the $\mathcal{Z}$-subalgebra of $U_{\mathcal{Z}}$ generated by the E_i (resp. F_i) for $i \in [1,r]$. Since $U_{\mathcal{Z}}^+$ contains for each $\beta \in \Phi^+$ the root vector E_β, it follows that $U_{\mathcal{Z}}^+$ is spanned over $\mathcal{Z}$ by the collection of PBW-monomials $\prod_{\beta\in\Phi^+} E_\beta^{n_\beta}$, $n_\beta \in \mathbb{N}$, and that these monomials form a $\mathcal{Z}$-basis for $U_{\mathcal{Z}}^+$; cf. [11, §1.7]. By symmetry, $U_{\mathcal{Z}}^-$ is also free over $\mathcal{Z}$. For $i \in [1,r]$, let A_i be the $\mathcal{Z}$-subalgebra of $U_{\mathcal{Z}}^0$ generated by $\{K_i^{\pm 1}, [K_i;0]\}$. Then $U_{\mathcal{Z}}^0 \cong A_1 \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} A_r$, so to prove the first claim it suffices to show that each A_i is $\mathcal{Z}$-free. Observe that $K_i^{-1} = K_i - (q_i - q_i^{-1})[K_i;0]$, where $q_i = q^{d_i}$, so A_i is generated as a $\mathcal{Z}$-algebra by K_i and $[K_i;0]$. The identity also shows that $K_i^2 = 1 + (q_i - q_i^{-1})K_i[K_i;0]$, so it follows that A_i is spanned over $\mathcal{Z}$ by the collection of elements $\{[K_i;0]^n, K_i[K_i;0]^m : n,m \in \mathbb{N}\}$. Now one can apply [11, (1.5.4)] and [16, 6.4(b2) and 6.7(i)(c)] to deduce that this set is linearly independent over $\mathcal{Z}$ and hence forms a $\mathcal{Z}$-basis for A_i. Thus we conclude that $U_{\mathcal{Z}}$ is free over $\mathcal{Z}$. The second claim of the lemma is now immediate. □

Lemma 3.2. *Let* B *be a noetherian* $\mathcal{Z}$*-algebra. Then* U_B *is noetherian.*

Proof sketch. The argument is due to Brown and Goodearl [6, §2.2]. In [10, §10.1], De Concini and Procesi define a sequence of degenerations

$$U_q = U^{(0)}, U^{(1)}, \ldots, U^{(2N)} \tag{3.1}$$

of the algebra U_q, each of which is the associated graded ring of the previous algebra with respect to a multiplicative $\mathbb{N}$-filtration. The definition of the degenerations relies on the commutation relations between the root vectors in U_q. Since the root vectors in U_q are elements of $U_{\mathcal{Z}}$ by our choice for the denominator set S, one can define a similar sequence of degenerations

$$U_B = U_B^{(0)}, U_B^{(1)}, \dots, U_B^{(2N)}$$

of the algebra U_B such that $U_B^{(2N)}$ is an iterated twisted polynomial ring over the torus U_B^0. The torus U_B^0 is generated as a B-algebra by the finite set of commuting elements $\{K_i^{\pm 1}, [K_i; 0] : i \in [1, r]\}$, where $[K_i; 0] := E_iF_i - F_iE_i$ (cf. [11, (1.5.4)]), so it is noetherian because B is noetherian. Then U_B is noetherian by [19, Theorems 1.2.9 and 1.6.9]. □

Corollary 3.3. *Let B be a $\mathcal{Z}$-algebra. There exists a resolution of the trivial U_B-module B by finitely-generated free U_B-modules:*

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to B \to 0.$$

Proof. First consider the case $B = \mathcal{Z}$. Set $P_{-1} = \mathcal{Z}$, $P_0 = U_{\mathcal{Z}}$, and let $P_0 \to P_{-1}$ be the augmentation map. Now given P_n with $n \geq 0$, let I_n be the kernel of the map $P_n \to P_{n-1}$. Since by induction P_n is a finitely-generated $U_{\mathcal{Z}}$-module, and since $U_{\mathcal{Z}}$ is noetherian by Lemma 3.2, the $U_{\mathcal{Z}}$-submodule I_n of P_n is also finitely-generated as a $U_{\mathcal{Z}}$-module. Then there exists a finitely-generated free $U_{\mathcal{Z}}$-module P_{n+1} mapping onto I_n. Take $P_{n+1} \to P_n$ to be the composite map $P_{n+1} \twoheadrightarrow I_n \hookrightarrow P_n$. We thus inductively construct the resolution $P_\bullet \to \mathcal{Z}$ of $\mathcal{Z}$ by finitely-generated free $U_{\mathcal{Z}}$-modules. Since $U_{\mathcal{Z}}$ is free over $\mathcal{Z}$ by Lemma 3.1, $P_\bullet \to \mathcal{Z}$ is a complex of free $\mathcal{Z}$-modules, and hence splits over $\mathcal{Z}$. It then follows for any $\mathcal{Z}$-algebra B that $P_\bullet \otimes_{\mathcal{Z}} B \to B$ is a resolution of B by finitely-generated free U_B-modules. □

3.2. **Base change and the universal coefficient theorem.** The crux of our argument for computing the cohomology ring $\mathrm{H}^\bullet(U_q, k)$ relies on the universal coefficient theorem, which we now recall.

Theorem 3.4 (Universal Coefficient Theorem for Homology [21, Theorem 7.55])**.** *Let R be a ring, A a left R-module, and (K, d) a chain complex of flat right R-modules such that the subcomplex of boundaries also consists of flat R-modules. Then for each $n \in \mathbb{Z}$, there exists a short exact sequence*

$$0 \to \mathrm{H}_n(K) \otimes_R A \xrightarrow{\lambda_n} \mathrm{H}_n(K \otimes_R A) \xrightarrow{\mu_n} \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(K), A) \to 0, \tag{3.2}$$

natural with respect to both K and A, such that $\lambda_n : \mathrm{cls}(z) \otimes a \mapsto \mathrm{cls}(z \otimes a)$.

We apply the universal coefficient theorem as follows:

Lemma 3.5. *Let B be a $\mathcal{Z}$-algebra, and Γ a B-algebra. Suppose B is a principal ideal domain. Then for each $n \in \mathbb{N}$, there exists a short exact sequence*

$$0 \to \mathrm{H}^n(U_B, B) \otimes_B \Gamma \xrightarrow{\lambda_n} \mathrm{H}^n(U_\Gamma, \Gamma) \xrightarrow{\mu_n} \mathrm{Tor}_1^B(\mathrm{H}^{n+1}(U_B, B), \Gamma) \to 0, \tag{3.3}$$

and the induced map $\lambda : \mathrm{H}^\bullet(U_B, B) \otimes_B \Gamma \to \mathrm{H}^\bullet(U_\Gamma, \Gamma)$ is an algebra homomorphism.

Proof. Let $P_\bullet \to B$ be a resolution of B by finitely-generated free U_B-modules as in Corollary 3.3, and set $K_n = \mathrm{Hom}_{U_B}(P_{-n}, B)$. Then the chain complex $K_\bullet$ consists of finitely-generated free B-modules. Since every submodule of a free module over a PID is again free, the subcomplex of boundaries in K is also free, hence flat, over B. Also, since P_n is free over U_B, there exists for each $n \in \mathbb{N}$ a natural isomorphism

$$\mathrm{Hom}_{U_B}(P_n, B) \otimes_B \Gamma \cong \mathrm{Hom}_{U_\Gamma}(P_n \otimes_B \Gamma, \Gamma). \tag{3.4}$$

Then applying the universal coefficient theorem with $R = B$, $A = \Gamma$, and K as above, one obtains the short exact sequence (3.3).

Now let $\alpha \in \mathrm{H}^a(U_B, B)$ and $\beta \in \mathrm{H}^b(U_B, B)$ be represented by cocycles $f_\alpha \in K_{-a}$ and $f_\beta \in K_{-b}$, respectively, and let $\Delta : P \to P \otimes_B P$ be a U_B-module chain map lifting the isomorphism $B \cong B \otimes_B B$. Then the product $\alpha\beta$ is represented by the cocycle $(f_\alpha \otimes_B f_\beta) \circ \Delta \in K_{-(a+b)}$. Observe that $\Delta \otimes \mathrm{id}_\Gamma : P \otimes_B \Gamma \to (P \otimes_B P) \otimes_B \Gamma \cong (P \otimes_B \Gamma) \otimes_\Gamma (P \otimes_B \Gamma)$ is a chain map lifting the isomorphism $\Gamma \cong \Gamma \otimes_\Gamma \Gamma$. Then making the identification (3.4), one sees for all $\gamma_\alpha, \gamma_\beta \in \Gamma$ that $\lambda(\alpha\beta \otimes_B \gamma_\alpha\gamma_\beta)$ and the product $\lambda_a(\alpha \otimes \gamma_\alpha)\lambda_b(\beta \otimes \gamma_\beta)$ are both represented by the cocycle

$$[(f_\alpha \otimes_B f_\beta) \circ \Delta] \otimes_B \gamma_\alpha\gamma_\beta \in \mathrm{Hom}_{U_B}(P_{a+b}, B) \otimes_B \Gamma,$$

and hence that λ is an algebra homomorphism. □

In Lemma 3.5 we assumed that B was a principal ideal domain to conclude that the subcomplex of boundaries in K was flat. This conclusion would also hold under the weaker assumption that B is right semihereditary, or perhaps under even weaker assumptions on B, but we will not require such a generalization in this paper.

We now collect some results useful for analyzing the Tor-group in (3.3).

Lemma 3.6. *Let B be a noetherian $\mathcal{Z}$-algebra. Then for each $n \in \mathbb{N}$, the cohomology group $\mathrm{H}^n(U_B, B)$ is a finitely-generated B-module.*

Proof. Let $K = \mathrm{Hom}_{U_B}(P_\bullet, B)$ be the complex of finitely-generated free B-modules considered in the proof of Lemma 3.5. Since B is noetherian, any subquotient of a finitely-generated B-module is again finitely-generated. In particular, $\mathrm{H}^n(U_B, B)$ is a B-module subquotient of K_{-n}, so is finitely-generated over B. □

Lemma 3.7. *Let B be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely-generated B-module. Then M is a free B-module if and only if $\mathrm{Tor}_1^B(M, B/\mathfrak{m}) = 0$.*

Proof. This follows from [4, II.3.2 Corollary 2 of Proposition 5]. □

In a similar vein, one has:

Lemma 3.8. *Let B be an integral domain, $b \in B$, and M a B-module. Then*

$$\mathrm{Tor}_1^B(M, B/bB) \cong \{m \in M : b.m = 0\}.$$

Proof. Compute the Tor-group using the resolution $0 \to B \overset{\times b}{\to} B \to B/bB \to 0$. □

3.3. **The integral form U_{A}.** We now define the integral form U_{A} and describe how we will apply the results of Section 3.2 to relate the cohomology theories for U_q, U_{A}, and $U(\mathfrak{g})$. To begin, set $\mathsf{A} = \mathbb{C}[q]_{(q-1)}$, the localization of $\mathbb{C}[q]$ at the maximal ideal generated by $q-1$. Then A is a local principal ideal domain, with quotient field $k = \mathbb{C}(q)$ and residue field $\mathbb{C}$. As in Section 1.2, we write $\mathbb{C}_1$ for the field $\mathbb{C}$ considered as an A-algebra via the map $q \mapsto 1$.

The field k is A-flat by [21, Corollary 3.50] because it is torsion-free, so applying Lemma 3.5 with $B = \mathsf{A}$ and $\Gamma = k$, we get for each $n \in \mathbb{N}$ the isomorphism

$$\mathrm{H}^n(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} k \cong \mathrm{H}^n(U_q, k). \tag{3.5}$$

On the other hand, $U_1 = U_{\mathsf{A}} \otimes_{\mathsf{A}} \mathbb{C}_1$, so applying Lemma 3.5 with $B = \mathsf{A}$ and $\Gamma = \mathbb{C}_1$, we get for each $n \in \mathbb{N}$ the short exact sequence

$$0 \to \mathrm{H}^n(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 \overset{\lambda_n}{\to} \mathrm{H}^n(U_1, \mathbb{C}) \to \mathrm{Tor}_1^{\mathsf{A}}(\mathrm{H}^{n+1}(U_{\mathsf{A}}, \mathsf{A}), \mathbb{C}_1) \to 0. \tag{3.6}$$

It follows from Lemmas 3.6 and 3.7 that the map λ_n is an isomorphism if and only if $\mathrm{H}^{n+1}(U_{\mathsf{A}}, \mathsf{A})$ is free as an A-module. In particular, if the algebra homomorphism $\lambda : \mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 \to \mathrm{H}^\bullet(U_1, \mathbb{C})$ is an isomorphism, then for each $n \in \mathbb{N}$, $\mathrm{H}^n(U_{\mathsf{A}}, \mathsf{A})$ must be A-free of rank $\dim_{\mathbb{C}} \mathrm{H}^n(U_1, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{H}^n(U(\mathfrak{g}), \mathbb{C})$.

Our strategy for computing $\mathrm{H}^\bullet(U_q, k)$ is now as follows. We first verify that the injective algebra homomorphism $\lambda : \mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 \to \mathrm{H}^\bullet(U_1, \mathbb{C})$ is an isomorphism, and hence that $\mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A})$ is A-free of rank $\dim_{\mathbb{C}} \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$, by showing that the odd degree homogeneous generators for $\mathrm{H}^\bullet(U_1, \mathbb{C}) \cong \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$ all lie in the image of λ. We verify this for $\mathfrak{g}$ not of type D_r or E_6 in Section 3.4, and for types D_r and E_6 in Sections 4.2 and 4.3. Next, using the fact that $\mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A})$ is A-free and that $\mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 \cong \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$ is an exterior algebra, we deduce in Section 4.1 that $\mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A})$ is an exterior algebra generated in the same odd degrees as is $\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$. Finally, we apply (3.5) to deduce the structure of $\mathrm{H}^\bullet(U_q, k)$.

3.4. **Cohomology for U_{A}.** Following the strategy outlined in Section 3.3, we first verify that λ is an isomorphism when $\mathfrak{g}$ is not of type D_r or E_6.

Theorem 3.9. *Suppose $\mathfrak{g}$ is not of type D_r or E_6. Then the injective algebra map*

$$\lambda : \mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 \to \mathrm{H}^\bullet(U_1, \mathbb{C})$$

is an isomorphism. In particular, $\mathrm{H}^\bullet(U_{\mathsf{A}}, \mathsf{A})$ is a finitely-generated free A-module.

Proof. We prove the theorem by showing that the odd-degree homogeneous generators for $\mathrm{H}^\bullet(U_1, \mathbb{C}) \cong \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$ described in Theorem 2.2 all lie in the image of λ. First suppose $\mathfrak{g}$ is of type A_1, A_2, B_2, C_2, E_7, E_8, F_4, or G_2, and let n be one of the odd degrees listed in Table 1. Using Theorem 2.2 and Table 1 one can check that $\mathrm{H}^{n+1}(U(\mathfrak{g}), \mathbb{C}) = 0$. Then (3.6) implies that

$$\mathrm{H}^{n+1}(U_{\mathsf{A}}, \mathsf{A})/(q-1)\,\mathrm{H}^{n+1}(U_{\mathsf{A}}, \mathsf{A}) \cong \mathrm{H}^{n+1}(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 = 0$$

and hence $\mathrm{H}^{n+1}(U_{\mathsf{A}}, \mathsf{A}) = 0$ by Nakayama's Lemma. Then $\lambda_n : \mathrm{H}^n(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 \to \mathrm{H}^n(U_1, \mathbb{C})$ is an isomorphism by (3.6), so for these Lie types we conclude that the odd-degree homogeneous generators for $\mathrm{H}^\bullet(U_1, \mathbb{C})$ all lie in the image of λ.

Now suppose that $\mathfrak{g}$ is of type X_r, with $X \in \{A, B, C\}$ and $r \geq 3$. Let $\mathfrak{g}' \subset \mathfrak{g}$ be the subalgebra of $\mathfrak{g}$ of type X_{r-1} as defined in cases (1) and (2) of Theorem 2.3. Define $U_q(\mathfrak{g}')$ and $U_{\mathsf{A}}(\mathfrak{g}')$ to be the subalgebras of U_q and U_{A}, respectively, generated by the set $\{E_i, F_i, K_i^{\pm 1} : i \in [2, r]\}$. Then $U_q(\mathfrak{g}')$ is isomorphic to the quantized enveloping algebra associated to $\mathfrak{g}'$, and $U_{\mathsf{A}}(\mathfrak{g}')$ is its corresponding integral form. By induction on the rank of $\mathfrak{g}$, we may assume for each $n \in \mathbb{N}$ that the space $\mathrm{H}^n(U_{\mathsf{A}}(\mathfrak{g}'), \mathsf{A})$ is A-free of rank $\dim_{\mathbb{C}} \mathrm{H}^n(U(\mathfrak{g}'), \mathbb{C})$. Let $n_1 < \cdots < n_r$ be the degrees listed in Table 1 of the homogeneous generators for $\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C}) \cong \mathrm{H}^\bullet(U_1, \mathbb{C})$. As in Theorem 2.3, write $\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C}) \cong \Lambda(x_{n_1}, \ldots, x_{n_r})$, with x_{n_i} of degree n_i, and set $z_i = x_{n_i}$. Let $j \in [1, r]$, and assume by induction that $z_1, \ldots, z_{j-1} \in \operatorname{im}(\lambda)$. To show that $z_j \in \operatorname{im}(\lambda)$, it suffices to show that $\mathrm{H}^{n_j+1}(U_{\mathsf{A}}, \mathsf{A})$ is A-free, since this implies by (3.6) that $\lambda_{n_j} : \mathrm{H}^{n_j}(U_{\mathsf{A}}, \mathsf{A}) \otimes_{\mathsf{A}} \mathbb{C}_1 \to \mathrm{H}^{n_j}(U_1, \mathbb{C})$ is an isomorphism.

By Theorem 2.2, the space $\mathrm{H}^{n_j+1}(U_1, \mathbb{C})$ is spanned by certain monomials in the generators $z_1, \ldots, z_r$, but since $n_i \neq 1$ for any i, no non-zero monomial can involve a generator z_i with $i \geq j$. Then $\mathrm{H}^{n_j+1}(U_1, \mathbb{C})$ is spanned by certain monomials in the generators $z_1, \ldots, z_{j-1} \in \operatorname{im}(\lambda)$, and it follows that these monomials are in the image of λ, and hence that λ_{n_j+1} is an isomorphism. Now consider the following diagram, where the vertical arrows are the corresponding restriction maps:

$$\begin{tikzcd} \mathrm{H}^{n_j+1}(U_{\mathsf{A}}(\mathfrak{g}),\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1 \arrow[r, "\lambda_{n_j+1}", "\sim"'] \arrow[d] & \mathrm{H}^{n_j+1}(U(\mathfrak{g}),\mathbb{C}) \arrow[d] \\ \mathrm{H}^{n_j+1}(U_{\mathsf{A}}(\mathfrak{g}'),\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1 \arrow[r, "\lambda_{n_j+1}", "\sim"'] & \mathrm{H}^{n_j+1}(U(\mathfrak{g}'),\mathbb{C}) \end{tikzcd} \tag{3.7}$$

The commutativity of the diagram follows from the fact that the universal coefficient theorem (Theorem 3.4) is natural with respect to the complex K. The bottom map in the diagram is an isomorphism by induction on the rank of the Lie algebra. The right-hand restriction map is also an isomorphism, since by Theorem 2.3 the homogeneous generators $z_1,\dots,z_{j-1}$ for $\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$ can be chosen so that the restriction map $\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})\to\mathrm{H}^\bullet(U(\mathfrak{g}'),\mathbb{C})$ maps them onto the corresponding generators for $\mathrm{H}^\bullet(U(\mathfrak{g}'),\mathbb{C})$. This implies that the left-hand restriction map is an isomorphism as well, hence that the map

$$\mathrm{H}^{n_j+1}(U_{\mathsf{A}}(\mathfrak{g}),\mathsf{A})\to\mathrm{H}^{n_j+1}(U_{\mathsf{A}}(\mathfrak{g}'),\mathsf{A})/(q-1)\,\mathrm{H}^{n_j+1}(U_{\mathsf{A}}(\mathfrak{g}'),\mathsf{A})$$

is surjective. Then the restriction map $\mathrm{H}^{n_j+1}(U_{\mathsf{A}},\mathsf{A})\to\mathrm{H}^{n_j+1}(U_{\mathsf{A}}(\mathfrak{g}'),\mathsf{A})$ is surjective by Nakayama's Lemma. By induction on the rank of the Lie algebra, the space $\mathrm{H}^{n_j+1}(U_{\mathsf{A}}(\mathfrak{g}'),\mathsf{A})$ is A-free of rank $\dim_{\mathbb{C}}\mathrm{H}^{n_j+1}(U(\mathfrak{g}'),\mathbb{C})=\dim_{\mathbb{C}}\mathrm{H}^{n_j+1}(U(\mathfrak{g}),\mathbb{C})$. Then the restriction map $\mathrm{H}^{n_j+1}(U_{\mathsf{A}},\mathsf{A})\to\mathrm{H}^{n_j+1}(U_{\mathsf{A}}(\mathfrak{g}'),\mathsf{A})$ is a split surjection of A-modules, and $\mathrm{H}^{n_j+1}(U_{\mathsf{A}},\mathsf{A})$ has A-rank at least $\dim_{\mathbb{C}}\mathrm{H}^{n_j+1}(U(\mathfrak{g}),\mathbb{C})$. Now

$$\begin{aligned}\dim_{\mathbb{C}}\mathrm{H}^{n_j+1}(U(\mathfrak{g}),\mathbb{C}) &\le \dim_k \mathrm{H}^{n_j+1}(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}k \qquad \text{by the bound on the } \mathsf{A}\text{-rank},\\ &\le \dim_{\mathbb{C}}\mathrm{H}^{n_j+1}(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1\\ &=\dim_{\mathbb{C}}\mathrm{H}^{n_j+1}(U(\mathfrak{g}),\mathbb{C}),\end{aligned}$$

so we conclude that $\mathrm{H}^{n_j+1}(U_{\mathsf{A}},\mathsf{A})$ is A-free by [2, Lemma 1.21]. □

4. Cohomology for the quantized enveloping algebra U_q

4.1. **Cohomology ring structure.** We now deduce the structure of $\mathrm{H}^\bullet(U_q,k)$ in any case for which $\lambda:\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1\to\mathrm{H}^\bullet(U_1,\mathbb{C})$ is an isomorphism.

Theorem 4.1. *Suppose that $\lambda:\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1\to\mathrm{H}^\bullet(U_1,\mathbb{C})$ is an isomorphism. Then the cohomology rings $\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})$ and $\mathrm{H}^\bullet(U_q(\mathfrak{g}),k)$ are exterior algebras generated by homogeneous elements in the odd degrees listed in Table* 1.

Proof. Since λ is an isomorphism, we have for each $n\in\mathbb{N}$ that $\mathrm{H}^n(U_{\mathsf{A}},\mathsf{A})$ is a free A-module of rank $\dim_{\mathbb{C}}\mathrm{H}^n(U(\mathfrak{g}),\mathbb{C})$ by the discussion in Section 3.3. Choose homogeneous elements $z_1,\dots,z_r\in\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})$ such that their images under λ in $\mathrm{H}^\bullet(U_1,\mathbb{C})\cong\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$ are the homogeneous generators described in Theorem 2.2. Since U_{A} is a Hopf algebra over the commutative ring A, the cohomology ring $\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})$ is graded-commutative [17, Corollary VIII.4.3]. The elements $z_1,\dots,z_r\in\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})$ are each homogeneous of odd degree, so $z_i^2=0$ for each $i\in[1,r]$, and there exists a well-defined map $\varphi:\Lambda(z_1,\dots,z_r)\to\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})$ of graded A-algebras. The induced map $\varphi\otimes_{\mathsf{A}}\mathbb{C}_1:\Lambda(z_1,\dots,z_r)\otimes_{\mathsf{A}}\mathbb{C}_1\to\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1$ is surjective by the choice of the z_i, so we conclude by Nakayama's Lemma that φ is surjective, hence a graded algebra isomorphism because $\Lambda(z_1,\dots,z_r)$ and $\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})$ are each A-free of the same finite rank. Extending scalars to k, we obtain via (3.5) the graded algebra isomorphism $\varphi\otimes_{\mathsf{A}}k:\Lambda(z_1,\dots,z_r)\otimes_{\mathsf{A}}k\xrightarrow{\sim}\mathrm{H}^\bullet(U_q(\mathfrak{g}),k)$. □

4.2. **Type D.** To extend Theorem 3.9 to the case when $\mathfrak{g}$ is of type D_r, we consider cohomological restriction maps corresponding not only to a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ of type D_{r-1} but also to a Lie subalgebra $\mathfrak{g}''$ of $\mathfrak{g}$ of type A_{r-1}. In the latter case, we also require the explicit understanding of the ring structure for $\mathrm{H}^\bullet(U_q(\mathfrak{g}''), k)$ that comes from Theorem 4.1.

Theorem 4.2. *The conclusion of Theorem* 3.9 *holds if* $\mathfrak{g}$ *is of type* D_r.

Proof. Suppose $\mathfrak{g}$ is of type D_r with $r \geq 4$. The overall strategy is similar to that in the proof of Theorem 3.9 for types A, B, and C, though some subtleties arise because the right-hand column of (3.7) need not be an isomorphism when $\mathfrak{g}$ is of type D. As in the proof of Theorem 3.9, we consider a subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of type D_{r-1}, as defined in case (3) of Theorem 2.3, and also a subalgebra $\mathfrak{g}'' \subset \mathfrak{g}$ of type A_{r-1}, as defined in case (4) of Theorem 2.3. (If $r = 4$, then $\mathfrak{g}'$ is of type A_3, and cases (3) and (4) of Theorem 2.3 coincide.) For $j \in [1, r-1]$ set $n_j = 4j - 1$, and set $n_r = 2r - 1$, so that $n_1, \ldots, n_r$ are the degrees listed in Table 1 for type D_r.

Our first step is to show for all $n \in [1, 2r]$ that $\mathrm{H}^n(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \cong \mathrm{H}^n(U_1, \mathbb{C})$. Since $\mathrm{H}^\bullet(U_1, \mathbb{C})$ is an exterior algebra generated in the odd degrees $n_1, \ldots, n_r$, this is equivalent to showing $\mathrm{H}^{n_j}(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \cong \mathrm{H}^{n_j}(U_1, \mathbb{C})$ whenever $n_j \leq 2r - 1$. First let $j \in [1, r]$ with $n_j \leq 2r-3$. It follows from Theorem 2.3 that the restriction map $\mathrm{H}^{n_j+1}(U(\mathfrak{g}), \mathbb{C}) \to \mathrm{H}^{n_j+1}(U(\mathfrak{g}'), \mathbb{C})$ is an isomorphism; cf. the analysis of (3.7). Also, by induction on the rank of $\mathfrak{g}$, we may assume for all $n \in [1, 2(r-1)]$ that $\mathrm{H}^n(U_\mathsf{A}(\mathfrak{g}'), \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \cong \mathrm{H}^n(U(\mathfrak{g}'), \mathbb{C})$, and hence that $\mathrm{H}^n(U_\mathsf{A}(\mathfrak{g}'), \mathsf{A})$ is A-free of rank $\dim_\mathbb{C} \mathrm{H}^n(U(\mathfrak{g}'), \mathbb{C})$; cf. Section 3.3. Now one can imitate the proof of Theorem 3.9, arguing by induction on the rank and the degree, to show for all $n_j \leq 2r - 3$ that $\mathrm{H}^{n_j}(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \cong \mathrm{H}^{n_j}(U_1, \mathbb{C})$. Then to complete the first step, we must now show that $\mathrm{H}^{2r-1}(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \cong \mathrm{H}^{2r-1}(U_1, \mathbb{C})$.

Given $y \in \mathrm{H}^\bullet(U_\mathsf{A}, \mathsf{A})$, set $\overline{y} = \lambda(y \otimes_\mathsf{A} 1) \in \mathrm{H}^\bullet(U_1, \mathbb{C})$. By the previous paragraph, we can choose $y_1, \ldots, y_s \in \mathrm{H}^\bullet(U_\mathsf{A}, \mathsf{A})$ such that $\overline{y}_1, \ldots, \overline{y}_s \in \mathrm{H}^\bullet(U_1, \mathbb{C})$ are representatives for the homogeneous generators for $\mathrm{H}^\bullet(U_1, \mathbb{C})$ of degrees less than or equal to $2r-3$. Then $\mathrm{H}^{2r}(U_1, \mathbb{C})$ is spanned over $\mathbb{C}$ by certain monomials in the vectors $\overline{y}_1, \ldots, \overline{y}_s$. Let $m_1, \ldots, m_t \in \mathrm{H}^{2r}(U_\mathsf{A}, \mathsf{A})$ be monomials in the y_i such that $\overline{m}_1, \ldots, \overline{m}_t$ form a basis for $\mathrm{H}^{2r}(U_1, \mathbb{C})$. We want to show that $\dim_k \mathrm{H}^{2r}(U_q, k) \geq t$, for this implies by (3.5) and [2, Lemma 1.21] that $\mathrm{H}^{2r}(U_\mathsf{A}, A)$ is A-free, and hence that $\mathrm{H}^{2r-1}(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \cong \mathrm{H}^{2r-1}(U_1, \mathbb{C})$ by (3.3).

Let $\rho : \mathrm{H}^\bullet(U_q, k) \to \mathrm{H}^\bullet(U_q(\mathfrak{g}''), k)$ be the restriction map. Given $y \in \mathrm{H}^\bullet(U_\mathsf{A}, \mathsf{A})$, let $\widetilde{y}$ denote its image in $\mathrm{H}^\bullet(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} k \cong \mathrm{H}^\bullet(U_q, k)$. By Theorems 3.9 and 4.1, $\mathrm{H}^\bullet(U_q(\mathfrak{g}''), k)$ is an exterior algebra generated by homogeneous elements of certain odd degrees. Moreover, it follows from Theorem 2.3 and the proof of Theorem 4.1 that we can take certain of the generators for $\mathrm{H}^\bullet(U_q(\mathfrak{g}''), k)$ to be the vectors $\rho(\widetilde{y}_1), \ldots, \rho(\widetilde{y}_s)$. This implies that the vectors $\rho(\widetilde{m}_1), \ldots, \rho(\widetilde{m}_t) \in \mathrm{H}^{2r}(U_q(\mathfrak{g}''), k)$ are linearly independent, and hence $\widetilde{m}_1, \ldots, \widetilde{m}_t \in \mathrm{H}^{2r}(U_q, k)$ are as well. We then conclude that $\dim_k \mathrm{H}^{2r}(U_q, k) \geq t$, which completes the first step of the proof.

We have shown for all $a \in \mathbb{N}$ that if $\mathfrak{g}$ is of type D_a, then $\mathrm{H}^n(U_\mathsf{A}, \mathsf{A}) \otimes_\mathsf{A} \mathbb{C}_1 \cong \mathrm{H}^n(U_1, \mathbb{C})$ for $n \in [1, 2a]$. Write $\mathrm{H}^\bullet(U_1, \mathbb{C}) \cong \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C}) \cong \Lambda(x_3, \ldots, x_{4r-5}, \widetilde{x}_{2r-1})$ as in Theorem 2.3. Suppose $n_i = \deg(x_i) > 2r - 1$; we must show that $x_i \in \mathrm{im}(\lambda)$. Set $m = 2r - 2$, and let $\mathfrak{g}_m$ be the finite-dimensional simple complex Lie algebra of type D_m. The inclusion of Dynkin diagrams $D_r \hookrightarrow D_m$ induces an inclusion of algebras $U_\mathsf{A} \hookrightarrow U_\mathsf{A}(\mathfrak{g}_m)$; cf. Section 3.4. We thus have the following commutative diagram, where the vertical arrows are the corresponding restriction maps:

$$\begin{tikzcd} \mathrm{H}^{n_i}(U_{\mathsf{A}}(\mathfrak{g}_m),\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1 \arrow[r, "\lambda_{n_i}"] \arrow[d] & \mathrm{H}^{n_i}(U(\mathfrak{g}_m),\mathbb{C}) \arrow[d] \\ \mathrm{H}^{n_i}(U_{\mathsf{A}}(\mathfrak{g}),\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1 \arrow[r, "\lambda_{n_i}"] & \mathrm{H}^{n_i}(U(\mathfrak{g}),\mathbb{C}) \end{tikzcd} \tag{4.1}$$

Since $n_i \leq 4r-5 < 2m$, the top row of (4.1) is an isomorphism by the first step of the proof. Also, since $n_i \leq 4r-5 \leq 4m-9$, it follows from Theorem 2.3 that x_i is in the image of the restriction map $\mathrm{H}^\bullet(U(\mathfrak{g}_m),\mathbb{C}) \to \mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$. Then from the commutativity of (4.1) we conclude that $x_i \in \mathrm{im}(\lambda)$. This completes the proof. $\square$

Corollary 4.3. *The conclusion of Theorem* 4.1 *holds if* $\mathfrak{g}$ *is of type* D_r*.*

4.3. **Type E_6.** To extend Theorem 3.9 to the case when $\mathfrak{g}$ is of type E_6, we consider restriction maps like those in cases (5) and (6) of Theorem 2.3.

Theorem 4.4. *Suppose* $\mathfrak{g}$ *is of type* E_6*. Then* $\mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1 \cong \mathrm{H}^\bullet(U_1,\mathbb{C})$*.*

Proof sketch. The strategy is similar to the proofs of Theorems 3.9 and 4.2. The generators for $\mathrm{H}^\bullet(U_1,\mathbb{C})$ are in degrees 3, 9, 11, 15, 17, and 23. One can check using Theorem 2.2 that $\mathrm{H}^n(U_1,\mathbb{C}) = 0$ for $n \in \{4,10,16\}$, so $\mathrm{H}^n(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1 \cong \mathrm{H}^n(U_1,\mathbb{C})$ if $n \in \{3,9,15\}$. The description of the restriction map from E_7 to E_6 in case (6) of Theorem 2.3 implies that the generators of degrees 11 and 23 are also in $\mathrm{im}(\lambda)$; cf. the analysis of (4.1). Then it remains to show that $\mathrm{H}^{17}(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1 \cong \mathrm{H}^{17}(U_1,\mathbb{C})$, or equivalently that $\mathrm{H}^{18}(U_{\mathsf{A}},\mathsf{A})$ is A-free. We can choose $y_3 \in \mathrm{H}^3(U_{\mathsf{A}},\mathsf{A})$ and $y_{15} \in \mathrm{H}^{15}(U_{\mathsf{A}},\mathsf{A})$ such that the product $\overline{y}_3\overline{y}_{15}$ spans $\mathrm{H}^{18}(U_1,\mathbb{C})$. Let $\mathfrak{g}' \subset \mathfrak{g}$ be the subalgebra of type D_5 as defined in case (5) of Theorem 2.3, and let $\rho : \mathrm{H}^\bullet(U_{\mathsf{A}},\mathsf{A}) \to \mathrm{H}^\bullet(U_{\mathsf{A}}(\mathfrak{g}'),\mathsf{A})$ be the corresponding restriction map. Then the argument in the third paragraph of the proof of Theorem 3.9 shows that ρ is surjective in degrees 3 and 15. This implies by the proof of Theorem 4.1 that $\mathrm{H}^{18}(U_{\mathsf{A}}(\mathfrak{g}'),\mathsf{A})\otimes_{\mathsf{A}} k \cong \mathrm{H}^{18}(U_q(\mathfrak{g}'),k) \cong k$ is spanned by $\rho(y_3y_{15})$. Then the product $y_3y_{15} \in \mathrm{H}^{18}(U_{\mathsf{A}},\mathsf{A})$ must span a one-dimensional subspace of $\mathrm{H}^{18}(U_q,k)$. Now

$$1 \leq \dim_k \mathrm{H}^{18}(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}} k \leq \dim_{\mathbb{C}} \mathrm{H}^{18}(U_{\mathsf{A}},\mathsf{A})\otimes_{\mathsf{A}}\mathbb{C}_1 \leq \dim_{\mathbb{C}} \mathrm{H}^{18}(U_1,\mathbb{C}) = 1,$$

so $\mathrm{H}^{18}(U_{\mathsf{A}},\mathsf{A})$ must be A-free or rank 1 by [2, Lemma 1.21]. $\square$

Here is the main result of our computations:

Theorem 4.5. *The cohomology ring* $\mathrm{H}^\bullet(U_q,k)$ *is an exterior algebra over a graded subspace with odd gradation. Explicitly,* $\mathrm{H}^\bullet(U_q,k)$ *is generated as an exterior algebra by homogeneous elements in the same odd degrees as for* $\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$*.*

4.4. **The third cohomology group.** A famous theorem of Chevalley and Eilenberg states that $\mathrm{H}^3(U(\mathfrak{g}),\mathbb{C}) \neq 0$ [9, Theorem 21.1]. They prove the non-vanishing of $\mathrm{H}^3(U(\mathfrak{g}),\mathbb{C})$ by showing that the Killing form on $\mathfrak{g}$ gives rise to a non-vanishing invariant 3-cochain in $\mathfrak{g}$. Our analysis gives us:

Corollary 4.6. *Let* $\mathfrak{g}$ *be a finite-dimensional simple complex Lie algebra. Then* $\dim_k \mathrm{H}^3(U_q(\mathfrak{g}),k) = 1$*.*

It is an interesting question whether the non-vanishing of $\mathrm{H}^3(U_q(\mathfrak{g}),k)$ could also be established in a manner similar to that of Chevalley and Eilenberg, perhaps by using the non-degenerate inner product on $U_q(\mathfrak{g})$ constructed by Rosso [20].

5. Cohomology for the specializations U_ε

5.1. **Generic behavior.** Recall the set S defined in Section 1.2. Set $T = S \cup \{(q+1)\}$, and set $\mathcal{A} = T^{-1}\mathbb{C}[q,q^{-1}]$. We call $\varepsilon \in \mathbb{C}$ a *bad root of unity* if $\varepsilon = 1$ or if ε is the root of some polynomial in T. Define the set $\mathbb{C}_\mathfrak{g} \subset \mathbb{C}$ by

$$\mathbb{C}_\mathfrak{g} = \left\{\varepsilon \in \mathbb{C}^\times : \varepsilon \text{ is not a bad root of unity}\right\}.$$

Then for all $\varepsilon \in \mathbb{C}_\mathfrak{g}$, the field $\mathbb{C}$ is an $\mathcal{A}$-algebra via the map $q \mapsto \varepsilon$, and we can apply the results of Section 3.2 with $B = \mathcal{A}$ and $\Gamma = \mathbb{C}_\varepsilon$. Moreover, up to multiplication by units, every prime element in $\mathcal{A}$ has the form $(q-1)$ or $(q-\varepsilon)$ for some $\varepsilon \in \mathbb{C}_\mathfrak{g}$.

Proposition 5.1. *The ring* $\mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A})$ *is a finitely-generated* $\mathcal{A}$*-module.*

Proof. For each $n \in \mathbb{N}$, the space $\mathrm{H}^n(U_\mathcal{A},\mathcal{A})$ is a finitely-generated $\mathcal{A}$-module by Lemma 3.6. Set $d = \dim_\mathbb{C} \mathfrak{g}$. Then for all $\varepsilon \in \mathbb{C}_\mathfrak{g}$, the ring $\mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$ satisfies the Poincaré duality $\mathrm{H}^n(U_\varepsilon,\mathbb{C}) \cong \mathrm{H}_{d-n}(U_\varepsilon,\mathbb{C})$ by [8, Corollary 3.2.2]. In particular, $\mathrm{H}^n(U_\varepsilon,\mathbb{C}) = 0$ for all $n > d$. This implies by Lemma 3.5 with $B = \mathcal{A}$ and $\Gamma = \mathbb{C}_\varepsilon$ that $\mathrm{H}^n(U_\mathcal{A},\mathcal{A})\otimes_\mathcal{A}\mathbb{C}_\varepsilon = 0$ for all $n > d$. Lemma 3.5 also implies that $\mathrm{H}^n(U_\mathcal{A},\mathcal{A})\otimes_\mathcal{A} \mathbb{C}_1 = 0$ for $n > d$, since $\mathrm{H}^n(U_1,\mathbb{C}) \cong \mathrm{H}^n(U(\mathfrak{g}),\mathbb{C}) = 0$ for $n > d$. Then it follows from the fundamental theorem for finitely-generated modules over a principal ideal domain that $\mathrm{H}^n(U_\mathcal{A},\mathcal{A}) = 0$ for all $n > d$. Then $\mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A}) = \bigoplus_{n=1}^{n=d} \mathrm{H}^n(U_\mathcal{A},\mathcal{A})$, so $\mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A})$ is a finitely-generated $\mathcal{A}$-module. $\square$

Corollary 5.2. *For all but finitely many* $\varepsilon \in \mathbb{C}_\mathfrak{g}$, $\mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A}) \otimes_\mathcal{A} \mathbb{C}_\varepsilon \cong \mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$, *and for all such* $\varepsilon \in \mathbb{C}_\mathfrak{g}$, $\mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$ *is generated as an exterior algebra by homogeneous elements in the same odd degrees as for* $\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$.

Proof. Set $S' = \{(q-\varepsilon) \in \mathcal{A} : \mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A}) \text{ has } (q-\varepsilon)\text{-torsion}\}$. It follows from Proposition 5.1 and the fundamental theorem for finitely-generated modules over a principal ideal domain that S' is finite. Set $\mathcal{B} = (S')^{-1}\mathcal{A}$. Since $\mathcal{B}$ is flat over $\mathcal{A}$, we have $\mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A}) \otimes_\mathcal{A} \mathcal{B} \cong \mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B})$ by Lemma 3.5, and we deduce that $\mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B})$ is a free $\mathcal{B}$-module, all of the torsion having been eliminated by the choice of denominator set. Since $\mathsf{A} = \mathbb{C}[q]_{(q-1)}$ is a localization of $\mathcal{B}$, we similarly have $\mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B})\otimes_\mathcal{B}\mathsf{A} \cong \mathrm{H}^\bullet(U_\mathsf{A},\mathsf{A})$, and we can choose the odd-degree generators $z_1,\ldots,z_r$ for the exterior algebra $\mathrm{H}^\bullet(U_\mathsf{A},\mathsf{A})$ to be elements of the subspace $\mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B})$. Then there exists an algebra map of free $\mathcal{B}$-modules $\varphi : \Lambda(z_1,\ldots,z_r) \to \mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B})$.

Set $W = \mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B})/\operatorname{im}(\varphi)$. Since $\Lambda(x_1,\ldots,x_r)$ and $\mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B})$ are each free modules of the same finite rank over the principal ideal domain $\mathcal{B}$, W is a finitely-generated torsion $\mathcal{B}$-module. Let $\varepsilon \in \mathbb{C}_\mathfrak{g}$ be such that $(q-\varepsilon) \notin S'$ and W has no $(q-\varepsilon)$-torsion. Then $\mathrm{Tor}_1^\mathcal{B}(W,\mathbb{C}_\varepsilon) = \mathrm{Tor}_1^\mathcal{B}(W,\mathcal{B}/(q-\varepsilon)\mathcal{B}) = 0$ by Lemma 3.8, so it follows from the long exact sequence for $\mathrm{Tor}_1^\mathcal{B}(-,\mathbb{C}_\varepsilon)$ applied to the sequence

$$0 \to \Lambda(z_1,\ldots,z_r) \xrightarrow{\varphi} \mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B}) \to W \to 0$$

that the algebra map $\varphi\otimes_\mathcal{B}\mathbb{C}_\varepsilon : \Lambda(x_1,\ldots,x_r)\otimes_\mathcal{B}\mathbb{C}_\varepsilon \to \mathrm{H}^\bullet(U_\mathcal{B},\mathcal{B})\otimes_\mathcal{B}\mathbb{C}_\varepsilon$ is injective. Then by dimension comparison $\varphi \otimes_\mathcal{B} \mathbb{C}_\varepsilon$ must also be surjective, hence an algebra isomorphism. Thus, the conclusion of the corollary holds for all $\varepsilon \in \mathbb{C}_\mathfrak{g}$ such that $\mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A})$ and W are each $(q-\varepsilon)$-torsion free, and fails for only the finitely many $\varepsilon \in \mathbb{C}_\mathfrak{g}$ such that one of $\mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A})$ or W has $(q-\varepsilon)$-torsion. $\square$

While Corollary 5.2 states for almost all values $\varepsilon \in \mathbb{C}_\mathfrak{g}$ that $\mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$ is an exterior algebra over an r-dimensional graded subspace, it unfortunately does not give any indication of the values for which this condition fails. We can at least say that the only values for which $\mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$ might not be an exterior algebra are those ε that are algebraic over $\mathbb{Q}$. Indeed, let $B = S^{-1}\mathbb{Q}[q,q^{-1}]$, with S as defined in Section 1.2. Then for each $n \in \mathbb{N}$, the space $\mathrm{H}^n(U_B,B)$ is a finitely-generated B-module by Lemma 3.6, and $\mathrm{H}^\bullet(U_B,B)\otimes_B \mathcal{A} \cong \mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A})$. This shows that $\mathrm{H}^\bullet(U_\mathcal{A},\mathcal{A})$ has $(q-\varepsilon)$-torsion if and only if there exists an irreducible polynomial $f \in \mathbb{Q}[q]$ such that $(q-\varepsilon)$ divides f in $\mathbb{C}[q]$ and $\mathrm{H}^\bullet(U_B,B)$ has f-torsion. We summarize this discussion in the following proposition:

Proposition 5.3. *If* $\varepsilon \in \mathbb{C}_\mathfrak{g}$ *is transcendental over* $\mathbb{Q}$*, then* $\mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$ *is an exterior algebra generated by homogeneous elements in the same odd degrees as for* $\mathrm{H}^\bullet(\mathfrak{g},\mathbb{C})$.

5.2. **Roots of unity.** Let p be a prime, and let h be the Coxeter number of the root system associated to $\mathfrak{g}$. We can show that the conclusion of Corollary 5.2 holds for U_ε provided ε is a primitive p-th root of unity and $p > 3(h-1)$.

Theorem 5.4. *Let* $\varepsilon \in \mathbb{C}$ *be a primitive* p*-th root of unity with* $p > 3(h-1)$*. Then* $\mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$ *is an exterior algebra generated by homogeneous elements in the same odd degrees as for* $\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$.

Proof sketch. The theorem is established by a sequence of arguments completely analogous to those used for the case when the parameter of the quantized enveloping algebra is an indeterminant, except that instead of relating $\mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$ to Lie algebra cohomology in characteristic zero, we relate $\mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$ to Lie algebra cohomology in characteristic p. Let U_q' be the algebra over $\mathbb{Q}(q)$ defined by the same generators and relations as for U_q, and let U_ε' be the algebra over $\mathbb{Q}(\varepsilon)$ obtained by replacing q in the definition of U_q' by ε. Then $U_\varepsilon' \otimes_{\mathbb{Q}(\varepsilon)} \mathbb{C} \cong U_\varepsilon$ and $\mathrm{H}^\bullet(U_\varepsilon',\mathbb{Q}(\varepsilon))\otimes_{\mathbb{Q}(\varepsilon)}\mathbb{C} \cong \mathrm{H}^\bullet(U_\varepsilon,\mathbb{C})$, so it suffices to show that $\mathrm{H}^\bullet(U_\varepsilon',\mathbb{Q}(\varepsilon))$ is an exterior algebra generated by homogeneous elements in the same odd degrees as for $\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$.

Let $\mathbb{F}_p$ be the field with p elements, and consider the map $\pi : \mathbb{Z}[q] \to \mathbb{F}_p$ that takes $q \mapsto 1$. Let $\phi_p(q) = q^{p-1} + \cdots + q + 1$ be the p-th cyclotomic polynomial. Then $\phi_p(1) = p$, and π factors through a map $\pi' : \mathbb{Z}[\varepsilon] \cong \mathbb{Z}[q]/(\phi_p) \to \mathbb{F}_p$. Let $\mathcal{Z}'$ be the localization of $\mathbb{Z}[\varepsilon]$ at the maximal ideal $\ker \pi'$. The ring $\mathbb{Z}[\varepsilon]$ is a noetherian Dedekind domain (because the ring of integers in an algebraic number field is always a Dedekind domain), hence so is the localization $\mathcal{Z}'$. A local Dedekind domain is a principal ideal domain, so we can apply the results of Sections 3.1–3.2 to the $\mathcal{Z}$-algebra $\mathcal{Z}'$, its quotient field $\mathbb{Q}(\varepsilon)$, and its residue field $\mathbb{F}_p$.

Let $\mathfrak{g}_{\mathbb{F}_p}$ be the Lie algebra over $\mathbb{F}_p$ obtained by the extension of scalars from a Chevalley basis for $\mathfrak{g}$. Then $U_{\mathcal{Z}'} \otimes_{\mathcal{Z}'} \mathbb{F}_p$ is a central extension of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{F}_p})$ by the group algebra over $\mathbb{F}_p$ for the finite group $G = (\mathbb{Z}/2\mathbb{Z})^r$. Since p is odd, the group algebra $\mathbb{F}_pG$ is a semisimple ring. Then as in Lemma 2.1, we get $\mathrm{H}^\bullet(U_{\mathcal{Z}'}\otimes_{\mathcal{Z}'}\mathbb{F}_p,\mathbb{F}_p) \cong \mathrm{H}^\bullet(U(\mathfrak{g}_{\mathbb{F}_p}),\mathbb{F}_p)$. Since $p > 3(h-1)$, the latter ring is an exterior algebra generated by homogeneous elements in the same odd degrees as for $\mathrm{H}^\bullet(U(\mathfrak{g}),\mathbb{C})$ [13, Theorem 1.2]. Now one argues as in Sections 3.4–4.3 to show that $\mathrm{H}^\bullet(U_{\mathcal{Z}'},\mathcal{Z}')$ is a finitely-generated free $\mathcal{Z}'$-module and that $\mathrm{H}^\bullet(U_{\mathcal{Z}'},\mathcal{Z}')$ and $\mathrm{H}^\bullet(U_{\mathcal{Z}'},\mathcal{Z}')\otimes_{\mathcal{Z}'}\mathbb{Q}(\varepsilon) \cong \mathrm{H}^\bullet(U_\varepsilon',\mathbb{Q}(\varepsilon))$ are exterior algebras over graded subspaces concentrated in the correct odd degrees. □

The lower bound of $3(h-1)$ in the above theorem is not sharp. The bound is made in order to guarantee that the cohomology ring $\mathrm{H}^\bullet(U(\mathfrak{g}_{\mathbb{F}_p}),\mathbb{F}_p)$ is an exterior algebra generated in the correct degrees. We have conducted computer calculations to compute the structure of $\mathrm{H}^\bullet(U(\mathfrak{g}_{\mathbb{F}_p}),\mathbb{F}_p)$ when p is small and when $\mathfrak{g}$ is of type A_1,

A_2, B_2, or G_2, and have determined in these cases that it is sufficient to assume $p > h$. Though we suspect that $\mathrm{H}^\bullet(U(\mathfrak{g}_{\mathbb{F}_p}), \mathbb{F}_p)$ should be an exterior algebra provided only that $p > h$, we have no proof of this claim at this time.

5.3. **Conjectures.** If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\varepsilon \in \mathbb{C}_\mathfrak{g}$, then it follows from Poincaré duality and [18, Theorem 7.16 and Remark 7.17(1)] that $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ is an exterior algebra generated by a vector in degree 3. For higher ranks it is not clear (at least, it is not clear to the author) how to proceed in general, even for specific values of $\varepsilon \in \mathbb{C}^\times$. If $\varepsilon \in \mathbb{C}^\times$ is not a root of unity, then it is well-known that the categories of finite-dimensional type-1 modules for U_q and U_ε are both equivalent to the category of finite-dimensional $\mathfrak{g}$-modules. The BGG categories for U_q and U_ε are also both equivalent to the integral block of the BGG category $\mathcal{O}$ for $\mathfrak{g}$ [1, Remark 6.3]. One might then hope to extend this equivalence to a larger subcategory of infinite-dimensional U_ε-modules containing the trivial module and thereby prove the following conjecture:

Conjecture 5.5. *Suppose $\varepsilon \in \mathbb{C}(q)$ is not a root of unity. Then $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ is an exterior algebra generated in the same odd degrees as for $\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$.*

In establishing the fact that the inclusion map $\Lambda^\bullet(\mathfrak{g}^*)^\mathfrak{g} \to \Lambda^\bullet(\mathfrak{g}^*)$ induces an isomorphism $\Lambda(\mathfrak{g}^*)^\mathfrak{g} \cong \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$, one uses the complete reducibility of finite-dimensional $\mathfrak{g}$-modules to conclude that $\Lambda^\bullet(\mathfrak{g}^*)^\mathfrak{g}$ is a $\mathfrak{g}$-module summand in the space of cocycles in $\Lambda^\bullet(\mathfrak{g}^*)$, and hence that $\Lambda^\bullet(\mathfrak{g}^*)^\mathfrak{g} \cong \mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$. If one could explicitly construct a finite-dimensional complex P computing $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ such that each term in P was a U_ε-module (i.e., a quantum version of the Koszul complex), then one could try to imitate the classical approach, at least for ε not a root of unity, to try to understand the structure of the cohomology ring $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$.

While no one has yet constructed a quantum analogue for the Koszul complex, it may be possible to find a suitable substitute by considering the sequence of May spectral sequences arising from the algebra degenerations (3.1) of De Concini and Procesi. Indeed, we have successfully used this approach in [12, §5.4] to help deduce the ring structure of the cohomology ring for the nilpotent subalgebra U_ε^-.

Finally, based on the results of Theorem 5.4 and on the comments made in the last paragraph of Section 5.2, we offer the following conjecture for the structure of $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ when $\varepsilon \in \mathbb{C}$ is a root of unity:

Conjecture 5.6. *Let ℓ be an odd positive integer, with ℓ coprime to 3 if $\mathfrak{g}$ is of type G_2. Let $\varepsilon \in \mathbb{C}$ be a primitive ℓ-th root of unity, and suppose $\ell > h$. Then $\mathrm{H}^\bullet(U_\varepsilon, \mathbb{C})$ is an exterior algebra generated in the same odd degrees as for $\mathrm{H}^\bullet(U(\mathfrak{g}), \mathbb{C})$.*

Acknowledgements

The author expresses his gratitude to Brian Parshall for suggesting this problem, to Nick Kuhn for useful conversations on the cohomology of Lie groups, and to the referee for pointing out several flaws in an earlier version of the manuscript.

References

1. H. H. Andersen and V. Mazorchuk, *Category $\mathcal{O}$ for quantum groups*, preprint, 2011.
2. H. H. Andersen, P. Polo, and K. Wen, *Representations of quantum algebras*, Invent. Math. **104** (1991), no. 1, 1–59. MR1094046 (92e:17011)
3. D. W. Barnes, *Spectral sequence constructors in algebra and topology*, Mem. Amer. Math. Soc. **53** (1985), no. 317, viii+174. MR776177 (86e:55032)

4. N. Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation. MR1727221 (2001g:13001)
5. ______, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley. MR1890629 (2003a:17001)
6. K. A. Brown and K. R. Goodearl, *Homological aspects of Noetherian PI Hopf algebras of irreducible modules and maximal dimension*, J. Algebra **198** (1997), no. 1, 240–265. MR1482982 (99c:16036)
7. K. A. Brown and J. J. Zhang, *Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras*, J. Algebra **320** (2008), no. 5, 1814–1850. MR2437632 (2009j:16007)
8. S. Chemla, *Rigid dualizing complex for quantum enveloping algebras and algebras of generalized differential operators*, J. Algebra **276** (2004), no. 1, 80–102. MR2054387 (2005e:17022)
9. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124. MR0024908 (9,567a)
10. C. De Concini and C. Procesi, *Quantum groups*, D-modules, representation theory, and quantum groups (Venice, 1992), Lecture Notes in Math., vol. 1565, Springer, Berlin, 1993, pp. 31–140. MR1288995 (95j:17012)
11. C. De Concini and V. G. Kac, *Representations of quantum groups at roots of* 1, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA, 1990, pp. 471–506. MR1103601 (92g:17012)
12. C. M. Drupieski, D. K. Nakano, and N. V. Ngo, *Cohomology for infinitesimal unipotent algebraic and quantum groups*, Transform. Groups **17** (2012), no. 2, 393–416. MR2921071
13. E. M. Friedlander and B. J. Parshall, *Cohomology of Lie algebras and algebraic groups*, Amer. J. Math. **108** (1986), no. 1, 235–253 (1986). MR821318 (87c:20080)
14. W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology. Vol. III: Cohomology of principal bundles and homogeneous spaces*, Academic Press, New York, 1976, Pure and Applied Mathematics, Vol. 47-III. MR0400275 (53:4110)
15. N. Kowalzig and U. Krähmer, *Duality and products in algebraic (co)homology theories*, J. Algebra **323** (2010), no. 7, 2063–2081. MR2594665
16. G. Lusztig, *Quantum groups at roots of* 1, Geom. Dedicata **35** (1990), no. 1-3, 89–113. MR1066560 (91j:17018)
17. S. Mac Lane, *Homology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1975 edition. MR1344215 (96d:18001)
18. A. Masuoka, *Abelian and non-abelian second cohomologies of quantized enveloping algebras*, J. Algebra **320** (2008), no. 1, 1–47. MR2417975 (2009f:16016)
19. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Pure and Applied Mathematics (New York), John Wiley & Sons Ltd., Chichester, 1987, With the cooperation of L. W. Small, A Wiley-Interscience Publication. MR934572 (89j:16023)
20. M. Rosso, *Analogues de la forme de Killing et du théorème d'Harish-Chandra pour les groupes quantiques*, Ann. Sci. École Norm. Sup. (4) **23** (1990), no. 3, 445–467. MR1055444 (93e:17026)
21. J. J. Rotman, *An introduction to homological algebra*, second ed., Universitext, Springer, New York, 2009. MR2455920 (2009i:18011)
22. H. Samelson, *Topology of Lie groups*, Bull. Amer. Math. Soc. **58** (1952), 2–37. MR0045129 (13,533b)
23. H. Toda and T. Watanabe, *The integral cohomology ring of F_4/T and E_6/T*, J. Math. Kyoto Univ. **14** (1974), 257–286. MR0358847 (50:11306)
24. T. Watanabe, *The integral cohomology ring of the symmetric space EVII*, J. Math. Kyoto Univ. **15** (1975), no. 2, 363–385. MR0380857 (52:1754)

Department of Mathematics, University of Georgia, Athens, Georgia 30602-7403

E-mail address: cdrup@math.uga.edu

Current address: Department of Mathematical Sciences, DePaul University, Chicago, Illinois 60614

E-mail address: cdrupies@depaul.edu