HEISENBERG UNIQUENESS PAIRS IN THE PLANE.
THREE PARALLEL LINES

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Abstract. A Heisenberg uniqueness pair is a pair \((\Gamma, \Lambda)\), where \(\Gamma\) is a curve in the plane and \(\Lambda\) is a set in the plane, with the following property: any bounded Borel measure \(\mu\) in the plane supported on \(\Gamma\), which is absolutely continuous with respect to arc length and whose Fourier transform \(\hat{\mu}\) vanishes on \(\Lambda\), must automatically be the zero measure. We characterize the Heisenberg uniqueness pairs for \(\Gamma\) as being three parallel lines \(\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\}\) with \(\alpha < \beta < \gamma\), 
\((\gamma - \alpha) / (\beta - \alpha) \in \mathbb{N}\).

1. Introduction

The Heisenberg uncertainty principle states that both a function and its Fourier transform cannot be too localized at the same time (see [2] and [3]). M. Benedicks in [1] proved that given a nontrivial function \(f \in L^1(\mathbb{R}^n)\), the Lebesgue measure of the set of points where \(f \neq 0\) and the set of points where the Fourier transform \(\hat{f} \neq 0\) cannot be simultaneously finite. In this paper we consider a similar problem for measures supported on a subset of \(\mathbb{R}^2\).

Let \(\Gamma\) be a smooth curve in the plane \(\mathbb{R}^2\) and \(\Lambda\) a subset in \(\mathbb{R}^2\). In [4], Hedenmalm and Montes-Rodríguez posed the problem of deciding when it is true that 
\[\hat{\mu}\mid_{\Lambda} = 0 \text{ implies } \mu = 0\]
for any Borel measure \(\mu\) supported on \(\Gamma\) and absolutely continuous with respect to the arc length measure on \(\Gamma\), where 
\[\hat{\mu}(\xi, \eta) = \int_{\mathbb{R}^2} e^{\pi i ((x,y), (\xi, \eta))} d\mu(x,y)\]

If this is the case, then \((\Gamma, \Lambda)\) is called a Heisenberg Uniqueness Pair (HUP).

When \(\Gamma\) is the circle, Lev [7] and Sjölin [8] independently characterized the HUP for some “small” sets \(\Lambda\).

In [4] Hedenmalm and Montes-Rodríguez characterized the HUP in the cases:

- \(\Gamma\) the hyperbola \(xy = 1\) and \(\Lambda = (\alpha \mathbb{Z} \times \{0\}) \cup \{0\} \times \beta \mathbb{Z}\), for \(\alpha, \beta > 0\).
- \(\Gamma\) two parallel lines in \(\mathbb{R}^2\).
In this note we present a result generalizing this last case. We characterize the HUP for $\Gamma$ as being three parallel lines:

$$\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\} \text{ with } \alpha < \beta < \gamma, \ (\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}.$$ 

2. Three parallel lines

Given a set $E \subset \mathbb{R}$ and a point $\xi \in E$, let us define:

- $A_{\text{loc}}^{E,\xi} = \{\text{functions } \psi \text{ defined on } E \text{ such that there exist a small interval } I_\xi \text{ around } \xi \text{ and a function } \varphi \in L^1(\mathbb{R}) \text{ such that } \psi(\zeta) = \varphi(\zeta), \text{ for } \zeta \in I_\xi \cap E\}$.
- $P^{1,p}[A_{\text{loc}}^{E,\xi}] = \{\text{functions } \psi \text{ defined on } E \text{ such that there exist an interval } I_\xi \text{ around } \xi \text{ and functions } \varphi_0, \varphi_1 \in L^1(\mathbb{R}) \text{ with } \psi^p(\zeta) + \varphi_1(\zeta)\psi(\zeta) + \varphi_0(\zeta) = 0, \text{ for } \zeta \in I_\xi \cap E\}$.

Wiener’s lemma [5, p. 57] states that if $\psi \in A_{\text{loc}}^{E,\xi}$ and $\psi(\xi) \neq 0$, then $1/\psi \in A_{\text{loc}}^{E,\xi}$. Observe also that if $\psi \in A_{\text{loc}}^{E,\xi}$ then $\psi \in P^{1,p}[A_{\text{loc}}^{E,\xi}]$. This is easy to see only if $p$ is natural.

Due to invariance under translation and rescaling (see [4]) it will be sufficient to study the case when $\Gamma = \mathbb{R} \times \{0, 1, p\}$ for $p \in \mathbb{N}$, $p > 1$.

Given a set $\Lambda \subset \mathbb{R}^2$, we say that $\mu$ is an admissible measure if $\mu$ is a Borel measure in the plane absolutely continuous with respect to arc length with $\text{supp } \mu \subset \Gamma$ and $\widehat{\mu}|_{\Lambda} = 0$.

If $\mu$ is a measure absolutely continuous with respect to arc length on $\Gamma$, then there exist functions $f, g, h \in L^1(\mathbb{R})$ such that

$$\widehat{\mu}(\xi, \eta) = \hat{f}(\xi) + e^{\pi i \eta} \hat{g}(\xi) + e^{p\pi i \eta} \hat{h}(\xi), \text{ for any } (\xi, \eta) \in \mathbb{R}^2.$$ 

In particular an admissible measure can be written in this form. Observe also that $\widehat{\mu}$ is 2-periodic with respect to the second variable. So, for any set $\Lambda \subset \mathbb{R}^2$, we may consider the periodized set

$$\mathcal{P}(\Lambda) = \{(\xi, \eta) \text{ such that } (\xi, \eta + 2k) \in \Lambda \text{ for some } k \in \mathbb{Z}\},$$

and it follows that $(\Gamma, \Lambda)$ is a HUP if and only if $(\Gamma, \overline{\mathcal{P}(\Lambda)})$ is a HUP, where $\overline{\mathcal{P}(\Lambda)}$ stands for the closure of $\mathcal{P}(\Lambda)$ in $\mathbb{R}^2$.

We may think without loss of generality that $\Lambda$ is a closed set in $\mathbb{R}^2$, 2-periodic with respect to the second coordinate.

We then have the following result.

**Theorem 1.** Let $\Gamma = \mathbb{R} \times \{0, 1, p\}$, for some $p \in \mathbb{N}$, $p > 1$ and $\Lambda \subset \mathbb{R}^2$, closed and 2-periodic with respect to the second variable. Then $(\Gamma, \Lambda)$ is a Heisenberg uniqueness pair if and only if

$$(2.1) \quad \mathfrak{F} := \Pi^3(\Lambda) \cup (\Pi^2(\Lambda) \setminus \Pi^2(\Lambda)) \cup (\Pi^1(\Lambda) \setminus \Pi^1(\Lambda))$$

is dense in $\mathbb{R}$.

$\Pi(\Lambda)$ means the projection of $\Lambda$ on the axis $\mathbb{R} \times \{0\}$ and given a point $\xi \in \Pi(\Lambda)$, and $\text{Img}(\xi)$ corresponds to the set of points $\eta \in [0, 2)$ with $(\xi, \eta) \in \Lambda$. The sets in $\mathfrak{F}$ are defined as follows:

- $\Pi^1(\Lambda) = \{\xi \in \Pi(\Lambda) \text{ such that there is a unique } \eta_0 \in \text{Img}(\xi)\}$.
- $\Pi^2(\Lambda) = \{\xi \in \Pi(\Lambda) \text{ such that there are two different points } \eta_0, \eta_1 \in \text{Img}(\xi), \text{ and if there is another point } \eta_2 \in \text{Img}(\xi), \text{ then } \frac{e^{p\pi i \eta_2} - e^{\pi i \eta_0}}{e^{\pi i \eta_2} - e^{\pi i \eta_0}} = \frac{e^{p\pi i \eta_1} - e^{\pi i \eta_0}}{e^{\pi i \eta_1} - e^{\pi i \eta_0}}\}$. 

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\* \( \Pi^3(\Lambda) = \{ \xi \in \Pi(\Lambda) \text{ such that there are at least three different points } \eta_0, \eta_1, \eta_2 \in Img(\xi) \text{ with } e^{p\pi i \eta_1} - e^{p\pi i \eta_0} \neq e^{p\pi i \eta_2} - e^{p\pi i \eta_0} \}. \)

The following technical lemma is easy to prove and shows that the functions \( \tau \) and \( \Phi \) are well defined for \( \xi \in \Pi^3(\Lambda) \).

**Lemma 2.** Let \( x, y, z \in \mathbb{C} \) be different with
\[
\tau = \frac{y^p - x^p}{y - x} = \frac{z^p - x^p}{z - x};
\]
then
\[
\frac{z^p - y^p}{z - y} = \tau \quad \text{and} \quad \Phi = x\tau - x^p = y\tau - y^p = z\tau - z^p.
\]

Let \( \chi \) be a function defined on \( \Pi^1(\Lambda) \) as \( \chi(\xi) = e^{p\pi i \eta} \), where \( \eta \in Img(\xi) \). We define the set \( \Pi^{1*}(\Lambda) \) as

\* \( \Pi^{1*}(\Lambda) = \{ \xi \in \Pi^1(\Lambda) \text{ such that } \chi \in \mathcal{D}_{\mathcal{P}}[A^{\Pi^1(\Lambda)}_{loc}, \xi] \}. \)

Let \( \tau, \Phi \) be functions defined on \( \Pi^2(\Lambda) \) as
\[
\tau(\xi) = \frac{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}}{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}} \quad \text{and} \quad \Phi(\xi) = e^{p\pi i \eta_0} \frac{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}}{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}} - e^{p\pi i \eta_0},
\]
where \( \eta_0, \eta_1 \in Img(\xi) \). We define the set \( \Pi^{2*}(\Lambda) \) as

\* \( \Pi^{2*}(\Lambda) = \{ \xi \in \Pi^2(\Lambda) \text{ such that } \tau, \Phi \in A^{\Pi^2(\Lambda)}_{loc}, \xi \}. \)

The next lemma will be needed for the proof of the necessity of condition (2.1) in Theorem 1.

**Lemma 3.** Let \( I \) be an interval in \( \mathbb{R} \) with \( \Pi^2(\Lambda) \) dense in \( I \). Then there exists a subinterval \( I' \subset I \) with \( I' \subset \Pi^2(\Lambda) \cup \Pi^3(\Lambda) \).

**Proof.** Pick an arbitrary point \( \tilde{\xi} \in I \cap \Pi^2(\Lambda) \). Since \( \tau, \Phi \in A^{\Pi^2(\Lambda)}_{loc}, \tilde{\xi} \) and \( \Pi^2(\Lambda) \) is dense in \( I \), we can extend the functions \( \tau, \Phi \) continuously on a neighborhood of \( \tilde{\xi} \). Let \( \tilde{\eta} \neq \tilde{\eta} \in Img(\tilde{\xi}) \). Then
\[
|\tau(\tilde{\xi})| = \left| \frac{e^{p\pi i \tilde{\eta}} - e^{p\pi i \tilde{\eta}}}{e^{p\pi i \tilde{\eta}} - e^{p\pi i \tilde{\eta}}} \right| < p,
\]
and since \( \tau \) is continuous around \( \tilde{\xi} \), there exists a small interval \( I' \) around \( \tilde{\xi} \) with \( |\tau(\xi)| < p \) for \( \xi \in I' \). We will see that \( I' \subset \Pi^2(\Lambda) \cup \Pi^3(\Lambda) \).

Given \( \xi \in I' \), consider a sequence \( \{\xi_k\} \subset \Pi^2(\Lambda) \cap I' \) with \( \xi_k \to \xi \), and for each \( \xi_k \) let \( \eta_k \neq \eta_k \in Img(\xi_k) \). There exist subsequences \( \{\eta^*_k\} \) and \( \{\eta^*_k\} \) such that \( \eta^*_k \to \eta^* \) and \( \eta^*_k \to \eta^* \) for some \( \eta^*, \eta^* \in [0, 2] \). Since the set \( \Lambda \) is closed and 2-periodic with respect to the second coordinate, we can assume WLOG that \( \xi \in \Pi(\Lambda) \) with \( \eta^* \neq \eta^* \in Img(\xi) \). Otherwise,
\[
|\tau(\xi)| \leftarrow |\tau(\xi,\eta)| = \left| e^{(p-1)p\pi i \eta^*} + e^{(p-2)p\pi i \eta^*} + \ldots + e^{(p-1)p\pi i \eta^*} \right| = p
\]
which is a contradiction with the fact that \( \xi \in I' \).

So \( I' \subset \Pi^2(\Lambda) \cup \Pi^3(\Lambda) \), and since the extended functions \( \tau, \Phi \) are continuous on \( I' \), we also have that \( \xi \in \Pi^2(\Lambda) \) for any \( \xi \in \Pi^2(\Lambda) \cap I' \). Also, we can conclude that \( I' \subset \Pi^2(\Lambda) \cup \Pi^3(\Lambda) \). \( \square \)
3. Proof of the main result

This section is devoted to the proof of Theorem 1. The proof of the sufficiency of condition (2.1) is rather easy. Let μ be an admissible measure. Then there exist functions \( f, g, h \in L^1(\mathbb{R}) \) such that

\[
\hat{\mu}(\xi, \eta) = \hat{f}(\xi) + e^{\pi i \eta} \hat{g}(\xi) + e^{\pi i \eta} \hat{h}(\xi), \quad \text{for any } (\xi, \eta) \in \mathbb{R}^2.
\]

Since \( \mathfrak{f} \) is dense in \( \mathbb{R} \) we will be done if we show that \( \hat{f}(\xi) = \hat{g}(\xi) = \hat{h}(\xi) = 0 \) for any \( \xi \in \mathfrak{f} = \Pi^3(\Lambda) \cup (\Pi^2(\Lambda) \setminus \Pi^2(\Lambda)) \cup (\Pi^1(\Lambda) \setminus \Pi^1(\Lambda)). \)

If \( \xi \in \Pi^3(\Lambda) \), let \( \eta_0, \eta_1, \eta_2 \in \text{Im} g(\xi) \) be different. Since \( \hat{\mu}_{|\Lambda} = 0 \) and \( \frac{e^{\pi i \eta} - e^{\pi i \eta_0}}{e^{\pi i \eta_2} - e^{\pi i \eta_0}} \), it follows that \( \hat{f}(\xi) = \hat{g}(\xi) = \hat{h}(\xi) = 0. \)

If \( \xi \in \Pi^2(\Lambda) \), let \( \eta_0 \neq \eta_1 \in \text{Im} g(\xi) \). Since \( \hat{\mu}_{|\Lambda} = 0 \), then \( \hat{g}(\xi) = -\tau(\xi) \hat{h}(\xi) \) and \( \hat{f}(\xi) = \Phi(\xi) \hat{h}(\xi) \). Suppose \( \hat{h}(\xi) \neq 0 \). Then by Wiener’s lemma and Fubini’s theorem, \( \tau, \Phi \in \mathcal{A}_{\text{loc}}^{\Pi^2(\Lambda)} \), which implies that \( \xi \in \Pi^2(\Lambda) \). So if \( \xi \in \Pi^2(\Lambda) \setminus \Pi^2(\Lambda) \), then \( \hat{f}(\xi) = \hat{g}(\xi) = \hat{h}(\xi) = 0. \)

Finally, if \( \xi \in \Pi^1(\Lambda) \) and \( \eta_0 \in \text{Im} g(\xi) \), since \( \hat{\mu}_{|\Lambda} = 0 \), then \( \hat{f}(\xi) = \hat{\chi}(\xi) \hat{g}(\xi) + \chi^p(\xi) \hat{h}(\xi) = 0 \), where \( \chi(\xi) = e^{\pi i \eta_0} \). Suppose \( \hat{h}(\xi) \neq 0 \); then \( \xi \in P^1, p[\mathcal{A}_{\text{loc}}^{\Pi^1(\Lambda)}] \) and \( \hat{\mu}_{|\Lambda} = 0 \). Otherwise, if \( \hat{g}(\xi) = 0 \), then by Wiener’s lemma and Fubini’s theorem, \( \chi \in \mathcal{A}_{\text{loc}}^{\Pi^1(\Lambda)} \) and also \( \chi^p \in \mathcal{A}_{\text{loc}}^{\Pi^1(\Lambda)} \), so \( \xi \in P^1, p[\mathcal{A}_{\text{loc}}^{\Pi^1(\Lambda)}] \) and \( \hat{\mu}_{|\Lambda} = 0 \). This means that \( \xi \in \Pi^2(\Lambda) \). Then \( \hat{f}(\xi) = \hat{g}(\xi) = \hat{h}(\xi) = 0. \)

For the proof of the necessity of condition (2.1), suppose that the set \( \mathfrak{f} \) is not dense in \( \mathbb{R} \) and let us pick an open interval \( I \) that has empty intersection with \( \mathfrak{f} \), i.e.,

\[
\Pi(\Lambda) \cap I = (\Pi^1(\Lambda) \cup \Pi^2(\Lambda)) \cap I.
\]

We consider three cases:

- There exists a small interval \( I_\xi \subset I \) around \( \xi \in \Pi^1(\Lambda) \) such that all the points in \( I_\xi \cap \Pi(\Lambda) \) belong to \( \Pi^1(\Lambda) \). Since \( \chi \in P^1, p[\mathcal{A}_{\text{loc}}^{\Pi^1(\Lambda)}] \), there exist an interval \( I' \subset I_\xi \) around \( \xi \) and functions \( \varphi_0, \varphi_1 \in L^1(\mathbb{R}) \) such that

\[
\chi^p(\xi^*) + \varphi_1(\xi^*) \chi(\xi^*) + \varphi_0(\xi^*) = 0
\]

for any \( \xi^* \in I' \cap \Pi(\Lambda) \). Let \( h \in L^1(\mathbb{R}) \) with \( \hat{h}(\xi) \neq 0 \) and \( \text{supp } \hat{h} \subset I' \), and define \( f, g \in L^1(\mathbb{R}) \) via \( \hat{f} = \hat{h} \hat{\varphi}_0 \), and \( \hat{g} = \hat{h} \hat{\varphi}_1 \). Now,

\[
\hat{\mu}(\xi^*, \eta^*) = \hat{f}(\xi^*) + \hat{g}(\xi^*) \chi(\xi^*) + \hat{h}(\xi^*) \chi^p(\xi^*) = 0
\]

for \( \xi^* \in I' \cap \Pi^1(\Lambda), \eta^* \in \text{Im} g(\xi^*) \). Finally, since \( \text{supp } \hat{h} \subset I' \) and \( I' \cap \Pi(\Lambda) = I' \cap \Pi^1(\Lambda) \), we can conclude that \( \hat{\mu}_{|\Lambda} = 0 \), and we have that \( \mu \) is a nontrivial admissible measure. So \( (\Gamma, \Lambda) \) is not a Heisenberg uniqueness pair.

- There exists a small interval \( I_\xi \subset I \) around \( \xi \in \Pi^2(\Lambda) \) such that all the points in \( I_\xi \cap \Pi(\Lambda) \) belong to \( \Pi^2(\Lambda) \). Now there exists a small interval \( I' \subset I_\xi \) around \( \xi \) and functions \( \Phi_1, \tau_1 \in L^1(\mathbb{R}) \) such that \( \tau_1 = \tau \) and \( \Phi_1 = \Phi \) on \( I' \cap \Pi(\Lambda) \). Consider a function \( h \in L^1(\mathbb{R}) \) with \( \text{supp } \hat{h} \subset I' \) and \( \hat{h}(\xi) \neq 0 \), and define \( f, g \in L^1(\mathbb{R}) \) as

\[
g = -h * \tau_1 \quad \text{and} \quad f = h * \Phi_1.
\]
Now, given a point $\xi^* \in I' \cap \Pi^2(\Lambda)$, let $\eta^* \neq \vartheta^* \in \text{Img}(\xi^*)$. Since $\tau(\xi^*) = \frac{e^{\vartheta^*}}{e^{\vartheta^*} - e^{\vartheta^*}}$ and $\Phi(\xi) = e^{\vartheta^*} - e^{\vartheta^*} - e^{\vartheta^*}$, we have
\[
\hat{\mu}(\xi^*, \eta^*) = \hat{f}(\xi^*) + \hat{g}(\xi^*)e^{\vartheta^*} + \hat{h}(\xi^*)e^{\vartheta^*} = 0
\]
and also that $\hat{\mu}(\xi^*, \vartheta^*) = 0$. So, the corresponding measure $\mu$ is a nontrivial admissible measure and $(\Gamma, \Lambda)$ is not a Heisenberg uniqueness pair.

- All the intervals $I_3 \subset I$ contain points in $\Pi^1(\Lambda)$ and points in $\Pi^2(\Lambda)$. That is, the sets $\Pi^1(\Lambda)$ and $\Pi^2(\Lambda)$ are dense in $I \cap (\Pi^1(\Lambda) \cup \Pi^2(\Lambda)) = I \cap \Pi(\Lambda)$. But this is not possible. In fact, if $\Pi^2(\Lambda)$ is dense in $I$, by Lemma 3 there exists a subinterval $I' \subset I$ such that $I' \subset \Pi^2(\Lambda) \cup \Pi^3(\Lambda).

This finishes the proof of the theorem.

4. Examples and Further Results

Given a point $\xi \in \Pi(\Lambda)$ such that $\sharp\{\eta \in \text{Img}(\xi)\} \geq 3$, we will state a criteria to decide whether the point $\xi$ belongs to $\Pi^3(\Lambda)$ or to $\Pi^2(\Lambda)$. But before this we prove the following lemma.

**Lemma 4.** Given $C \subset \mathbb{C}$, there exist at most $p$ different points $\rho(k) \in [0, 2)$ such that for any $j \neq k$,
\[
x^p - y^p = C, \quad \text{where} \quad x = e^{\pi i \rho(k)}, \ y = e^{\pi i \rho(j)}.
\]

**Proof.** Observe that for fixed $C$, there exists a constant $C^* \in \mathbb{C}$ such that
\[
xC - x^p = C^*
\]
for any $x = e^{\pi i \rho(k)}$ solution of (4.1). Now it is obvious that there are at most $p$ different solutions $\rho(k) \in [0, 2)$ of the equation (4.2). \(\square\)

**Corollary 5.** Given a point $\xi \in \Pi(\Lambda)$, if $\sharp\{\eta \in \text{Img}(\xi)\} > p$, then $\xi \in \Pi^3(\Lambda)$.

In particular, if $\Gamma$ consists of three parallel equidistant lines in the plane $(p = 2)$, we have

- $\Pi^3(\Lambda) = \{ \xi \in \Lambda \text{ such that } \sharp\{\eta \in \text{Img}(\xi)\} \geq 3 \},$
- $\Pi^2(\Lambda) = \{ \xi \in \Lambda \text{ such that } \sharp\{\eta \in \text{Img}(\xi)\} = 2 \}.$

**Example 6.** The following example shows that Corollary 5 is sharp:

- Let $\Lambda = \mathbb{R} \times \{2k/p\}_{k=0,\ldots,p-1}$. Then for any $\xi \in \mathbb{R},$
  \[
  \sharp\{\eta \in \text{Img}(\xi)\} = p
  \]
  and $\xi \in \Pi^2(\Lambda)$. Observe that in this case, $(\Gamma, \Lambda)$ is not an HUP.

This lemma will be useful for another example.

**Lemma 7.** For any $z \in \mathbb{C}$ with $|z| < 1$, there exist $w_1, w_2 \in \mathbb{C}$ unimodular with $z = w_1 + w_2$.

**Proof.** Let $z = re^{i\sigma}$ and let $v \in [0, \pi/2]$ with $\cos v = r/2$. Let’s take
\[
w_1 = e^{i(v + \sigma)}, \quad w_2 = e^{i(-v + \sigma)}.
\]
Then,
\[
w_1 + w_2 = e^{i(v + \sigma)} + e^{i(-v + \sigma)} = e^{i\sigma}2\cos(v) = re^{i\sigma} = z,
\]
and this finishes the proof. \(\square\)
Example 8. Let \( p = 2 \). Let \( g \) be a bounded, continuous function with \(|g| < 1\) that is nowhere locally the Fourier transform of an \( L^1 \) function. There exists a set \( \Lambda \subset \mathbb{R} \times [0,2) \) such that \( \Pi(\Lambda) = \Pi^2(\Lambda) \) is dense in \( \mathbb{R} \) and the function \( \Phi \equiv g \) on \( \Pi^2(\Lambda) \). So, \( (\Gamma, \Lambda) \) is not an HUP.

Let’s first prove the existence of the function \( g \). Let \( E \) be a dense set of measure zero on the circle \( T \). By [6] there exists a continuous function \( f \) such that the Fourier series of \( f \) fails to converge on any point of \( E \). Now let \( g : \mathbb{R} \to \mathbb{C} \) be the \( 2 \)-periodic function defined as \( g(t) = f(e^{\pi it}) \). It is easy to see that this function \( g \) is continuous but it is not a Fourier transform of an \( L^1 \) function locally at any point. By a standard argument we can think that \( g \) is bounded with \(|g| < 1\).

Now we will define the set \( \Lambda \). By Lemma 7, for any \( \xi \in \mathbb{R} \) there exist \( w_1(\xi) = e^{\pi i \eta_0} \), \( w_2(\xi) = e^{\pi i \eta_1} \) with \( w_1(\xi) + w_2(\xi) = g(\xi) \). Observe also that there is a dense set \( \Psi \) of \( \mathbb{R} \) such that \( \eta_0 \neq \eta_1 \) for any \( \xi \in \Psi \). Otherwise the function \( g \) is constant on an interval, and we get a contradiction with the fact that \( g \) is not locally the Fourier transform of an \( L^1 \) function.

We define \( \Lambda = \{(\xi, \eta_0) \cup (\xi, \eta_1)\}_{\xi \in \Psi} \). Now \( \Pi(\Lambda) = \Pi^2(\Lambda) \) and \( \Phi(\xi) = e^{\pi i \eta_0} + e^{\pi i \eta_1} = g(\xi) \), for any \( \xi \in \Psi \).

Since \( \Phi \notin \mathcal{A}_{loc}^{\Pi^2(\Lambda),\xi} \) for any \( \xi \in \Pi^2(\Lambda) \), we have that \( \Pi(\Lambda) = \Pi^2(\Lambda) \), and so \( (\Gamma, \Lambda) \) is not an HUP.

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