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DERIVATIONS OF SUBHOMOGENEOUS C^*-ALGEBRAS ARE IMPLEMENTED BY LOCAL MULTIPLIERS

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Abstract. Let A be a subhomogeneous C^*-algebra. Then A contains an essential closed ideal J with the property that for every derivation δ of A there exists a multiplier $a \in M(J)$ such that $\delta = \mathrm{ad}(a)$ and $\|\delta\| = 2\|a\|$.

1. Introduction

It is still unknown whether every derivation of a C^*-algebra A becomes inner in its local multiplier algebra $M_{\mathrm{loc}}(A)$. An affirmative answer was given by Elliott [6] for AF-algebras and by Pedersen [10] for general separable C^*-algebras (or, more generally, for C^*-algebras in which every essential closed ideal is σ-unital). However, in the inseparable case the problem seems to be wide open. Therefore, it is natural to begin by looking at the simplest cases, such as subhomogeneous C^*-algebras. In this paper we provide a short argument that every derivation of a (possibly inseparable) subhomogeneous C^*-algebra is also implemented by a local multiplier. Moreover, we obtained the following result.

Theorem 1.1. *Let A be a subhomogeneous C^*-algebra. Then A contains an essential closed ideal J with the property that for every derivation δ of A there exists a multiplier $a \in M(J)$ such that $\delta = \mathrm{ad}(a)$ and $\|\delta\| = 2\|a\|$. In particular, every derivation of A is implemented by an element of the bounded symmetric algebra of quotients $Q_b(A)$ of A.*

2. Notation and preliminaries

Throughout this paper A will denote a C^*-algebra (unless otherwise stated) and $M(A)$ its multiplier algebra. By an *ideal* of A we always mean a closed two-sided ideal. Let I be an *essential ideal* of A (i.e. I has a non-zero intersection with every other closed non-zero ideal of A). If I' is another essential ideal of A which is contained in I, then $M(I)$ is canonically embedded as a C^*-subalgebra into $M(I')$ by restriction of multipliers to the smaller ideal. In this way, we obtain a directed system of C^*-algebras with isometric connecting morphisms, where I runs through the directed set of all essential ideals of A. Forming the algebraic direct limit of this directed family yields the pre-C^*-algebra $Q_b(A)$, which is called the *bounded symmetric algebra of quotients* of A. The completion of $Q_b(A)$ is called the *local multiplier algebra* of A, and it is denoted by $M_{\mathrm{loc}}(A)$ (see [2] for details).

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By $\hat{A}$ and $\mathrm{Prim}(A)$ we respectively denote the *spectrum* of A (i.e. the set of all classes of irreducible representations of A) and the *primitive spectrum* of A (i.e. the set of all primitive ideals of A), equipped with the Jacobson topology.

As usual, for an ideal I of A we identify the open subset $\{P \in \mathrm{Prim}(A) \ : \ I \not\subseteq P\}$ (resp. closed subset $\{P \in \mathrm{Prim}(A) \ : \ I \subseteq P\}$) of $\mathrm{Prim}(A)$ with $\mathrm{Prim}(I)$ (resp. $\mathrm{Prim}(A/I)$), using the homeomorphism $P \mapsto P \cap I$ (resp. $P \mapsto P/I$). Note that I is essential if and only if $\mathrm{Prim}(I)$ is dense in $\mathrm{Prim}(A)$.

For $a \in A$ we define a function

$$\check{a} : \mathrm{Prim}(A) \to \mathbb{R}_+, \quad \check{a}(P) := \|a + P\|.$$

Since $\check{a}$ is lower semi-continuous on $\mathrm{Prim}(A)$ [4, Proposition II.6.5.6], by [4, Corollary II.6.4.9] we have

$$\|a\| = \sup\{\check{a}(P) \ : \ P \in U\}, \tag{2.1}$$

for every dense subset U of $\mathrm{Prim}(A)$.

If all irreducible representations of A have the same finite dimension n, we say that A is *n-homogeneous.* In this case by [7, Section 3.2] $\mathrm{Prim}(A) = \hat{A}$ is a (locally compact) Hausdorff space, and there exists a locally trivial C^*-bundle E over $\mathrm{Prim}(A)$ with fibres isomorphic to the matrix algebra $\mathrm{M}_n(\mathbb{C})$ such that A is isomorphic to the C^*-algebra $\Gamma_0(E)$ of all continuous sections of E which vanish at infinity.

If A is a finite direct sum of homogeneous C^*-algebras, A is said to be *locally homogeneous*, and if

$$n := \sup\{\dim \pi \ : \ [\pi] \in \hat{A}\} < \infty,$$

A is said to be *n-subhomogeneous.* In this case by [11, 6.2.5], A has a finite *standard composition series*

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k = A \tag{2.2}$$

of ideals of A such that each quotient I_i/I_{i-1} is a homogeneous C^*-algebra. The ideal I_1 is called the *n-homogeneous ideal* of A (since it is the largest ideal of A which is n-homogeneous as a C^*-algebra).

A *derivation* of an algebra A is a linear map $\delta : A \to A$ satisfying the Leibniz rule

$$\delta(xy) = \delta(x)y + x\delta(y) \quad (x, y \in A).$$

If A is a subalgebra of an algebra B, then every element $a \in B$ which *derives* A (i.e. $ax - xa \in A$ for all $x \in A$) implements an *inner derivation* $\mathrm{ad}(a) : A \to A$ given by

$$\mathrm{ad}(a)(x) := ax - xa \quad (x \in A).$$

If A is a C^*-algebra, it is well known that every derivation δ of A is (completely) bounded, and it leaves every ideal of A invariant. For an ideal I of A, by δ_I (resp. $\delta|_I$) we denote the induced derivation of A/I, $\delta_I(x + I) = \delta(x) + I$ $(x \in A)$ (resp. the restriction derivation of I). Following [1] (see also [9, Remark 5.2]), we define a function

$$|\delta| : \mathrm{Prim}(A) \to \mathbb{R}_+, \quad |\delta|(P) := \|\delta_P\|.$$

Note that $|\delta|$ is lower semi-continuous on $\mathrm{Prim}(A)$ [1, Lemma 2.2], and by [2, Theorem 5.3.12] we have

$$\|\delta\| = \sup\{|\delta|(P) \ : P \in U\}, \tag{2.3}$$

for every dense subset U of $\mathrm{Prim}(A)$. If in addition $|\delta|$ is continuous on $\mathrm{Prim}(A)$, δ is said to be *smooth*. By [1, Theorem 2.4] each smooth derivation δ of A is implemented by a multiplier of A. Moreover, there exists a unique multiplier $a \in M(A)$ such that $\delta = \mathrm{ad}(a)$ and $|\delta|(P) = 2\check{a}(P)$ for all $P \in \mathrm{Prim}(A)$.

3. Results

Remark 3.1. It is well known that every derivation of a unital locally homogeneous C^*-algebra A is inner [12, Theorem 1] (see also [5]). In this case $\mathrm{Prim}(A)$ is Hausdorff, so [9, Corollary 5.8] implies that every derivation of A is smooth.

We shall first extend this result to the non-unital case.

Proposition 3.2. *If A is a locally homogeneous C^*-algebra, then every derivation δ of A is smooth. Therefore, there exists a unique multiplier $a \in M(A)$ such that $\delta = \mathrm{ad}(a)$ and $|\delta|(P) = 2\check{a}(P)$ for all $P \in \mathrm{Prim}(A)$.*

Proof. It is sufficient to prove the assertion when A is (say n-)homogeneous. Let E be a locally trivial C^*-bundle over $\mathrm{Prim}(A)$ such that $A = \Gamma_0(E)$. For an arbitrary point $P_0 \in \mathrm{Prim}(A)$ choose a compact neighborhood V of P_0 such that the restriction bundle $E|_V$ is trivial. Let I be the ideal of $\Gamma_0(E)$ consisting of all sections in $\Gamma_0(E)$ which vanish at points of V. Using the Tietze extension theorem for sections of Banach bundles [8, Theorem II.14.8], we can identify A/I with $\Gamma(E|_V) \cong C(V, \mathrm{M}_n(\mathbb{C}))$. By Remark 3.1 the induced derivation δ_I of A/I is smooth. Since

$$|\delta_I|(P) = \|(\delta_I)_P\| = \|\delta_P\| = |\delta|(P) \quad (P \in V = \mathrm{Prim}(A/I)),$$

we conclude that the function $|\delta|$ is continuous on V, so in particular it is continuous at $P_0 \in V$. Since $P_0 \in \mathrm{Prim}(A)$ was arbitrary, the proof is finished. □

Remark 3.3. One may wonder if for every derivation δ of a locally homogeneous C^*-algebra A (when extended to a derivation of $M(A)$), the function $|\delta|$ is in fact continuous on the whole space $\mathrm{Prim}(M(A))$. Obviously, this is true for derivations which are implemented by elements of A. However, R. Archbold and D. Somerset informed us that there are examples of homogeneous C^*-algebras A for which $\mathrm{Prim}(M(A))$ is non-Hausdorff [3, Theorem 2.1]. If A is such an algebra, then by [9, Corollary 5.8] there exists an element $a \in M(A)$ such that the function $|\mathrm{ad}(a)|$ is not continuous on $\mathrm{Prim}(M(A))$ (even though it is continuous when restricted to $\mathrm{Prim}(A)$).

Lemma 3.4. *Let A be a subhomogeneous C^*-algebra. Then A contains an essential locally homogeneous ideal J.*

Proof. We proceed by induction on the length $k = k(A)$ of the standard composition series (2.2) of A. Suppose that $k = 1$. In this case A is homogeneous, so we may let $J = A$. Let $k > 1$, and suppose that the assertion is true for all subhomogeneous C^*-algebras B which satisfy $k(B) < k$. Let A be an n-subhomogeneous C^*-algebra with $k(A) = k$, and let I be the n-homogeneous ideal of A. If I is essential, the proof is finished, so assume that I is not essential. Let $U := \mathrm{Prim}(I)$ and $U' := \mathrm{Prim}(A) \setminus \overline{U}$. By assumption, U' is an open non-empty subset of $\mathrm{Prim}(A)$, and let K be the ideal of A such that $K = \mathrm{Prim}(U')$. Then $k(K) < k$, so our induction hypothesis implies that K contains an essential locally homogeneous ideal I'. Obviously, $I \cap I' = \{0\}$. Therefore, $J := I \oplus I'$ is an essential locally homogeneous ideal of A. □

Proof of Theorem 1.1. By Lemma 3.4 A contains an essential locally homogeneous ideal J. Since δ leaves J invariant, by Proposition 3.2 there exists a unique multiplier $a \in M(J)$ such that $\delta|_J = \mathrm{ad}(a)$ and $|\delta|(P) = |\delta|_J|(P) = 2\check{a}(P)$ for all $P \in \mathrm{Prim}(J)$. Since J is essential in A, we have $A \subseteq M(J)$, and the restriction of $\mathrm{ad}(a)$ (when considered as a derivation of $M(J)$) on A coincides with δ. Hence, $\delta(x) = \mathrm{ad}(a)(x)$ for all $x \in A$. Finally, using (2.3) and (2.1), we conclude that

$$\begin{aligned} \|\delta\| &= \sup\{|\delta|(P) \ : \ P \in \mathrm{Prim}(J)\} \\ &= 2\sup\{\check{a}(P) \ : \ P \in \mathrm{Prim}(J)\} \\ &= 2\|a\|, \end{aligned}$$

since $\mathrm{Prim}(J)$ is a dense (open) subset of $\mathrm{Prim}(A)$ and $\mathrm{Prim}(M(J))$. □

Note that the final statement of Theorem 1.1 is not true for general C^*-algebras (see [1, Example 6.5]). However, we state the following question.

Problem 3.5. If every irreducible representation of a C^*-algebra A is finite dimensional, is every derivation of A implemented by an element of $Q_b(A)$?

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