ON HARMONIC NON-COMMUTATIVE $L^p$-OPERATORS ON LOCALLY COMPACT QUANTUM GROUPS

MEHRDAD KALANTAR

(Communicated by Marius Junge)

Abstract. For a locally compact quantum group $G$ with tracial Haar weight $\varphi$ and a quantum measure $\mu$ on $G$, we study the space $H^p_\mu(G)$ of $\mu$-harmonic operators in the non-commutative $L^p$-space $L^p(G)$ associated to the Haar weight $\varphi$. The main result states that if $\mu$ is non-degenerate, then $H^p_\mu(G)$ is trivial for all $1 \leq p < \infty$.

1. Introduction and preliminaries

Non-commutative Poisson boundaries of (discrete) quantum groups $G$ was first introduced and studied by Izumi in [6]. Motivated by the classical setting, in fact, he defined the Poisson boundary of $G$ associated to a ‘quantum measure’ $\mu$ as the space of $\mu$-harmonic ‘functions’, i.e., the fixed point space of the Markov operator associated to $\mu$. For discrete quantum groups, this was further studied by several authors (cf. [7], [14], [15]). Poisson boundaries in the locally compact quantum group setting was studied by Neufang, Ruan and the author in [9]. Quantum versions of several important classical results regarding harmonic functions were proved there. In particular, triviality of special classes of harmonic functions, such as $C_0$-functions, was proved.

Another important fact regarding classical harmonic functions on locally compact groups is that for $1 \leq p < \infty$, any $L^p$-harmonic function associated to an adapted probability measure is trivial. The main result of this paper is a quantum version of this result. But, in order to talk about $\mu$-harmonic elements in the non-commutative $L^p$-spaces, we first need to define the convolution action by $\mu$ on such spaces.

In his PhD thesis [4], Cooney studied the non-commutative $L^p$-spaces associated to the Haar weight $\varphi$ of a locally compact quantum group $G$. He mainly considered Haagerup’s version and could prove that in the Kac algebra setting, the convolution action of an ‘absolutely continuous quantum measure’ can be extended to the Haagerup non-commutative $L^p$-spaces. So, we cannot consider harmonic operators in the general setting of all locally compact quantum groups. Moreover, in the case of non-tracial $\varphi$, there are different ways to define the non-commutative $L^p$-spaces. Although all these spaces are isometrically isomorphic as Banach spaces, the identifications are not necessarily compatible with the quantum group structure, so it is not clear whether the space of $\mu$-harmonic $L^p$-operators is the same, as a Banach space, for all different definitions of non-commutative $L^p$-spaces.

Received by the editors January 29, 2012.

2010 Mathematics Subject Classification. Primary 46L52, 46L53, 46L65.

©2013 American Mathematical Society
Reverts to public domain 28 years from publication

3969
Therefore, in this paper, instead of restricting ourselves to the Kac algebra setting, we consider locally compact quantum groups $G$ whose Haar weight $\varphi$ is a trace. In this case, the convolution action is extended to the non-commutative $L^p$-spaces, and the main result of the paper states that in the case of a non-degenerate quantum measure $\mu$, for $1 \leq p < \infty$, any $\mu$-harmonic element which lies in the non-commutative $L^p$-space of $\varphi$ is trivial.

First, let us introduce our terminology and recall some results on locally compact quantum groups which we will be using in this paper. For more details, we refer the reader to [11].

A locally compact quantum group $G$ is a quadruple $(M, \Gamma, \varphi, \psi)$, where $M$ is a von Neumann algebra with a co-associative co-multiplication $\Gamma : M \to M \otimes M$, and $\varphi$ and $\psi$ are (normal faithful semi-finite) left and right Haar weights on $M$, respectively. We write $M^+_\varphi = \{ x \in M^+ : \varphi(x) < \infty \}$ and $\mathcal{N}_\varphi = \{ x \in M^+ : \varphi(x^*x) < \infty \}$, and we denote by $\Lambda_\varphi$ the inclusion of $\mathcal{N}_\varphi$ into the GNS Hilbert space $H_\varphi$ of $\varphi$. For each locally compact quantum group $G$, there exists a left fundamental unitary operator $W$ on $H_\varphi \otimes H_\varphi$ which satisfies the pentagonal relation and such that the co-multiplication $\Gamma$ on $M$ can be expressed as

$$\Gamma(x) = W^*(1 \otimes x)W \quad (x \in M).$$

There exists an anti-automorphism $R$ on $M$, called the unitary antipode, such that $R^2 = \iota$, and

$$\Gamma \circ R = \chi(R \otimes R) \circ \Gamma,$$

where $\chi(x \otimes y) = (y \otimes x)$ is the flip map. It can be easily seen that if $\varphi$ is a left Haar weight, then $\varphi R$ defines a right Haar weight on $M$.

Let $M_*$ be the predual of $M$. Then the pre-adjoint of $\Gamma$ induces on $M_*$ an associative completely contractive multiplication

$$\ast : M_* \otimes M_* \ni f_1 \otimes f_2 \longmapsto f_1 \ast f_2 = (f_1 \otimes f_2) \ast \Gamma \in M_*.$$

The left regular representation $\lambda : M_* \to \mathcal{B}(H_\varphi)$ is defined by

$$\lambda : M_* \ni f \longmapsto \lambda(f) = (f \otimes \iota)(W) \in \mathcal{B}(H_\varphi),$$

which is an injective and completely contractive algebra homomorphism from $M_*$ into $\mathcal{B}(H_\varphi)$. Then $\hat{M} = \{ \lambda(f) : f \in M_* \}''$ is the von Neumann algebra associated with the dual quantum group $\hat{G}$. It follows that $W \in M \hat{\otimes} \hat{M}$. We also define the completely contractive injection

$$\hat{\lambda} : \hat{M} \ni \hat{f} \longmapsto \hat{\lambda}(\hat{f}) = (\iota \otimes \hat{f})(W) \in M.$$

The reduced quantum group $C^*$-algebra

$$\mathcal{C}_0(G) = \overline{\lambda(L_1(\hat{G}))}^{\| \cdot \|}$$

is a weak* dense $C^*$-subalgebra of $M$. Let $\mathcal{M}(G)$ denote the operator dual $\mathcal{C}_0(G)^{\ast}$. There exists a completely contractive multiplication on $\mathcal{M}(G)$ given by the convolution

$$\ast : \mathcal{M}(G) \hat{\otimes} \mathcal{M}(G) \ni \mu \otimes \nu \longmapsto \mu \ast \nu = \mu(\iota \otimes \nu)\Gamma = \nu(\mu \otimes \iota)\Gamma \in \mathcal{M}(G)$$

such that $\mathcal{M}(G)$ contains $M_*$ as a norm closed two-sided ideal. Therefore, for each $\mu \in \mathcal{M}(G)$, we obtain a pair of completely bounded maps

$$f \longmapsto \mu \ast f \quad \text{and} \quad f \longmapsto f \ast \mu.$$
on \( M_\ast \) through the left and right convolution products of \( \mathcal{M}(G) \). The adjoint maps give the convolution actions \( x \mapsto \mu \ast x \) and \( x \mapsto x \ast \mu \) that are normal completely bounded maps on \( M \).

We denote by \( \mathcal{P}(G) \) the set of all states on \( \mathcal{C}_0(G) \) (i.e., ‘the quantum probability measures’). For any such element the convolution action is a \textit{Markov operator}, i.e., a unital normal completely positive map, on \( M \).

Now assume that the left Haar weight \( \varphi \) on \( G \) is a trace, and let \( \psi = \varphi R \) be the right Haar weight. We denote by \( \mathcal{L}^p(G) \) and \( \tilde{\mathcal{L}}^p(G) \) the non-commutative \( L^p \)-spaces associated to \( \varphi \) and \( \psi \), respectively. These spaces are obtained by taking the closure of \( \mathfrak{M}_\varphi \) and \( \mathfrak{M}_\psi \) under the norms \( \|x\| = \varphi(|x|^p)^{\frac{1}{p}} \) and \( \|x\| = \psi(|x|^p)^{\frac{1}{p}} \), respectively (see [12] for details). We denote by \( \mathcal{L}^\infty(G) \) the von Neumann algebra \( M \). Similarly to the classical case, one can also construct the non-commutative \( L^p \)-spaces using the complex interpolation method (cf. [5], [10], [13]). The map

\[
\mathfrak{M}_\varphi \ni x \mapsto \varphi \cdot x \in M_\ast
\]

extends to an isometric isomorphism between \( \mathcal{L}^1(G) \) and \( M_\ast \), where \( \langle \varphi \cdot x, y \rangle = \varphi(xy) \).

### 2. \( \mu \)-Harmonic Operators

We assume that \( \mu \in \mathcal{P}(G) \) throughout this section. By invariance of the left Haar weight \( \varphi \), we can easily see that \( \mathcal{L}^p(G) \cap \mathcal{L}^\infty(G) \) is invariant under the left convolution action by \( \mu \). Since \( \varphi = \psi R \) is a trace, by \([11], \text{Proposition 5.20}\) we have

\[
R((\iota \otimes \varphi)\Gamma(a)(1 \otimes b)) = (\iota \otimes \varphi)((1 \otimes a)\Gamma(b))
\]

for all \( a, b \in \mathfrak{N}_\varphi \). Therefore we obtain

\[
(\mu \ast (\varphi \cdot a), b) = \langle \mu, (\iota \otimes \varphi)((1 \otimes a)\Gamma(b)) \rangle = \langle \mu R, (\iota \otimes \varphi)((1 \otimes a)\Gamma(b)) \rangle
\]

\[
= \langle (\mu R) \ast (b \cdot \varphi), a \rangle = \langle (b \cdot \varphi), (\mu R) \ast a \rangle = \langle \varphi \cdot ((\mu R) \ast a), b \rangle.
\]

Since the map \( \mathfrak{M}_\varphi \ni x \mapsto \varphi \cdot x \mapsto (\mu R) \ast (\varphi \cdot x) = \varphi \cdot (\mu \ast x) \mapsto \mu \ast x \in \mathfrak{M}_\varphi \)

extends to an operator on \( \mathcal{L}^1(G) \) with the same norm as the convolution operator by \( \mu \) on \( \mathcal{L}^\infty(G) \). Now, interpolating between \( \mathcal{L}^1(G) \) and \( \mathcal{L}^\infty(G) \), we can extend the convolution action

\[
\mathcal{L}^p(G) \cap \mathcal{L}^\infty(G) \ni x \mapsto \mu \ast x \in \mathcal{L}^p(G) \cap \mathcal{L}^\infty(G)
\]

to \( \mathcal{L}^p(G) \). An operator \( x \in \mathcal{L}^p(G) \) is called \( \mu \)-harmonic if \( \mu \ast x = x \) and \( \mathcal{H}_\mu^p(G) = \{ x \in \mathcal{L}^p(G) : \mu \ast x = x \} \) is the space of \( \mu \)-harmonic operators. It is easy to see that \( \mathcal{H}_\mu^p(G) \) is a weak* closed subspace of \( \mathcal{L}^p(G) \) for all \( 1 < p \leq \infty \).

Similarly to the case \( p = \infty \), we have a projection \( E^p_\mu : \mathcal{L}^p(G) \to \mathcal{H}_\mu^p(G) \) constructed as follows. Let \( \mathcal{U} \) be a free ultra-filter on \( \mathbb{N} \), and define \( E^p_\mu : \mathcal{L}^p(G) \to \mathcal{L}^p(G) \) by the weak* limit

\[
E^p_\mu(x) = \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^{n} \mu^k \ast x.
\]

Then it is easy to see that \( E^p_\mu \circ E^p_\mu = E^p_\mu \) and that \( \mathcal{H}_\mu^p(G) = E^p_\mu(\mathcal{L}^p(G)) \). Moreover, by considering the convolution action on \( \mathcal{L}^p(G) \cap \mathcal{L}^\infty(G) \) and passing to limits, we can see that \( E^p_\mu \) is also positive.
Proposition 2.1. The unitary antipode $R$ extends to an isometric isomorphism 

$$R : \mathcal{L}^p(G) \to \hat{\mathcal{L}}^p(G)$$

such that $R(\mathcal{H}_R^p(G)) = \hat{\mathcal{H}}^p_{R'}$ for all $1 \leq p \leq \infty$.

Proof. Since $R$ is an anti-automorphism, we have

$$\psi(|R(a)|^p) = \psi(R(|a|^p)) = \varphi(|a|^p)$$

for all $a \in \mathfrak{M}_\varphi$. Therefore $R$ extends to an isometry from $\mathcal{L}^p(G)$ onto $\hat{\mathcal{L}}^p(G)$. Moreover, we have

$$R(\mu \ast a) = R((\mu \otimes \iota)\Gamma(a)) = R((\mu \otimes \iota)\Gamma(R^2(a)))$$

$$= R((\iota \otimes \mu)(R \otimes R)\Gamma(R(a)))$$

$$= (\iota \otimes \mu R)\Gamma(R(a)) = R(a) \ast \mu R,$$

which implies that $R(\mathcal{H}_R^p(G)) = \hat{\mathcal{H}}^p_{R'}$. \hfill \square

Therefore, for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we can identify each $\mathcal{L}^p(G)$ and $\hat{\mathcal{L}}^q(G)$ with the dual space of the other via

$$\langle a, b \rangle = \varphi(aR(b)) = \psi(R(a)b), \quad a \in \mathcal{L}^p(G), \quad b \in \hat{\mathcal{L}}^q(G).$$

Theorem 2.2. Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have linear isometric isomorphisms

$$\mathcal{H}_R^p(G)^* \cong \hat{\mathcal{H}}_R^q(G) \quad \text{and} \quad \mathcal{H}_R^q(G) \cong \hat{\mathcal{H}}_R^p(G)^*.$$

Proof. Denote

$$J^p_R(G) := \{x - \mu \ast x : x \in \mathcal{L}^p(G)\}^- \quad \text{and} \quad \hat{J}^q_R(G) := \{y - y \ast \mu : y \in \hat{\mathcal{L}}^q(G)\}^-.$$

Since

$$\langle x, y \ast \mu \rangle = \psi(R(x)(y \ast \mu)) = \psi(R(x)(\iota \otimes \mu)\Gamma(y)) = \mu((\psi \otimes \iota)(R(x) \otimes 1)\Gamma(y))$$

$$= \mu R((\psi \otimes \iota)(R(x)(y \otimes 1))) = \psi((\iota \otimes \mu R)\Gamma(R(x))y)$$

$$= \psi(R((\mu \otimes \iota)\Gamma(x))y) = \psi(R(\mu \ast x)y) = \langle \mu \ast x, y \rangle$$

for all $x \in \mathfrak{M}_\varphi$ and $y \in \mathfrak{M}_\psi$, it follows that $\mathcal{H}_R^p(G) = \hat{J}^q_R(G)^\perp$, and therefore

$$\mathcal{H}_R^p(G)^* = \frac{\hat{\mathcal{L}}^q(G)}{\mathcal{H}_R^p(G)^\perp} = \frac{\hat{\mathcal{L}}^q(G)}{\hat{J}^q_R(G)}.$$

In the following we show that the correspondence

$$\frac{\hat{\mathcal{L}}^q(G)}{\hat{J}^q_R(G)} \ni y + \hat{J}^q_R(G) \mapsto \hat{E}^q_\mu(y) \in \hat{\mathcal{H}}_R^q(G)$$
defines a linear isometric isomorphism. First we observe that
\[
\tilde{E}_\mu^q(y \star \mu - y) = \lim_{n \to \infty} (y \star \mu - y) \star \frac{1}{n} \sum_{k=1}^{n} \mu^k = 0
\]
for all \( y \in L^q(G) \), which implies that the above map is well-defined. It is obviously onto. To check the injectivity, first note that
\[
y - y \star \mu^k = (y - y \star \mu) + (y \star \mu - y \star \mu^2) + (y \star \mu^{k-1} - y \star \mu^k) \in \tilde{J}^q_\mu(G),
\]
for all \( k \in \mathbb{N} \). Now suppose that \( \tilde{E}_\mu^q(y) = 0 \). Then, by the above and by the weak* closeness of \( \tilde{J}^q_\mu(G) \), we have
\[
y = y - \tilde{E}_\mu^q(y) = y - \left( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} y \star \mu^k \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (y - y \star \mu^k) \in \tilde{J}^q_\mu(G),
\]
and therefore the injectivity of the map follows. Moreover, since \( \tilde{E}_\mu^q \) is an idempotent, it follows that
\[
y + \tilde{J}^q_\mu(G) = \tilde{E}_\mu^q(y) + \tilde{J}^q_\mu(G).
\]
Therefore
\[
\|y + \tilde{J}^q_\mu(G)\| \leq \|\tilde{E}_\mu^q(y)\|.
\]
On the other hand, we have
\[
\|\tilde{E}_\mu^q(y)\| = \sup \{ |\langle \tilde{E}_\mu^q(y) , x \rangle| : x \in L^p(G) , \|x\| \leq 1 \}
= \sup \{ |\langle y , E_\mu^p(x) \rangle| : x \in L^p(G) , \|x\| \leq 1 \}
\leq \|y + \tilde{J}^q_\mu(G)\|.
\]
This shows that the map is isometric and so yields the first identification. The second identification is proved along similar lines. \( \square \)

**Proposition 2.3.** For \( 1 < p \leq \infty \) the space \( \mathcal{H}^p_\mu(G) \) is generated by its positive elements.

**Proof.** By considering the polar decomposition, we observe that \( L^p(G) \cap L^\infty(G) \) is self-adjoint. Let \( x \in \mathcal{H}^p_\mu(G) \), and \( L^p(G) \cap L^\infty(G) \ni x_n \to x \) in \( L^p(G) \). Using the continuity of the adjoint on \( L^p(G) \), we obtain
\[
\mu \star x^* = \lim_n \mu \star x_n^* = \lim_n (\mu \star x_n)^* = (\lim_n \mu \star x_n)^* = x^*,
\]
where the limits are taken in \( L^p(G) \). Therefore, \( \mathcal{H}^p_\mu(G) \) is self-adjoint and so is generated by its self-adjoint elements. Now, let \( x \) be a self-adjoint element in \( L^p(G) \), and let \( x = x_+ - x_- \) where both \( x_+ \) and \( x_- \) are in \( L^p(G) \). Then we have
\[
x = E_\mu^p(x) = E_\mu^p(x_+) - E_\mu^p(x_-),
\]
which yields the result by positivity of the map \( E_\mu^p \). \( \square \)
Main Theorem: Case $1 < p < \infty$. A state $\mu \in \mathcal{P}(G)$ is called non-degenerate on $C_0(G)$ if for every non-zero element $x \in C_0(G)^+$ there exists $n \in \mathbb{N}$ such that $\langle x, \mu^n \rangle \neq 0$.

**Theorem 2.4.** Let $G$ be a non-compact, locally compact quantum group with a tracial (left) Haar weight $\varphi$, and let $\mu \in \mathcal{P}(G)$ be non-degenerate. Then for all $1 < p < \infty$ we have
\[
\mathcal{H}_\mu^p(G) = \{0\}.
\]

**Proof.** First let $1 < p \leq 2$, and suppose $0 \leq x \in \mathcal{H}_\mu^p(G)$ with $\|x\|_p = 1$. Define
\[
\tilde{\mu} := \sum_{i=n}^{\infty} \frac{\mu^n_i}{2^n}.
\]
Since $\mu$ is non-degenerate, $\tilde{\mu}$ is faithful, and $\tilde{\mu} \ast x = x$. Now, let $q \geq 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Using the duality between $L^p(G)$ and $\hat{L}^q(G)$, we assign to each pair $a \in L^p(G)$ and $b \in \hat{L}^q(G)$ an element $\Omega_{a,b} \in L^\infty(G)$ defined by
\[
\langle f, \Omega_{a,b} \rangle = \langle f \ast a, b \rangle \quad (f \in M_s).
\]
We clearly have $\|\Omega_{a,b}\| \leq \|a\|_p \|b\|_q$. Now, choose $y \in \hat{L}^q(G)$, $\|y\|_q = 1$, such that $\langle x, y \rangle = 1$. We claim that $\Omega_{x,y} \in C_0(G)$ (in fact, $\Omega_{a,b} \in C_0(G)$ for all $a \in L^p(G)$ and $b \in \hat{L}^q(G)$). To see this, assume that
\[
x = \int_0^{\infty} \lambda \, d\lambda
\]
is the spectral decomposition of $x$, and let
\[
x_n = \int_{\frac{1}{2}}^{n} \lambda \, d\lambda.
\]
Then $x_n \in L^p(G) \cap L^\infty(G) \subseteq \mathcal{H}_\varphi$, $\|x_n\|_p \leq \|x\|_p$, and $\|x - x_n\|_p \to 0$. Also let $y_n \in \mathcal{H}_\varphi$ be such that $\|y_n - y\|_q \to 0$.

Denote by $\omega_{\eta,\zeta}$ the vector functional associated with $\eta, \zeta \in H_\varphi$. Then, for $f \in M_s$, we have
\[
\langle f, \Omega_{x_n, y_n} \rangle = \langle f \ast x_n, y_n \rangle = \langle \lambda(f) \Lambda_\varphi(x_n), \Lambda_\varphi(R(y_n)) \rangle = \langle \omega_{\Lambda_\varphi(x_n), \Lambda_\varphi(R(y_n))} \lambda(f) \rangle = \langle f, \hat{\lambda}(\omega_{\Lambda_\varphi(x_n), \Lambda_\varphi(R(y_n))}) \rangle;
\]
which implies that $\Omega_{x_n, y_n} = \hat{\lambda}(\omega_{\Lambda_\varphi(x_n), \Lambda_\varphi(R(y_n))}) \in C_0(G)$. Moreover, it follows that
\[
\|\Omega_{x,y} - \Omega_{x_n, y_n}\|_\infty \leq \|\Omega_{x-x_n, y}\|_\infty + \|\Omega_{x_n, y-y_n}\|_\infty \leq \|x - x_n\|_p \|y\|_q + \|x_n\|_p \|y - y_n\|_q \to 0.
\]
This shows that $\Omega_{x,y} \in C_0(G)$, as claimed. But then we have $\|\Omega_{x,y}\| \leq \|x\|_p \|y\|_q = 1$, and
\[
\langle \tilde{\mu}, \Omega_{x,y} \rangle = \langle \tilde{\mu} \ast x, y \rangle = \langle x, y \rangle = 1.
\]
Since $\tilde{\mu}$ is faithful, it follows that $\Omega_{x,y} = 1$, and therefore $1 \in C_0(G)$, which contradicts our assumption of $G$ being non-compact, so $x = 0$. This shows, by Proposition 2.3 that $\mathcal{H}_\mu^p(G) = \{0\}$ for all $1 < p \leq 2$. Now, a similar argument yields $\mathcal{H}_\mu^q(G) = 0$ for all $1 < q \leq 2$, which implies by Theorem 2.2 that $\mathcal{H}_\mu^p(G) = \mathcal{H}_\mu^q(G)^* = \{0\}$ for all $2 \leq p < \infty$. \[\square\]
Main Theorem: Case $p = 1$. Since $L^1(G)$ is not a dual Banach space, our proof for $1 < p < \infty$ does not work in this case, and so we have to treat this case separately. We do this by first proving a similar result for $M_\omega$ and then using the identification of the latter with $L^1(G)$. Note that for the following theorem we do not assume that the Haar weight is a trace.

Theorem 2.5. Let $G$ be a non-compact, locally compact quantum group, and let $\mu \in \mathcal{P}(G)$ be non-degenerate. If $\omega \in \mathcal{M}(G)$ is such that $\mu \ast \omega = \omega$, then $\omega = 0$.

Proof. Assume that $\mu \ast \omega = \omega$, and let $\tilde{\mu}$ be as in the proof of Theorem 2.4. So, $\tilde{\mu}$ is faithful, and $\mu \ast \omega = \omega$. Therefore we have

$$\lambda(\tilde{\mu})\lambda(\omega)\xi = \lambda(\tilde{\mu} \ast \omega)\xi = \lambda(\omega)\xi$$

for all $\xi \in H_\varphi$. Now if $\omega \neq 0$, there exists $\xi \in H_\varphi$ such that $\|\lambda(\omega)\xi\| = 1$. Denote by $\hat{\omega}$ the restriction of $\omega\lambda(\omega)\xi$ to $\hat{M}$. Then $\|\hat{\omega}\| = 1$, and

$$\langle \tilde{\mu} , \lambda(\hat{\omega}) \rangle = \langle \lambda(\tilde{\mu}) , \hat{\omega} \rangle = \langle \lambda(\tilde{\mu})\lambda(\omega)\xi , \lambda(\omega)\xi \rangle = \langle \lambda(\omega)\xi , \lambda(\omega)\xi \rangle = 1.$$  

Since $\|\lambda(\hat{\omega})\| \leq 1$ and $\tilde{\mu}$ is faithful, it follows that $\hat{\omega} = 1$. But this implies that $1 \in C_0(G)$, which contradicts our assumption of $G$ being non-compact. Hence, $\omega = 0$.  

Theorem 2.6. Let $G$ be a non-compact, locally compact quantum group with a tracial (left) Haar weight $\varphi$, and let $\mu \in \mathcal{P}(G)$ be non-degenerate. Then

$$H^1_\mu(G) = \{0\}.$$  

Proof. Let $x \in H^1_\mu(G)$. We have that $\mu R \in \mathcal{P}(G)$ is non-degenerate, and from equations (2.1) and (2.2) we get

$$\mu R \ast (\varphi \cdot x) = \varphi \cdot (\mu \ast x) = \varphi \cdot x.$$  

Hence, $\varphi \cdot x = 0$ by Theorem 2.5 and therefore $x = 0$.  

Remark 2.7. The statements of Theorems 2.4 and 2.6 are not true in general for the case $p = \infty$. Any non-degenerate probability measure on a non-amenable discrete group is a counterexample [8].  

Compact Case. We conclude by proving the triviality of $\mu$-harmonic operators in the compact quantum group setting.

Theorem 2.8. Let $G$ be a compact quantum group with tracial Haar state, and let $\mu \in \mathcal{P}(G)$ be non-degenerate. Then $H^p_\mu(G) = C_1$ for all $1 \leq p \leq \infty$.

Proof. The case $p = \infty$ was proved in the general case in [9]. Let $1 \leq p < \infty$, and assume that $x \in H^p_\mu(G)$, $x \notin C_1$ and $\|x\|_p = 1$. Then there exists $y \in \tilde{L}^q(G)$ with $\|y\|_q = 1$ such that $\langle x, y \rangle = 1$ (we let $q = \infty$ for $p = 1$) and $\langle 1, y \rangle = 0$. Then from the proof of Theorem 2.4 (which we can also apply to the case of $p = 1$ and $q = \infty$, since $L^\infty(G) \subseteq L^2(G)$ for a compact quantum group) we have $\Omega_{x,y} = 1$ and

$$(2.3) \quad \langle \varphi \ast x, y \rangle = \langle \varphi , 1 \rangle = 1,$$

where $\varphi$ is the Haar state on $G$. Now, let $x_n \in L^\infty(G)$ be such that $\|x_n - x\|_p \to 0$. Then

$$\langle \varphi \ast x, y \rangle = \lim_n \langle \varphi \ast x_n, y \rangle = \lim_n \langle \varphi , x_n \rangle \langle 1, y \rangle = 0.$$  

But this contradicts (2.3), and therefore $x = 0$. Hence, $H^p_\mu(G) = C_1$.  

□
Remark 2.9. All of our results in this paper can be proved, by slight modifications of the arguments, for a state $\mu$ on the universal $C^*$-algebra $C_u(\mathbb{G})$.

References


School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6

E-mail address: mkalanta@math.carleton.ca