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L^p-NUCLEAR PSEUDO-DIFFERENTIAL OPERATORS ON $\mathbb{Z}$ AND $\mathbb{S}^1$

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ABSTRACT. Conditions for pseudo-differential operators from $L^{p_1}(\mathbb{Z})$ into $L^{p_2}(\mathbb{Z})$ and from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ to be nuclear are presented for $1 \le p_1$, $p_2 < \infty$. In the cases when $p_1 = p_2$, the trace formulas are given.

1. Introduction

Although all results in this paper are presented for $\mathbb{Z}$ and $\mathbb{S}^1$ only, extensions to the lattice $\mathbb{Z}^n$ and the torus $\mathbb{T}^n$ are valid.

Let $\mathbb{Z}$ be the set of all integers and let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$. Then for every sequence in $L^p(\mathbb{Z})$, $1 \le p < \infty$, we define the sequence $T_\sigma a$ formally by

$$(T_\sigma a)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \sigma(n,\theta)(\mathcal{F}_{\mathbb{Z}} a)(\theta)\, d\theta, \quad n \in \mathbb{Z},$$

where $\mathcal{F}_{\mathbb{Z}} a$ is the Fourier transform of a given by

$$(\mathcal{F}_{\mathbb{Z}} a)(\theta) = \sum_{n=-\infty}^{\infty} a(n) e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Let $\mathbb{S}^1$ be the unit circle centered at the origin and let τ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then for all f in $L^p(\mathbb{S}^1)$, $1 \le p < \infty$, we define the function $T_\tau f$ on $\mathbb{S}^1$ formally by

$$(T_\tau f)(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \tau(\theta, n) \hat{f}(n), \quad \theta \in [-\pi, \pi],$$

where $\hat{f}$, sometimes denoted by $\mathcal{F}_{\mathbb{S}^1} f$, is the Fourier transform of f defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta, \quad n \in \mathbb{Z}.$$

Sufficient conditions on σ are given in [10] to guarantee that the pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$, $1 \le p < \infty$, is a bounded linear operator. Under additional mild conditions on σ, the bounded linear operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$, $1 \le p < \infty$, turns out to be compact.

Sufficient conditions on τ can be found in [11] in order to ensure that the pseudo-differential operator $T_\tau : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is a bounded linear operator for $1 \le p < \infty$.

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In the case when $p = 2$, a simple necessary and sufficient condition for $T_\tau : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ and $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ to be Hilbert–Schmidt is that, respectively, $\tau \in L^2(\mathbb{S}^1 \times \mathbb{Z})$ and $\sigma \in L^2(\mathbb{Z} \times \mathbb{S}^1)$.

The results hitherto described can be found in the book [14]. Related results can be found in [12, 13].

The aim of this paper is to give a systematic investigation when the pseudo-differential operators $T_\sigma : L^{p_1}(\mathbb{Z}) \to L^{p_2}(\mathbb{Z})$ and $T_\tau : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ are nuclear for $1 \le p_1, p_2 < \infty$. If they are nuclear, we find formulas for the traces when $p_1 = p_2$.

The main tools that we use are the following results in [3, 4], which are the L^p-analogs of the L^2-results in [1, 2].

Theorem 1.1. *Let (X_1, μ_1) and (X_2, μ_2) be σ-finite measure spaces. A bounded linear operator $A : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2)$, $1 \le p_1, p_2 < \infty$, is nuclear if and only if there exist sequences $\{g_n\}_{n=1}^\infty$ in $L^{p_2}(X_2, \mu_2)$ and $\{h_n\}_{n=1}^\infty$ in $L^{p_1'}(X_1, \mu_1)$ such that*

$$\sum_{n=1}^\infty \|g_n\|_{L^{p_2}(X_2,\mu_2)} \|h_n\|_{L^{p_1'}(X_1,\mu_1)} < \infty$$

and for all f in $L^{p_1}(X_1, \mu_1)$,

$$(Af)(x) = \int_{X_1} \left(\sum_{n=1}^\infty g_n(x) h_n(y) \right) f(y)\, d\mu_1(y)$$

for almost all x in X_1.

The function k defined by

$$k(x, y) = \sum_{n=1}^\infty g_n(x) h_n(y), \quad x \in X_1, y \in X_2,$$

is known as a kernel of the nuclear operator $A : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2)$.

Theorem 1.2. *Let (X, μ) be a σ-finite measure space. If $A : L^p(X, \mu) \to L^p(X, \mu)$, $1 \le p < \infty$, is a nuclear operator, then the trace $\mathrm{tr}(A)$ of $A : L^p(X, \mu) \to L^p(X, \mu)$ is given by*

$$\mathrm{tr}(A) = \int_X k(x, x)\, d\mu(x),$$

where k is the kernel induced by Theorem 1.1.

Remark 1.3. Since k is defined only up to a set of measure zero on $X \times X$, the function $k(x, x)$ on the diagonal of $X \times X$ is not well-defined, resulting in a multitude of traces for a given nuclear operator. This caveat, however, does not come up in the case of the counting measure on $\mathbb{Z}$.

The following theorem of Grothendieck [6, 7] gives a subclass of nuclear operators known as 2/3-nuclear operators.

Theorem 1.4. *Let (X, μ) be a σ-finite measure space and let $1 \le p < \infty$. If there exist sequences $\{g_n\}_{n=1}^\infty$ in $L^p(X, \mu)$ and $\{h_n\}_{n=1}^\infty$ in $L^{p'}(X, \mu)$ such that*

$$\sum_{n=1}^\infty \|g_n\|_{L^p(X,\mu)}^{2/3} \|h_n\|_{L^{p'}(X,\mu)}^{2/3} < \infty$$

and for all f in $L^p(X,\mu)$,

$$(Af)(x) = \int_X \left(\sum_{n=1}^{\infty} g_n(x)\, h_n(y)\right) f(y)\, d\mu(y)$$

for almost all x in X, then $A : L^p(X,\mu) \to L^p(X,\mu)$ is a nuclear operator. If $\{\lambda_j\}_{j=1}^{\infty}$ is the set of all eigenvalues of $A : L^p(X,\mu) \to L^p(X,\mu)$ with multiplicities counted, then

$$\operatorname{tr}(A) = \sum_{j=1}^{\infty} \lambda_j, \tag{1.1}$$

where the series is absloutely convergent.

Remark 1.5. It is important to point out that for 2/3-nuclear operators, Lidskii's formula (1.1) to the effect that the trace is the sum of all the eigenvalues with multiplicities counted is valid [5, 9, 8]. The book [5] contains a lucid exposition of nuclear operators on Banach spaces and Lidskii's formula.

In Section 2 we give results on the nuclearity of pseudo-differential operators from $L^{p_1}(\mathbb{Z})$ into $L^{p_2}(\mathbb{Z})$, where $1 \le p_1, p_2 < \infty$, and the trace formulas when $p_1 = p_2$. Analogous results for the unit circle $\mathbb{S}^1$ are given in Section 3.

2. Pseudo-differential operators on $\mathbb{Z}$

To give conditions for nuclearity, we define for all n in $\mathbb{Z}$ the function e_n on $\mathbb{Z}$ by

$$e_n(m) = \delta_{nm},$$

where δ_{nm} is the Kronecker delta. Then we have the following theorem.

Theorem 2.1. *Let $k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be a function such that*

$$\sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} |k(n,m)|^{p_2}\right)^{1/p_2} < \infty.$$

Let $A : L^{p_1}(\mathbb{Z}) \to L^{p_2}(\mathbb{Z})$, $1 \le p_1, p_2 < \infty$, be the linear operator defined by

$$e_m(Ae_n) = k(n,m), \quad n, m \in \mathbb{Z},$$

where e_m is considered to be an element in $L^{p_2'}(\mathbb{Z})$. Then $A : L^{p_1}(\mathbb{Z}) \to L^{p_2}(\mathbb{Z})$ is a nuclear operator. Moreover if $p_1 = p_2$, then

$$\operatorname{tr}(A) = \sum_{n=-\infty}^{\infty} k(n,n).$$

Proof. For all a in $L^{p_1}(\mathbb{Z})$, we have

$$a = \sum_{n=-\infty}^{\infty} e_n(a) e_n$$

and

$$Aa = \sum_{n=-\infty}^{\infty} e_n(a) A e_n.$$

So, by Minkowski's inequality,

$$\begin{aligned}\|Aa\|_{L^{p_2}(\mathbb{Z})} &\leq \sum_{n=-\infty}^{\infty} \|e_n(a)Ae_n\|_{L^{p_2}(\mathbb{Z})} \\ &\leq \sup_{n\in\mathbb{Z}} |a_n| \sum_{n=-\infty}^{\infty} \|Ae_n\|_{L^{p_2}(\mathbb{Z})} \\ &= \sup_{n\in\mathbb{Z}} |a_n| \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} |k(n,m)|^{p_2}\right)^{1/p_2} < \infty.\end{aligned}$$

So, $A : L^{p_1}(\mathbb{Z}) \to L^{p_2}(\mathbb{Z})$ is a bounded linear operator. Since

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \|Ae_n\|_{L^{p_2}(\mathbb{Z})}\|e_n\|_{L^{p_1}(\mathbb{Z})} &= \sum_{n=-\infty}^{\infty} \|Ae_n\|_{L^{p_2}(\mathbb{Z})} \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} |k(n,m)|^{p_2}\right)^{1/p_2} < \infty,\end{aligned}$$

it follows from Theorem 1.1 that $A : L^{p_1}(\mathbb{Z}) \to L^{p_2}(\mathbb{Z})$ is a nuclear operator, and by Theorem 1.2,

$$\operatorname{tr}(A) = \sum_{n=-\infty}^{\infty} k(n,n).$$

□

The preceding theorem and Theorem 1.4 together imply the following theorem.

Theorem 2.2. *Let $k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be a function such that*

$$\sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} |k(n,m)|^{p_2}\right)^{2/(3p_2)} < \infty.$$

Then the linear operator $A : L^{p_1}(\mathbb{Z}) \to L^{p_2}(\mathbb{Z})$ is a 2/3-nuclear operator.

We can now come back to pseudo-differential operators on $\mathbb{Z}$.

Theorem 2.3. *Let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$ such that we can find a function c in $L^1(\mathbb{Z})$ and a function w in $L^p(\mathbb{Z})$, $1 < p < \infty$, for which*

$$|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m)| \leq |c(n)|\,|w(m)|, \quad n,m \in \mathbb{Z}.$$

Then the pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a nuclear operator. Moreover,

$$\operatorname{tr}(T_\sigma) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \sigma(n,\theta)\,d\theta.$$

Proof. Let $k : \mathbb{Z} \times \mathbb{Z}$ be the function defined by

$$k(n,m) = (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, n-m), \quad n,m \in \mathbb{Z}.$$

Then

$$\sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} |(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m)|^p\right)^{1/p} = \sum_{n=-\infty}^{\infty} |c(n)| \left(\sum_{m=-\infty}^{\infty} |w(m)|^p\right)^{1/p} < \infty.$$

So, by Theorem 2.1, $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a nuclear operator and

$$\operatorname{tr}(T_\sigma) = \sum_{n=-\infty}^{\infty} (\mathcal{F}_{\mathbb{S}^1}\sigma)(n,0) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} \sigma(n,\theta)\, d\theta.$$

□

If we use Theorem 2.2 instead of Theorem 2.1, then we can have Lidskii's formula as well.

Theorem 2.4. *Let σ be a measurable function on $\mathbb{Z}\times\mathbb{S}^1$ such that we can find a function c in $L^{2/3}(\mathbb{Z})$ and a function w in $L^p(\mathbb{Z})$, $1 \le p < \infty$, for which*

$$|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m)| \le |c(n)|\,|w(m)|, \quad n,m\in\mathbb{Z}.$$

Then the pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a $2/3$-nuclear operator. Moreover,

$$\operatorname{tr}(T_\sigma) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} \sigma(n,\theta)\, d\theta,$$

which is Lidskii's formula for the sum of all eigenvalues of $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ with multiplicities counted.

Proof. Again, let $k : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ be the function defined by

$$k(n,m) = (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, n-m), \quad n,m\in\mathbb{Z}.$$

Then

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty} |(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m)|^p\right)^{2/(3p)} &= \sum_{n=-\infty}^{\infty} |c(n)|^{2/3}\left(\sum_{m=-\infty}^{\infty}|w(m)|^p\right)^{2/(3p)} \\ &< \infty.\end{aligned}$$

So, by Theorem 2.2, the proof is complete. □

3. Pseudo-differential operators on $\mathbb{S}^1$

We begin this section with the kernel representation of pseudo-differential operators on $\mathbb{S}^1$.

For all f in $L^p(\mathbb{S}^1)$, we get for all θ in $[-\pi,\pi]$,

$$\begin{aligned}(T_\tau f)(\theta) &= \sum_{n=-\infty}^{\infty} e^{in\theta}\tau(\theta,n)\hat{f}(n) \\ &= \sum_{n=-\infty}^{\infty} e^{in\theta}\tau(\theta,n)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\phi} f(\phi)\, d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)}\tau(\theta,n)\right) f(\phi)\, d\phi \\ &= \int_{-\pi}^{\pi} k(\theta,\phi) f(\phi)\, d\phi,\end{aligned}$$

where

$$k(\theta,\phi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)}\tau(\theta,n), \quad \theta,\phi\in[-\pi,\pi]. \tag{3.1}$$

A sufficient condition on the symbol τ for the corresponding pseudo-differential operator $T_\tau : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 \le p_1, p_2 < \infty$, to be nuclear is given in the following theorem.

Theorem 3.1. *Let τ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ such that*

$$\sum_{n=-\infty}^{\infty} \|\tau(\cdot, n)\|_{L^{p_2}(\mathbb{S}^1)} < \infty.$$

Then $T_\tau : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ is a nuclear operator.

Proof. Let $\{g_n\}_{n=-\infty}^{\infty}$ and $\{h_n\}_{n=-\infty}^{\infty}$ be sequences given by

$$g_n(\theta) = e^{in\theta}\tau(\theta, n), \quad \theta \in [-\pi, \pi], n \in \mathbb{Z},$$

and

$$h_n(\phi) = \frac{1}{2\pi} e^{-in\phi}, \quad \phi \in [-\pi, \pi], n \in \mathbb{Z}.$$

Then for all n in $\mathbb{Z}$,

$$g_n \in L^{p_2}(\mathbb{S}^1)$$

and

$$h_n \in L^{p_1'}(\mathbb{S}^1).$$

So,

$$\sum_{n=-\infty}^{\infty} \|g_n\|_{L^{p_2}(\mathbb{S}^1)} \|h_n\|_{L^{p_1'}(\mathbb{S}^1)} = \sum_{n=-\infty}^{\infty} \|\tau(\cdot, n)\|_{L^{p_2}(\mathbb{S}^1)} < \infty.$$

Thus, by Theorem 1.1, the pseudo-differential operator $T_\tau : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ that has kernel k given by (3.1) is nuclear. □

This proof gives the following theorem.

Theorem 3.2. *Let τ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ such that*

$$\sum_{n=-\infty}^{\infty} \|\tau(\cdot, n)\|_{L^p(\mathbb{S}^1)}^{2/3} < \infty.$$

Then the pseudo-differential operator $T_\tau : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is 2/3-nuclear. Moreover,

$$\operatorname{tr}(T_\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \tau(\theta, n)\, d\theta,$$

which is also equal to the sum of all eigenvalues of $T_\tau : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ with multiplicities counted.

Example 3.3. In polar coordinates, the Laplacian $\Delta_{\mathbb{S}^1}$ on $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{\partial^2}{\partial\theta^2}.$$

The one-parameter semigroup generated by $\Delta_{\mathbb{S}^1}$ is $e^{t\Delta_{\mathbb{S}^1}}$, $t > 0$. In fact, $\Delta_{\mathbb{S}^1} : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$, $1 \le p < \infty$, is given by

$$\begin{aligned}(e^{\Delta_{\mathbb{S}^1}} f)(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{in(\theta-\phi)} f(\phi)\, d\phi \\ &= \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-n^2 t} \hat{f}(n), \quad \theta \in [-\pi, \pi],\end{aligned}$$

for all f in $L^p(\mathbb{S}^1)$. Thus, for $t > 0$, $e^{t\Delta_S} : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is a pseudo-differential operator on $\mathbb{S}^1$ with the symbol τ_t given by

$$\tau_t(\theta, n) = e^{-n^2 t}, \quad \theta \in [-\pi, \pi], n \in \mathbb{Z},$$

and with k_t given by

$$k_t(\theta, \phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)} e^{-n^2 t}, \quad \theta, \phi \in [-\pi, \pi].$$

Now, for $t > 0$, let $\{g_n\}_{n=-\infty}^{\infty}$ and $\{h_n^t\}_{n=-\infty}^{\infty}$ be sequences in, respectively, $L^p(\mathbb{S}^1)$ and $L^{p'}(\mathbb{S}^1)$ given by

$$g_n(\phi) = e^{in\phi}, \quad \phi \in [-\pi, \pi],$$

and

$$h_n^t(\phi) = e^{-n^2 t}, \quad \phi \in [-\pi, \pi].$$

Thus, for $t > 0$,

$$\sum_{n=-\infty}^{\infty} \|g_n\|_{L^p(\mathbb{S}^1)}^{2/3} \|h_n^t\|_{L^{p'}(\mathbb{S}^1)}^{2/3} < \infty.$$

So, by Theorem 1.4, $e^{t\Delta_{\mathbb{S}^1}} : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is nuclear and

$$\operatorname{tr}(e^{t\Delta_{\mathbb{S}^1}}) = \int_{-\pi}^{\pi} k_t(\phi, \phi)\, d\phi = \sum_{n=-\infty}^{\infty} e^{-n^2 t},$$

thus confirming Lidskii's theorem.

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