

RIGIDITY THEOREMS OF HYPERSURFACES IN LOCALLY SYMMETRIC RIEMANNIAN MANIFOLD

SHICHENG ZHANG AND BAOQIANG WU

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ABSTRACT. In this paper, the linear Weingarten hypersurfaces in a locally symmetric Riemannian manifold are investigated and the rigidity theorems are proved by the operator \square introduced by S. Y. Cheng and S. T. Yau, which is a generalization of main results obtained by several authors.

1. INTRODUCTION

When the ambient manifolds possess very nice symmetry, for example, the sphere, many results have been obtained in the study of the minimal hypersurface and the hypersurface with constant mean curvature or constant scalar curvature in these ambient manifolds. (One can refer to [1]-[7]). Recently, Q.M. Cheng and H. Nakagawa [8], and H.W. Xu [9] independently proved the optimal rigidity theorem for the hypersurface of constant mean curvature in a sphere.

In order to study hypersurfaces with constant scalar curvature, Cheng and Yau [12] introduced a new self-adjoint differential operator \square acting on C^2 -functions defined on Riemannian manifolds. As a by-product of this approach they were able to classify closed hypersurface M with constant normalized scalar curvature R satisfying $R \geq c$ and non-negative sectional curvatures immersed in complete and simply connected $(n + 1)$ -dimensional Riemannian manifolds of constant sectional curvature c , which will be denoted by $Q^{n+1}(c)$ and are also known as space forms.

By using the Cheng-Yau technique, X. Liu and H. Li [7] also obtained some rigidity theorems for hypersurface with constant scalar curvature. Therefore, it is important and natural to extend the Riemannian space forms to the locally symmetric Riemannian manifolds.

Let the ambient manifold N^{n+1} be a locally symmetric Riemannian manifold with sectional curvature K_N and M be an n -dimensional complete hypersurface with constant mean curvature H in N^{n+1} . When $\frac{1}{2} < \delta \leq K_N \leq 1$ (δ is a constant) at all points $x \in M$ and the squared norm of the second fundamental form S satisfies $S < n + \frac{n^3}{2(n-1)H^2} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$, S. Shu [13] and S. Ding [14] have obtained that the hypersurface M is a totally umbilical hypersurface,

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respectively. H.W. Xu [15] has also obtained the same result when M is an n -dimensional closed minimal hypersurface with constant mean curvature H in N^{n+1} and sectional curvature K_N satisfying the condition $\delta \leq K_N \leq 1$ at all points $x \in M$ and the squared norm of the second fundamental form S satisfies $S \leq (2\delta - 1)n$.

Now let us introduce the notion of linear Weingarten hypersurfaces in an $(n + 1)$ -dimensional locally symmetric Riemannian manifold N^{n+1} as follows:

Definition 1.1. Let M be a hypersurface in an $(n + 1)$ -dimensional locally symmetric Riemannian manifold N^{n+1} . We call M a linear Weingarten hypersurface if $cR + dH + e = 0$, where c, d and e are constants such that $c^2 + d^2 \neq 0$, R and H respectively denote the scalar curvature and the mean curvature of M .

Remark 1.1. When the constant $d = 0$ in Definition 1.1, a linear Weingarten hypersurface M reduces to a hypersurface with constant scalar curvature. When the constant $c = 0$ in Definition 1.1, a linear Weingarten hypersurface M reduces to a hypersurface with constant mean curvature. In such a sense, the linear Weingarten hypersurfaces can be regarded as a natural generalization of hypersurfaces with constant scalar curvature or with constant mean curvature.

In this paper, let the ambient manifold N^{n+1} be a locally symmetric Riemannian manifold with sectional curvature K_N satisfying the condition $K(u \wedge v) \geq c_2$ (c_2 is a constant), where $u, v \in TM$ and $K(\omega \wedge v) = c_1$ (c_1 is a constant), where $\omega \in T^\perp M$ and $v \in TM$; we shall say the manifold N^{n+1} satisfies condition (*). We study the linear Weingarten hypersurfaces in a locally symmetric Riemannian manifold.

As in the computation of ([10, 11]), we compute the scalar curvature of a point in N^{n+1} ,

$$K = \sum_{A=1}^{n+1} K_{AA} = 2 \sum_{i=1}^n K_{n+1in+1i} + \sum_{i,j=1}^n K_{ijji} = 2nc_1 + \sum_{i,j=1}^n K_{ijji}.$$

It is known that K is constant when N^{n+1} is locally symmetric, so $\sum_{i,j=1}^n K_{ijji}$ is constant.

From (2.3), we denote

$$(1.1) \quad n(n - 1)P = n^2H^2 - S = n(n - 1)r - \sum_{i,j=1}^n K_{ijji},$$

where r is a normalized scalar curvature of M . By investigating Cheng and Yau’s operator \square given in [12] and using some new estimations, we are able to study the compact or complete linear Weingarten hypersurfaces in a locally symmetric Riemannian manifold as follows:

Theorem 1.1. *Let M be an n -dimensional compact hypersurface in a locally symmetric Riemannian manifold N^{n+1} satisfying condition (*). If the squared norm of the second fundamental form S and the mean curvature H satisfy the conditions*

$$n\left(\frac{P - b}{a}\right)^2 \leq S \leq \frac{n}{(n - 2)(nP + 2c)}[n(n - 1)P^2 + 4c(n - 1)P + nc^2]$$

and $P = aH + b$, $(n - 1)a^2 + 4nb \geq 0$, then

(1) $S = n(\frac{P-b}{a})^2$ and M is a totally umbilical hypersurface or

(2) M is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple, and $S = \frac{n}{(n-2)(nP+2c)}[n(n-1)P^2 + 4c(n-1)P + nc^2]$, where $c = 2c_2 - c_1 \geq 0$.

Remark 1.2. Since $n(n - 1)P = n(n - 1)r - \sum_{i,j=1}^n K_{ijji}$, a hypersurface M in Theorem 1.1 satisfying $P = aH + b$ is just a linear Weingarten hypersurface in Definition 1.1.

Remark 1.3. If $c_1 = 1$, $c_2 = 1$, i.e. the locally symmetric Riemannian manifold N^{n+1} is the unit sphere $S^{n+1}(1)$, our Theorem 1.1 reduces to Theorem 1.3 in [16].

Remark 1.4. If $c_1 = 1$, $c_2 = 1$, when the constant a in the above identically vanishes, our Theorem 1.1 reduces to Theorem 2 in [17] and the main theorem in [7].

Theorem 1.2. *Let M be an n -dimensional complete hypersurface in a locally symmetric Riemannian manifold N^{n+1} satisfying condition (*). If the squared norm of the second fundamental form S satisfies the conditions $S < 2\sqrt{n-1}c$ and $P = aH + b$, $a \leq 0$, $(n - 1)a^2 + 4nb \geq 0$, then $S = nH^2$ and M is a totally umbilical hypersurface, where $c = 2c_2 - c_1 \geq 0$.*

Example. We consider the Riemannian product manifold

$$N^{n+1} = S^n(r) \times R^1 = \{(x^1, x^2, \dots, x^{n+2}) \in R^{n+2}, \sum_{A=1}^{n+1} (x^A)^2 = r^2\}.$$

Its sectional curvature is given by

$$K(u_i, u_j) = \frac{1}{r^2}, \quad K(u_i, u_{n+1}) = 0 \quad (i, j = 1, \dots, n).$$

Then,

$$M_1 = \{(x^1, x^2, \dots, x^{n+1}, c) \in R^{n+2}\} \subset N^{n+1}$$

is a totally umbilical hypersurface in a locally symmetric Riemannian manifold N^{n+1} , and

$$M_2 = \{(0, x^2, \dots, x^{n+2}) \in R^{n+2}\} \subset N^{n+1}$$

is an isoparametric hypersurface in a locally symmetric Riemannian manifold N^{n+1} , respectively.

2. PRELIMINARIES

If M is a hypersurface in N^{n+1} , let $\{e_1, e_2, \dots, e_{n+1}\}$ be a local frame of orthonormal vector fields in N^{n+1} such that, restricted to M , the vectors $\{e_1, e_2, \dots, e_n\}$ are tangent to M and the vector e_{n+1} is normal to M . Let $\{\omega^1, \omega^2, \dots, \omega^{n+1}\}$ be its dual frame field. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n + 1, \quad 1 \leq i, j, k, \dots \leq n.$$

Let K_{ABCD} and R_{ijkl} be the components of the curvature tensors of N^{n+1} and M , respectively. Let $h = h_{ij}$ be the second fundamental form of M ; the square of the norm of h is denoted by $S = \sum_{i,j=1}^n (h_{ij})^2$.

It is well known that for an arbitrary hypersurface M of N^{n+1} , we have

$$(2.1) \quad d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega^k \wedge \omega^l,$$

$$(2.2) \quad R_{ijkl} = K_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}.$$

The components R_{ij} of Ricci tensor and the normalized scalar curvature r of M are given by

$$(2.3) \quad \begin{aligned} R_{ij} &= \sum_{k=1}^n K_{kij k} + nHh_{ij} - \sum_{k=1}^n h_{ik}h_{kj}, \\ n(n-1)r &= \sum_{i,j=1}^n K_{ijji} + (nH)^2 - S. \end{aligned}$$

We denote the first and second covariant derivatives of h_{ij} by h_{ijk} and h_{ijkl} respectively, which are defined as in [1]. Following [1] and [19], we have

$$(2.4) \quad h_{ijk} - h_{ikj} = -K_{n+1ijk},$$

$$(2.5) \quad \sum_k h_{ijk} \omega_k = dh_{ij} - \sum_k h_{ik} \omega_{kj} - \sum_k h_{kj} \omega_{ki},$$

and the Ricci formula

$$(2.6) \quad h_{ijkl} - h_{ijlk} = \sum_s h_{sj} R_{sikl} + \sum_s h_{is} R_{sjkl}.$$

Let $K_{n+1ijk,l}$ be the covariant derivative of K_{n+1ijk} as the section of $T^\perp M \otimes T^*M \otimes T^*M \otimes T^*M$ and let $K_{ABCD,E}$ be the covariant derivative of K_{ABCD} as the curvature tensor of N^{n+1} . Restricted to M we have

$$(2.7) \quad \sum_l K_{n+1ijkl} \omega_l = dK_{n+1ijk} + \sum_s K_{n+1sjk} \omega_{is} + \sum_s K_{n+1ijs} \omega_{ks}$$

and

$$(2.8) \quad K_{n+1ijk,l} = K_{n+1ijkl} - K_{n+1in+1k}h_{jl} - K_{n+1ijn+1}h_{kl} + \sum_m K_{mijk}h_{ml}.$$

The mean curvature of M is given by $H = \frac{1}{n} \sum_{i=1}^n h_{ii}e_{n+1}$, and the Laplacian Δh_{ij} of the second fundamental form h of M is defined by $\Delta h_{ij} = \sum_{k=1}^n h_{ijkk}$:

$$(2.9) \quad \begin{aligned} \Delta h_{ij} &= \sum_k h_{kkij} + nHK_{n+1in+1j} - \sum_k K_{n+1kn+1k}h_{ij} + nH \sum_k h_{ik}h_{kj} - Sh_{ij} \\ &+ \sum_k (K_{lki k}h_{lj} + K_{lkj k}h_{li} + 2K_{lij k}h_{lk}) - \sum_k (K_{n+1ijk,k} + K_{n+1kik,j}). \end{aligned}$$

Since N^{n+1} is complete and locally symmetric, we have

$$K_{ABCD,E} = 0,$$

for all A, B, C, D, E . This together with (2.2), (2.7) and (2.8) implies

$$\begin{aligned}
 \frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ijkk}h_{ij} \\
 &= nH\text{tr}H_{n+1}^3 - S^2 + \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k,l} 2(h_{ij}h_{kl}K_{lijk} + h_{li}h_{ij}K_{lkjk}) \\
 (2.10) \quad &+ \sum_{i,j,k} h_{ijkk}h_{ij} + \sum_{i,j} nHh_{ij}K_{n+1ijn+1} - S \sum_k K_{n+1kn+1k}.
 \end{aligned}$$

Following Cheng-Yau [12], we introduce a modified operator \square acting on any C^2 -function f by

$$(2.11) \quad \square f = \sum_{i,j} (nH\delta_{ij} - h_{ij})f_{ij} - \frac{n-1}{2}a\Delta f.$$

It follows [12] that the operator \square is self-adjoint relative to the L^2 inner product of M , i.e.

$$\int_M f \square g = \int_M g \square f.$$

Choose a local frame of orthonormal vector fields $\{e_i\}$ so that at an arbitrary point x of M , $h_{ij} = \lambda_i\delta_{ij}$; then at the point x , and by use of (1.1) and (2.10), we have

$$\begin{aligned}
 \square(nH) &= nH\Delta(nH) - \sum_l \lambda_i(nH)_{ii} - \frac{n-1}{2}a\Delta(nH) \\
 &= \frac{1}{2}\Delta(n(n-1)r - \sum_{i,j=1}^n K_{ijji}) + \frac{1}{2}\Delta S \\
 &\quad - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii} - \frac{1}{2}\Delta(n(n-1)r) \\
 (2.12) \quad &= \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii}.
 \end{aligned}$$

Putting (2.10) into (2.12), we obtain

$$\begin{aligned}
 \square(nH) &= nH\Delta(nH) - \sum_l \lambda_i(nH)_{ii} \\
 (2.13) \quad &= \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + X + Y + Z,
 \end{aligned}$$

where

$$\begin{aligned}
 X &= nH\text{tr}H_{n+1}^3 - S^2, \\
 Y &= \sum_{i,j,k,l} 2(h_{ij}h_{kl}K_{lijk} + h_{li}h_{ij}K_{lkjk}), \\
 Z &= \sum_{i,j} nHh_{ij}K_{n+1ijn+1} - S \sum_k K_{n+1kn+1k}.
 \end{aligned}$$

3. PROOF OF THEOREMS

The following lemmas are useful in the proofs of Theorems 1.1 and 1.2.

Lemma 3.1. *Let M be an n -dimensional compact hypersurface in a locally symmetric Riemannian Manifold N^{n+1} satisfying condition $(*)$. If $P = aH + b$, $a, b \in R$ and $(n - 1)a^2 + 4nb \geq 0$, then we have*

$$|\nabla h|^2 = \sum_{\alpha, i, j, k} (h_{ijk}^\alpha)^2 \geq n^2 |\nabla H|^2.$$

Proof. From the Gauss equation, we have

$$S = n^2 H^2 - n(n - 1)P = n^2 H^2 - n(n - 1)(aH + b).$$

Taking the covariant derivative of the above equation, we have

$$2 \sum_{\alpha, i, j} h_{ij}^\alpha h_{ijk}^\alpha = 2n^2 H H_k - n(n - 1)aH_k.$$

Therefore,

$$4|h|^2 |\nabla h|^2 \geq 4 \sum_k \left(h_{ij}^\alpha h_{ijk}^\alpha \right)^2 = [2n^2 H - n(n - 1)a]^2 |\nabla H|^2.$$

On the other hand,

$$\begin{aligned} & [2n^2 H - n(n - 1)a]^2 - 4n^2 S \\ &= 4n^4 H^2 + n^2(n - 1)^2 a^2 - 4n^3(n - 1)Ha \\ &\quad - 4n^3[nH^2 - (n - 1)(aH + b)] \\ &= n^2(n - 1)[(n - 1)a^2 + 4nb] \\ &\geq 0. \end{aligned}$$

It follows that

$$|\nabla h|^2 \geq n^2 |\nabla H|^2.$$

□

Lemma 3.2 ([19]). *Let μ_1, \dots, μ_n be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = B$, where $B = \text{const.} \geq 0$. Then*

$$\left| \sum_i \mu_i^3 \right| \leq \frac{n - 2}{\sqrt{n(n - 1)}} B^{\frac{3}{2}},$$

and equality holds if and only if

$$\mu_1 = \dots = \mu_{n-1} = -\sqrt{\frac{1}{n(n - 1)}} B, \quad \mu_n = \sqrt{\frac{n - 1}{n}} B.$$

Choose a local frame of orthonormal vector fields $\{e_i\}$ so that at an arbitrary point x of M , $h_{ij} = \lambda_i \delta_{ij}$; then at the point x we have

$$S = \sum_i \lambda_i^2.$$

Putting $\mu_j = H - \lambda_j$, we obtain

$$(3.1) \quad \sum_j \mu_j = 0, \quad |\phi|^2 = \sum_j \mu_j^2 = S - nH^2,$$

and

$$(3.2) \quad \sum_i \lambda_i^3 = nH^3 + 3H \sum_i \mu_i^2 - \sum_i \mu_i^3.$$

So

$$\begin{aligned} nH \operatorname{tr} H_{n+1}^3 &= nH(nH^3 + 3H \sum_i \mu_i^2 - \sum_i \mu_i^3) \\ &\geq 3nH^2|\phi|^2 + n^2H^4 - n|H| \frac{n-2}{\sqrt{n(n-1)}}|\phi|^3, \end{aligned}$$

and we obtain

Lemma 3.3. $X \geq |\phi|^2 \left[nH^2 - |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| \right].$

Lemma 3.4 ([15]). $Y \geq 2nc_2|\phi|^2.$

Lemma 3.5. $Z = c_1n(nH^2 - S) = -c_1n|\phi|^2.$

Proof.

$$\begin{aligned} Z &= \sum_{i,j} nHh_{ij}K_{n+1ijn+1} - S \sum_k K_{n+1kn+1k} \\ &= nH \sum_i \lambda_i K_{n+1in+1i} - S \sum_i K_{n+1in+1i} \\ &= \sum_i (S - nH\lambda_i)K_{n+1in+1i} = c_1n(nH^2 - S). \end{aligned}$$

This proves Lemma 3.5. □

From the assumption of Theorem 1.1 and Lemmas 3.1, 3.3, 3.4 and 3.5, we obtain

$$(3.3) \quad \square(nH) \geq |\phi|^2 \left[nc - |\phi|^2 - n|H| \frac{n-2}{\sqrt{n(n-1)}}|\phi| + nH^2 \right],$$

where $c = 2c_2 - c_1.$

The next lemmas have essential roles in the proof of Theorem 1.2.

Lemma 3.6 ([20]). *Let M be an n -dimensional complete Riemannian manifold whose sectional curvature is bounded from below and let $f : M \rightarrow \mathbb{R}$ be a smooth function which is bounded from above on M . Then there is a sequence of points $\{p_k\}$ in M such that*

$$\lim_{k \rightarrow \infty} f(p_k) = \sup f; \quad \lim_{k \rightarrow \infty} |\nabla f(p_k)| = 0; \quad \lim_{k \rightarrow \infty} \sup(\Delta f(p_k)) \leq 0.$$

Lemma 3.7. *Let M be a complete hypersurface in an $(n + 1)$ -dimensional locally symmetric Riemannian manifold N^{n+1} with bounded mean curvature. If $P = aH + b$, $(n - 1)a^2 + 4nb \geq 0$, then there is a sequence of points $\{p_k\} \in M$ such that*

$$\lim_{k \rightarrow \infty} nH(p_k) = n \sup H; \quad \lim_{k \rightarrow \infty} |\nabla nH(p_k)| = 0; \quad \lim_{k \rightarrow \infty} \sup(\square(nH)(p_k)) \leq 0.$$

Proof. Choose a local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ at $p \in M$ such that $h_{ij} = \lambda_j \delta_{ij}$. Then,

$$\square(nH) = \sum_i \left((nH - \lambda_i^{n+1}) - \frac{n-1}{2}a \right) (nH)_{ii}.$$

If $H \equiv 0$, Theorem 1.1 is obvious. Let us suppose that H is not identically zero. By changing the orientation of M if necessary, we may assume $H > 0$.

From (1.1), we have

$$\begin{aligned} n(n-1)P &= n^2H^2 - S, \\ (\lambda_i^{n+1})^2 &\leq S = n^2H^2 - n(n-1)(aH + b) \\ &= (nH)^2 - (n-1)a(nH) - n(n-1)b \\ &= \left(nH - \frac{1}{2}(n-1)a\right)^2 - \frac{1}{4}(n-1)\left((n-1)a^2 + 4nb\right) \\ &\leq \left(nH - \frac{1}{2}(n-1)a\right)^2, \end{aligned}$$

and so

$$(3.4) \quad |\lambda_i^{n+1}| \leq \left|nH - \frac{1}{2}(n-1)a\right|.$$

From (2.2) and (3.4), we show that

$$R_{ijij} = K_{ijij} + h_{ii}^{n+1}h_{jj}^{n+1} - (h_{ij}^{n+1})^2 \geq c_2 - \left(nH - \frac{1}{2}(n-1)a\right)^2.$$

Since H is bounded, we may apply Lemma 3.6 to nH , obtaining a sequence of points $\{p_k\} \in M$ such that

$$(3.5) \quad \lim_{k \rightarrow \infty} nH(p_k) = n \sup H, \quad \lim_{k \rightarrow \infty} |\nabla nH(p_k)| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \sup((nH)_{ii}(p_k)) \leq 0.$$

Since H is bounded, taking subsequences if necessary, we can arrive at a sequence $\{p_k\} \in M$ and such that $H(p_k) \geq 0$. Thus from (3.4), we get

$$\begin{aligned} 0 &\leq nH(p_k) - \frac{1}{2}(n-1)a - |\lambda_i^{n+1}| \\ &\leq nH(p_k) - \frac{1}{2}(n-1)a - \lambda_i^{n+1} \\ &\leq nH(p_k) - \frac{1}{2}(n-1)a + |\lambda_i^{n+1}| \\ (3.6) \quad &\leq 2nH(p_k) - (n-1)a. \end{aligned}$$

Using once more the fact that H is bounded, from (3.6) and $a \leq 0$ we infer that $nH(p_k) - \frac{1}{2}(n-1)a - \lambda_i^{n+1}$ is non-negative and bounded. By applying $\square(nH)$ at p_k , taking the limit and using (3.5) and (3.6), we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup(\square(nH)(p_k)) &\leq \sum_i \lim_{k \rightarrow \infty} \sup\left[\left(nH(p_k) - \frac{1}{2}(n-1)a - \lambda_i^{n+1}\right)(nH)_{ii}(p_k)\right] \\ &\leq 0. \end{aligned}$$

□

Proof of Theorem 1.1. Since M is a compact hypersurface in locally symmetric Riemannian manifold N^{n+1} , from (3.3) we have

$$(3.7) \quad 0 \geq \int_M (S - nH^2) \left[nc - (S - nH^2) - n|H| \frac{n-2}{\sqrt{n(n-1)}} \sqrt{S - nH^2} + nH^2 \right] dv.$$

On the other hand, since

$$S \leq \frac{n}{(n-2)(nP+2c)}[n(n-1)P^2 + 4c(n-1)P + nc^2],$$

we get

$$\frac{n-2}{n} \sqrt{(n(n-1)P+S)(S-nP)} \leq nc + 2(n-1)P + \frac{n-2}{n} S.$$

From (1.1) and (3.1), we have

$$(3.8) \quad S - nH^2 = \frac{n-1}{n}(S - nP).$$

So, we obtain

$$(3.9) \quad \begin{aligned} &nc - nH^2 + (S - nH^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{S - nH^2} \\ &= nc + 2(n-1)P + \frac{n-2}{n}S - \frac{n-2}{n}\sqrt{(n(n-1)P+S)(S-nP)}. \end{aligned}$$

This implies that

$$(3.10) \quad (S - nH^2) \left[nc - (S - nH^2) - n|H| \frac{n-2}{\sqrt{n(n-1)}} \sqrt{S - nH^2} + nH^2 \right] \geq 0.$$

By (3.4) and (3.7), we obtain either $S - nH^2 = S - n(\frac{P-b}{a})^2 = 0$ and M is totally umbilical or $S = \frac{n}{(n-2)(nP+2c)}[n(n-1)P^2 + 4c(n-1)P + nc^2]$.

If M is not totally umbilical, we can see that $S = \frac{n}{(n-2)(nP+2c)}[n(n-1)P^2 + 4c(n-1)P + nc^2]$. Hence equalities hold in Lemma 3.1 and Lemma 3.2, and it follows that $\lambda_i = const.$ for all i and $(n-1)$ of the λ_i 's are equal. After re-numbering if necessary, we can assume that

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n-1}, \lambda_1 \neq \lambda_n.$$

Therefore, M is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple. This completes the proof of Theorem 1.1. \square

Proof of Theorem 1.2. Now we assume $P = aH + b$ and $(n-1)a^2 + 4nb \geq 0$. For a real number $d = \frac{n+2\sqrt{n-1}}{n-2}\sqrt{n} > 0$, we have

$$(3.11) \quad 2|H||\phi| \leq dH^2 + \frac{1}{d}|\phi|^2.$$

From (3.3) and (3.11), we obtain

$$\begin{aligned} \square(nH) \geq |\phi|^2 \left[nc + nH^2 \left(2 - \frac{(n-2)d}{2\sqrt{n(n-1)}} + \frac{n(n-2)}{2\sqrt{n(n-1)}d} \right) \right. \\ \left. - S \left(1 + \frac{n(n-2)}{2\sqrt{n(n-1)}d} \right) \right], \end{aligned}$$

that is,

$$(3.12) \quad \square(nH) \geq |\phi|^2 \left(nc - \frac{n}{2\sqrt{n-1}}S \right).$$

From the assumption $S < 2\sqrt{n-1}c$ and according to Lemma 3.7, there exists a sequence of points $\{p_k\}$ in M such that

$$(3.13) \quad \lim_{k \rightarrow \infty} \sup(\square(nH)(p_k)) \leq 0, \quad \lim_{k \rightarrow \infty} nH(p_k) = \sup(nH).$$

From (1.1) and (3.1), we have

$$|\phi|^2 = n(n-1)(H^2 - aH - b).$$

Notice that $\lim_{k \rightarrow \infty} nH(p_k) = \sup(nH)$, so we have

$$\lim_{k \rightarrow \infty} |\phi|^2(p_k) = \sup |\phi|^2, \quad \lim_{k \rightarrow \infty} S(p_k) = \sup S.$$

Evaluating (3.12) at the points p_k of the sequence, taking the limit and using (3.13), we obtain

$$(3.14) \quad 0 \geq \limsup_{k \rightarrow \infty} (\square(nH)(p_k)) \geq \sup |\phi|^2 (nc - \frac{n}{2\sqrt{n-1}} \sup S).$$

If $\sup S < 2\sqrt{n-1}c$, then we have $\sup |\phi|^2 = 0$, that is, $|\phi|^2 = 0$. Thus, we infer that $S = nH^2$ and M is a totally umbilical hypersurface.

This completes the proof of Theorem 1.2. \square

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SCHOOL OF MATHEMATICS, JIANGSU NORMAL UNIVERSITY, XUZHOU, JIANGSU 221116, PEOPLE'S REPUBLIC OF CHINA

E-mail address: zhangshicheng@jsnu.edu.cn

SCHOOL OF MATHEMATICS, JIANGSU NORMAL UNIVERSITY, XUZHOU, JIANGSU 221116, PEOPLE'S REPUBLIC OF CHINA

E-mail address: wubaoqiang@jsnu.edu.cn