ON IDENTITIES OF INFINITE DIMENSIONAL LIE SUPeralgebras

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Abstract. We study codimension growth of infinite dimensional Lie superalgebras over an algebraically closed field of characteristic zero. We prove that if a Lie superalgebra \( L \) is a Grassmann envelope of a finite dimensional simple Lie algebra, then the PI-exponent of \( L \) exists and is a positive integer.

1. Introduction

We shall consider algebras over a field \( F \) of characteristic zero. One of the approaches in the investigation of associative and non-associative algebras is to study numerical invariants associated with their identical relations. Given an algebra \( A \), we can associate to it the sequence of its codimensions \( \{c_n(A)\}_{n \in \mathbb{N}} \) (all notions and definitions will be given in the next section).

This sequence gives some information not only about the identities of \( A \) but also about the structure of \( A \). For example, \( A \) is nilpotent if and only if \( c_n(A) = 0 \) for all large enough \( n \). If \( A \) is an associative non-nilpotent \( F \)-algebra, then \( A \) is commutative if and only if \( c_n(A) = 1 \) for all \( n \geq 1 \).

For an associative algebra \( A \) with a non-trivial polynomial identity, the sequence \( c_n(A) \) is exponentially bounded by the celebrated Regev Theorem [20], while \( c_n(A) = n! \) if \( A \) does not satisfy any non-trivial polynomial identity. In the non-associative case the sequence of codimensions may have even faster growth. For example, if \( A \) is an absolutely free algebra, then

\[
c_n(A) = a_n n!
\]

where

\[
a_n = \frac{1}{2} \left( \frac{2n - 2}{n - 1} \right)
\]

is the Catalan number, i.e. the number of all possible arrangements of brackets in the word of length \( n \).
For a Lie algebra $L$ the sequence $\{c_n(L)\}_{n \in \mathbb{N}}$ is in general not exponentially bounded, even if $L$ satisfies non-trivial Lie identities (see for example [18]). Nevertheless, a class of Lie algebras with exponentially bounded codimensions is sufficiently wide. It includes, in particular, all finite dimensional algebras [1][11], Kac-Moody algebras [23][24], infinite dimensional simple Lie algebras of Cartan type [15], Virasoro algebra, and many others.

In the case when $\{c_n(A)\}_{n \in \mathbb{N}}$ is exponentially bounded, the upper and lower limits of the sequence $\{\sqrt[n]{c_n(A)}\}_{n \in \mathbb{N}}$ exist and a natural question arises: does the ordinary limit

$$\lim_{n \to \infty} \sqrt[n]{c_n(A)}$$

exist? In the case of existence we call this limit $exp(A)$ or the PI-exponent of $A$.

Amitsur conjectured in the 1980’s that for any associative P.I. algebra such a limit exists and that it is a non-negative integer. This conjecture was confirmed first for verbally prime P.I. algebras in [4][21] and later for the general case in [8][9]. For Lie algebras a series of positive results was obtained for finite dimensional algebras [6][7][25], for algebras with nilpotent commutator subalgebras [17], for affine Kac-Moody algebras [23][24], and for some other classes (see [16]). For Lie superalgebras there exist only partial results [26][27][30][31].

On the other hand, it was shown in [28] that there exists a Lie algebra $L$ with

$$3.1 < \liminf_{n \to \infty} \sqrt[n]{c_n(L)} \leq \limsup_{n \to \infty} \sqrt[n]{c_n(L)} < 3.9.$$  

This algebra $L$ is soluble and almost nilpotent; i.e. it contains a nilpotent ideal of finite codimension. In the general non-associative case there exists, for any real number $\alpha > 1$, an algebra $A_\alpha$ such that

$$\lim_{n \to \infty} \sqrt[n]{c_n(A_\alpha)} = \alpha$$

(see [5]). Note also that by a recent result [12] there exist finite dimensional Lie superalgebras with a fractional limit $\sqrt[n]{c_n(L)}$.

In the present paper we shall study Grassmann envelopes of finite dimensional simple Lie algebras. Our main result is the following theorem:

**Theorem 1.** Let $L_0 \oplus L_1$ be a finite dimensional simple Lie algebra over an algebraically closed field $F$ of characteristic zero with some $\mathbb{Z}_2$-grading. Also, let $\bar{L} = L_0 \otimes G_0 \oplus L_1 \otimes G_1$ be the Grassmann envelope of $L$. Then the limit

$$exp(\bar{L}) = \lim_{n \to \infty} \sqrt[n]{c_n(\bar{L})}$$

exists and is a positive integer. Moreover, $exp(\bar{L}) = \dim L$.

Another result of our paper concerns graded identities. Since any Lie superalgebra $L$ is $\mathbb{Z}_2$-graded, one can consider $\mathbb{Z}_2$-graded identities of $L$ and the corresponding graded codimensions $c_n^\nu(L)$. We shall prove that graded codimensions have similar properties.
Theorem 2. Let \( L = L_0 \oplus L_1 \) be a finite dimensional simple Lie algebra over an algebraically closed field \( F \) of characteristic zero with some \( \mathbb{Z}_2 \)-grading. Also, let \( \bar{L} = L_0 \otimes G_0 \oplus L_1 \otimes G_1 \) be a Grassmann envelope of \( L \). Then the limit

\[
\exp^{gr}(\bar{L}) = \lim_{n \to \infty} \sqrt[n]{c_n^{gr}(\bar{L})}
\]

exists and is a non-negative integer. Moreover, \( \exp^{gr}(\bar{L}) = \dim L \).

In other words, both PI-exponent \( \exp(\bar{L}) \) and graded PI-exponent \( \exp^{gr}(\bar{L}) \) exist, they are integers, and they coincide. Note that for an arbitrary \( \mathbb{Z}_2 \)-graded algebra the growth of ordinary codimensions and graded codimensions may differ. For example, if \( A = M_k(F) \otimes F\mathbb{Z}_2 \) with the canonical \( \mathbb{Z}_2 \)-grading induced from group algebra \( F\mathbb{Z}_2 \), where \( M_k(F) \) is a full \( k \times k \) matrix algebra, then \( \exp(A) = k^2 \), while \( \exp^{gr}(A) = 2k^2 \) (see [10] for details). In the Lie case one can take \( L = L_0 \oplus L_1 \) to be a two dimensional metabelian algebra with \( L_0 = \langle e \rangle, L_1 = \langle f \rangle \) and with only one non-trivial product \([e,f] = f\). Then \( c_n(L) = n - 1 \) for all \( n \geq 2 \); hence \( \exp(L) = 1 \). On the other hand, \( \exp^{gr}(L) = 2 \).

2. The main constructions and definitions

Let \( A \) be an arbitrary non-associative algebra over a field \( F \) and let \( F\{X\} \) be an absolutely free \( F \)-algebra with a countable generating set \( X \). A polynomial \( f = f(x_1, \ldots, x_n) \) is said to be an identity of \( A \) if \( f(a_1, \ldots, a_n) = 0 \) for any \( a_1, \ldots, a_n \in A \). The set of all identities of \( L \) forms a T-ideal \( Id(A) \) in \( F\{X\} \), that is, an ideal which is stable under all endomorphisms of \( F\{X\} \). Denote by \( P_n = P_n(x_1, \ldots, x_n) \) the subspace of all multilinear polynomials on \( x_1, \ldots, x_n \) in \( F\{X\} \). Then \( P_n \cap Id(A) \) is a subspace of all multilinear identities of \( A \) of degree \( n \). In the case when \( \text{char } F = 0 \), the T-ideal \( Id(A) \) is completely determined by the subspaces \( \{P_n \cap Id(A)\}, n = 1, 2, \ldots \).

For estimating how many identities an algebra \( A \) can have, one can define the so-called \( n \)-th codimension of the identities of \( A \) or, for short, the codimension of \( A \):

\[
c_n(A) = \dim \frac{P_n}{P_n \cap Id(A)}, \quad n = 1, 2, \ldots
\]

As was mentioned above, the class of associative and non-associative algebras with exponentially bounded sequence \( \{c_n(A)\} \) is sufficiently wide. In the case when \( c_n(A) < a^n \) for some real \( a \), one can define the lower and the upper PI-exponents of \( A \) as follows:

\[
\exp(A) = \lim_{n \to \infty} \sqrt[n]{c_n(A)}, \quad \exp(A) = \lim_{n \to \infty} \sqrt[n]{c_n(A)}
\]

and the ordinary PI-exponent as follows:

\[
\exp(A) = \lim_{n \to \infty} \sqrt[n]{c_n(A)}
\]

provided that \( \exp(A) = \overline{\exp}(A) \).

For \( \mathbb{Z}_2 \)-graded algebras one can also consider graded identities. Let \( X \) and \( Y \) be two infinite sets of variables and let \( F\{X \cup Y\} \) be an absolutely free algebra generated by \( X \cup Y \). If we suppose that all elements of \( X \) are even and all elements of \( Y \) are odd, i.e. \( \deg(x) = 0, \deg(y) = 1 \) for any \( x \in X, y \in Y \), then \( F\{X \cup Y\} \) can be
naturally endowed by a $\mathbb{Z}_2$-grading. A polynomial $f = f(x_1, \ldots, x_m, y_1, \ldots, y_n) \in F\{X \cup Y\}$ is said to be a graded identity of a superalgebra $A = A_0 \oplus A_1$ if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$, for all $a_1, \ldots, a_m \in A_0, b_1, \ldots, b_n \in A_1$. Fix $0 \leq k \leq n$ and denote by $P_{k,n-k}$ the subspace of $F\{X \cup Y\}$ spanned by all multilinear polynomials in $x_1, \ldots, x_k \in X, y_1, \ldots, y_{n-k} \in Y$. Then $P_{k,n-k} \cap \text{Id}(A)$ is the set of all multilinear polynomial identities of the superalgebra $A = A_0 \oplus A_1$ in $k$ even and $n-k$ odd variables.

One of the equivalent definitions of graded codimensions of $A$ is

$$c_{n}^{gr}(A) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(A),$$

where

$$c_{k,n-k}(A) = \dim \frac{P_{k,n-k}}{P_{k,n-k} \cap \text{Id}(A)}.$$

Starting from a $\mathbb{Z}_2$-graded algebra of some class (Lie, Jordan alternative, etc.), one can construct a $\mathbb{Z}_2$-graded algebra of a different class using the notion of a Grassmann envelope – they play an exceptional role in PI-theory. For example, any variety of associative algebras is generated by the Grassmann envelope of some finite dimensional associative superalgebra \[14\]. In the Lie case any so-called special variety is generated by the Grassmann envelope of a finitely generated Lie superalgebra \[22\].

We recall this construction for the Lie and the super Lie cases. Let $G$ be the Grassmann algebra generated by 1 and the infinite set $\{e_1, e_2, \ldots\}$ satisfying the following relations: $e_i e_j = -e_j e_i$, $i, j = 1, 2, \ldots$. It is known that $G$ has a natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$, where

$$G_0 = \text{Span}\langle e_{i_1} \cdots e_{i_n} | n = 2k, k = 0, 1, \ldots \rangle,$$

$$G_1 = \text{Span}\langle e_{i_1} \cdots e_{i_n} | n = 2k+1, k = 0, 1, \ldots \rangle.$$

Given a Lie algebra $L$ with a $\mathbb{Z}_2$-grading $L = L_0 \oplus L_1$, its Grassmann envelope

$$G(L) = L_0 \otimes G_0 \oplus L_1 \otimes G_1 \subset L \otimes G$$

is a Lie superalgebra. Vice versa, if $L = L_0 \oplus L_1$ is a Lie superalgebra, then $G(L)$ is an ordinary Lie algebra with a $\mathbb{Z}_2$-grading.

3. Cocharacters of Grassmann envelopes

The main tool in studying codimension asymptotics is the representation theory of symmetric groups. We refer the reader to \[13\] for details. The symmetric group $S_n$ acts naturally on multilinear polynomials in $F\{X\}$ as

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Hence $P_n$ is an $FS_n$-module and $P_n \cap \text{Id}(L)$, and also

$$P_n(L) = \frac{P_n}{P_n \cap \text{Id}(L)}$$

are $FS_n$-modules. The $S_n$-character $\chi(P_n(L))$ is called the $n$-th cocharacter of $L$ and we shall write

$$\chi_n(L) = \chi(P_n(L)).$$
Recall that any irreducible $FS_n$-module corresponds to a partition $\lambda$ of $n$, $\lambda \vdash n$, $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_1 \geq \cdots \geq \lambda_k$ are positive integers and $\lambda_1 + \cdots + \lambda_k = n$. By the Maschke Theorem, any finite dimensional $FS_n$-module $M$ decomposes into a direct sum of irreducible components, and hence its character $\chi(M)$ has a decomposition

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where $m_\lambda$ are non-negative integers. In particular, for the algebra $L$ we have

$$\chi(L) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda.$$

Integers $m_\lambda$ in (3) are called multiplicities of $\chi_\lambda$ in $\chi_n(L)$, and $d_\lambda = \deg \chi_\lambda = \chi_\lambda(1)$ are the dimensions of the corresponding irreducible representations. Therefore

$$c_n(L) = \dim P_n(L) = \sum_{\lambda \vdash n} m_\lambda d_\lambda. \quad (4)$$

For any partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ one can construct the Young diagram $D_\lambda$ containing $\lambda_1$ boxes in the first row, $\lambda_2$ boxes in the second row, and so on:

$$D_\lambda = \begin{array}{cccc}
\vdots & \vdots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \cdots \\
\end{array}$$

Given integers $k, l, d \geq 0$, we define the partition

$$h(k, l, d) = (l + d, \ldots, l + d, l, \ldots, l)$$

of $n = kl + d(k + l)$. The Young diagram associated with $h(k, l, d)$ is hook-shaped, and we define $H(k, l)$, an infinite hook, as the union of all $D_\lambda$ with $\lambda = h(k, l, d)$, $d = 1, 2, \ldots$. For short we shall say that a partition $\lambda \vdash n$ lies in the hook $H(k, l)$, $\lambda \in H(k, l)$, if $D_\lambda \subset H(k, l)$. In other words, $\lambda \in H(k, l)$ if $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\lambda_{k+1} \leq l$. According to this definition we shall say that the cocharacter of $L$ lies in the hook $H(k, l)$ if $m_\lambda = 0$ in (3) as soon as $\lambda \not\in H(k, l)$.

A special case of $H(k, l)$ is an infinite strip $H(k, 0)$. In this case $\lambda \in H(k, 0)$ if $\lambda_{k+1} = 0$.

The following fact is well-known, and we state it without proof.

**Lemma 1.** Let $L$ be a finite dimensional algebra, $\dim L = d < \infty$. Then $\chi_n(L)$ lies in the hook $H(d, 0)$ for all $n \geq 1$. \qed

Another important numerical invariant of the identities of $L$ is the colength $l_n(L)$. By definition

$$l_n(L) = \sum_{\lambda \vdash n} m_\lambda, \quad (5)$$

where $m_\lambda$ are taken from (3). It easily follows from (1) and (5) that

$$\max\{d_\lambda | m_\lambda \neq 0\} \leq c_n(L) \leq l_n(L) \cdot \max\{d_\lambda | m_\lambda \neq 0\}. \quad (6)$$
For studying graded identities of $L = L_0 \oplus L_1$ we need to act separately on even
and odd variables. More precisely, the space $P_{k,n-k} = P_{k,n-k}(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ is an $S_k \times S_{n-k}$-module where symmetric groups $S_k, S_{n-k}$ act on $x_1, \ldots, x_k$
and $y_1, \ldots, y_{n-k}$, respectively. Any irreducible $S_k \times S_{n-k}$-module is a tensor product
of an $S_k$-module and an $S_{n-k}$-module and corresponds to the pair $\lambda, \mu$ of partitions,
$\lambda \vdash k, \mu \vdash n - k$. As before, the subspace $P_{n-k} \cap \text{Id}(L)$ is an $S_k \times S_{n-k}$-stable
subspace, and one can consider the quotient
\[ P_{k,n-k}(L) = \frac{P_{k,n-k}}{P_{k,n-k} \cap \text{Id}(L)} \]
as an $S_k \times S_{n-k}$-module. Its $S_k \times S_{n-k}$-character $\chi_{k,n-k}(L) = \chi(P_{k,n-k}(L))$ is
decomposed into irreducible components,
\begin{equation}
\chi_{k,n-k}(L) = \sum_{\lambda \vdash k, \mu \vdash n - k} m_{\lambda,\mu} \chi_{\lambda,\mu},
\end{equation}
and we define the $(k, n - k)$-colength of $L$ as
\[ l_{k,n-k}(L) = \sum_{\lambda \vdash k, \mu \vdash n - k} m_{\lambda,\mu} \]
with $m_{\lambda,\mu}$ taken from (7).

First, we prove some relations between graded and non-graded numerical in-
variants. We begin by recalling the correspondence between multilinear homoge-
nous polynomials in a free $\mathbb{Z}_2$-graded Lie algebra and in a free Lie superalgebra.
Let $f = f(x_1, \ldots, x_k, y_1, \ldots, y_m)$ be a non-associative polynomial multilinear on
$x_1, \ldots, x_k$, $y_1, \ldots, y_m$, where $x_1, \ldots, x_k$ are supposed to be even indeterminates
and $y_1, \ldots, y_m$ are supposed to be odd. Then $f$ is a linear combination of mono-
mials from $P_{k,m}$. Let $M = M(x_1, \ldots, x_k, y_1, \ldots, y_m)$ be such a monomial. We fix
positions of $y_1, \ldots, y_m$ in $M$ and write $M$ for short in the following form:
\[ M = X_0 y_{\sigma(1)} X_1 \cdots X_{m-1} y_{\sigma(m)} X_m, \]
where $X_0, \ldots, X_m$ are some words (possibly empty) consisting of left and right
brackets and indeterminates $x_1, \ldots, x_k$. Now we define a monomial $\widetilde{M}$ on even
indeterminates $x_1, \ldots, x_k$ and odd indeterminates $y_1, \ldots, y_m$ from the free Lie
superalgebra as
\[ \widetilde{M} = \text{sgn}(\sigma) X_0 y_{\sigma(1)} X_1 \cdots X_{m-1} y_{\sigma(m)} X_m. \]
Extending this map $\sim$ by linearity, we obtain a linear isomorphism $P_{k,m} \rightarrow P_{k,m}$
of two subspaces of a $\mathbb{Z}_2$-graded free Lie algebra and a free Lie superalgebra,
respectively. Although the monomials in $P_{k,m}$ are not linearly independent, it easily
follows from the Jacobi and the super-Jacobi identities that the map $\sim$ is well-
defined. Similarly, we can define the inverse map from a free Lie superalgebra to a
free $\mathbb{Z}_2$-graded Lie algebra.

Following the same argument as in the associative case (see [10] Lemma 3.4.7),
we obtain the following result for any $\mathbb{Z}_2$-graded Lie algebra $L$ and its Grassmann
envelope $G(L) = G_0 \otimes L_0 \oplus G_1 \otimes L_1$.

**Lemma 2.** Let $f \in P_{k,m}$ be a multilinear polynomial in a free Lie algebra $L$. Then
\[ \bullet \ f \text{ is a graded identity of } L \text{ if and only if } \widetilde{f} \text{ is a graded identity of } G(L) \text{ and} \]
\[ \bullet \ \widetilde{f} = f. \]
The next lemma is an obvious generalization of Lemma 1.

**Lemma 3.** Let \( L = L_0 \oplus L_1 \) be a finite dimensional Lie algebra, \( \dim L_0 = k, \dim L_1 = l \), and let
\[
\chi_{q,n-q}(L) = \sum_{\lambda \vdash q} m_{\lambda,\mu} \chi_{\lambda,\mu}
\]
be its \((q,n-q)\)-graded cocharacter. If \( m_{\lambda,\mu} \neq 0 \), then \( \lambda \in H(k,0) \) and \( \mu \in H(l,0) \).

□

Using this remark we restrict the shape of the graded cocharacter of the Grassmann envelope \( G(L) \).

**Lemma 4.** Let \( L = L_0 \oplus L_1 \) be a finite dimensional Lie algebra, \( \dim L_0 = k, \dim L_1 = l \), and let \( \tilde{L} \) be its Grassmann envelope. If
\[
\chi_{q,n-q}(\tilde{L}) = \sum_{\lambda \vdash q} m_{\lambda,\mu} \chi_{\lambda,\mu}
\]
and \( m_{\lambda,\mu} \neq 0 \) in (8), then \( \lambda \in H(k,0) \) and \( \mu \in H(0,l) \).

**Proof.** Suppose \( m_{\lambda,\mu} \neq 0 \) in (8) for some \( \lambda \vdash q, \mu \vdash n - q \). Then there exists a multilinear polynomial \( g = g(x_1, \ldots, x_q, y_1, \ldots, y_{n-q}) \) such that
\[
f = e_{T_\lambda} e_{T_\mu} g(x_1, \ldots, y_{n-q})
\]
is not a graded identity of \( \tilde{L} \), where \( e_{T_\lambda} \in FS_q, e_{T_\mu} \in FS_{n-q} \) are essential idempotents generating minimal left ideals in \( FS_q, FS_{n-q} \), respectively. Inclusion \( \lambda \in H(k,0) \) immediately follows by Lemma 3 since \( L \) and \( G(L) \) have the same cocharacters on even indeterminates. Since \( e_{T_\lambda} \) and \( e_{T_\mu} \) commute, applying Lemma 4.8.6 from [10] we get
\[
\tilde{f} = ae_{T_\lambda} g,
\]
where \( a \in I_{\mu'} \). Here \( \mu' \) is the conjugated to \( \mu \) partition of \( n - q \) and \( I_{\mu'} \) is the minimal two-sided ideal of \( FS_{n-q} \) generated by \( e_{T_{\mu'}} \). That is, \( I_{\mu'} \) has the character \( r \cdot \chi_{\mu'} \), where \( r = d_{\mu'} = \deg \mu' \).

By Lemma 2, \( \tilde{f} \) is not a graded identity of \( G(L) \). Since \( \tilde{h} = h \) for any \( h \in P_{q,n-q} \), we see that \( \tilde{f} \) is not a graded identity of \( L \) and \( \mu' \in H(l,0) \) by Lemma 3. In other words, the number of rows of the Young diagram \( D_{\mu'} \) does not exceed \( l \). This number equals the number of columns of \( D_{\mu} \); hence \( \mu \in H(0,l) \), and we are done.

□

Using the previous lemma we restrict the shape of the non-graded cocharacter of \( G(L) \).

**Lemma 5.** Let \( L = L_0 \oplus L_1 \) be a finite dimensional Lie algebra, \( \dim L_0 = k, \dim L_1 = l \), and let
\[
\chi(\tilde{L}) = \sum_{\lambda \vdash n} m_{\lambda,\lambda} \chi_{\lambda}
\]
be the \( n \)-th (non-graded) cocharacter of \( L = G(L) \). Then \( m_\lambda \neq 0 \) only if \( \lambda \in H(k,l) \).
Lemma 6. Let $G(L) = \bar{L} = \bar{L}_0 \oplus \bar{L}_1$ be the Grassmann envelope of a finite dimensional Lie algebra $L = L_0 \oplus L_1$ with $\dim L_0 = k, \dim L_1 = l$. Then its colength sequence $\{l_n(\bar{L})\}$ is polynomially bounded.

Proof. We use the notation $\{z_1, z_2, \ldots\}$ for non-graded indeterminates here since $\{x_1, x_2, \ldots\}$ were even variables in the previous statements.

Let

$$\chi(\bar{L}) = \sum_{\lambda \vdash n} m_{\lambda} \chi_{\lambda}$$

be the $n$-th cocharacter of $\bar{L}$. By Lemma 5 we have $\lambda \in H(k,l)$ as soon as $m_{\lambda} \neq 0$ in (9). Fix $\lambda \vdash n$ with $m_{\lambda} = m \neq 0$ and consider the $FS_n$-submodule

$$W_1 \oplus \cdots \oplus W_m \subseteq P_n(\bar{L})$$

with $\chi(W_i) = \chi_{\lambda}$, for all $i = 1, \ldots, m$.

We shall prove that

$$m \leq (k + l)2^{2kl}n^{k^2 + l^2}$$

in (10). Denote by $\lambda'_1, \ldots, \lambda'_l$ the heights of the first $l$ columns of the Young diagram $D_{\lambda}$. Clearly, it suffices to prove inequality (11) only for $\lambda$ with $\lambda_k > l$ and $\lambda'_l > k$.

Otherwise, $\lambda \in H(k',l')$ with $k' \leq k, l' \leq l$ and $k' + l' < k + l$.

Denote

$$\mu_1 = \lambda'_1 - k, \ldots, \mu_l = \lambda'_l - k.$$

Then $\lambda_1 + \cdots + \lambda_k + \mu_1 + \cdots + \mu_l = n$.

It is well-known (see, for example, [29]) that one can choose multilinear $f_1 \in W_1, \ldots, f_m \in W_m$ such that $FS_n f_1 = W_1, \ldots, FS_n f_m = W_m$ and each $f_i, i = 1, \ldots, m$, is symmetric on $k$ sets of indeterminates of orders $\lambda_1, \ldots, \lambda_k$ and is alternating on $l$ sets of orders $\mu_1, \ldots, \mu_l$.

According to this decomposition into symmetric and alternating sets we rename $z_1, \ldots, z_n$ as follows:

$$(z_1, \ldots, z_n) = \{z_1^1, \ldots, z_1^{\lambda_1}, \ldots, z_k^1, \ldots, z_k^{\lambda_k}, z_{\lambda_k+1}^1, \ldots, z_{\lambda_k}^{\mu_1}, \ldots, z_l^1, \ldots, z_l^{\mu_l}\},$$

where each $f_i$ is symmetric on any set $\{z_{\lambda_j}^s, \ldots, z_{\lambda_j}^s\}, j = 1, \ldots, k$, and is alternating on any set $\{z_{\mu_j}^s, \ldots, z_{\mu_j}^s\}, s = 1, \ldots, l$. 

Proof. Suppose $f \in P_n$ is not an identity of $\bar{L}$. Since $f$ is multilinear we may assume that $f(x_1, \ldots, x_q, y_1, \ldots, y_{n-q}) \in P_{q,n-q}$ is not an identity of $\bar{L}$ for some $0 \leq q \leq n$. Moreover, we can consider only the case when a graded polynomial $f$ generates in $P_{q,n-q}$ an irreducible $S_q \times S_{n-q}$-submodule $M$ with the character $(\chi_\lambda, \chi_{\mu}), \lambda \vdash q, \mu \vdash n - q$.

Now we lift the $S_q \times S_{n-q}$-action up to an $S_n$-action and consider a decomposition of $FS_n M$ into irreducible components:

$$\chi(FS_n M) = \sum_{\nu \vdash n} m_{\nu} \chi_{\nu}.$$
We shall find \( \delta_1, \ldots, \delta_m \in F \) such that
\[
f = \delta_1 f_1 + \cdots + \delta_m f_m
\]
is an identity of \( \tilde{L} \) if \( \{ \} \) does not hold. Note that for any \( \delta_1, \ldots, \delta_m \in F \) a polynomial \( f \) is also symmetric on each subset \( \{ z_1^i, \ldots, z_k^i \}, 1 \leq i \leq k, \) and alternating on each subset \( \{ z_1^s, \ldots, z_k^s \}, s = 1, \ldots, l. \)

Let \( E = \{ e_1, \ldots, e_{k+l} \} \) be a homogeneous basis of \( L \) with \( E_0 = \{ e_1, \ldots, e_k \} \subset L_0, \)
\( E_1 = \{ e_{k+1}, \ldots, e_{k+l} \} \subset L_1. \) Then \( f \) is an identity of \( \tilde{L} \) if and only if \( \varphi(f) = 0 \) for any evaluation \( \varphi: Z \rightarrow \tilde{L} \) such that \( \varphi(z_i) = g_i \otimes a_i, 1 \leq i \leq n, \) where \( a_i \) is a basis element from \( E \) and \( g_i \in G \) has the same parity as \( a_i \) and \( g_1 \cdots g_n \neq 0 \) in \( G. \)

Note also that \( \varphi(z_i) = g_i \otimes a_i, 1 \leq i \leq n, \) provided that \( g_1 \cdots g_n \neq 0. \)

Using these two remarks we shall find an upper bound for the number of evaluations for asking the question whether \( f \) is an identity of \( \tilde{L} \) or not.

First consider one symmetric subset \( Z_1 = \{ z_1^1, \ldots, z_k^1 \}. \) If \( \varphi(z_i^1) = g \otimes e, \varphi(z_j^1) = h \otimes e, \) for some \( i \neq j \) with \( e \in E_1, \) then \( \varphi(f) = 0, \) as follows from the symmetry on \( Z_1. \) Hence we need to check only evaluations with at most \( l \) odd values \( \varphi(z_i^1) = g_1 \otimes e_{t_1}, \ldots, \varphi(z_i^k) = g_l \otimes e_{t_l}, \) where \( e_{t_1}, \ldots, e_{t_l} \in E_1 \) are distinct. Since \( Z_1 \) is the symmetric set of variables, the result of evaluation \( \varphi \) does not depend (up to the sign) on the choice of \( i_1, \ldots, i_r. \) Hence we have \( \binom{l}{r} \) possibilities.

Given \( 0 \leq r \leq l, \) we estimate the number of evaluations of remaining \( \lambda_1 - r \) variables in the even component of \( \tilde{L}. \) First, let \( r = 0 \) and \( \varphi(z_i^1) = g_i \otimes a_i, a_i \in E_0, 1 \leq i \leq \lambda_1. \) If \( e_1 \) appears in the row \( (a_1, \ldots, a_{\lambda_1}) \) exactly \( \alpha_1 \) times, \( e_2 \) appears \( \alpha_2 \) times, and so on, then the result of such substitution depends only on \( \alpha_1, \ldots, \alpha_k \) since \( f \) is symmetric on \( Z_1. \) Hence we have no more than \( (\lambda_1 + 1)^k \) variants since \( 0 \leq \alpha_1, \ldots, \alpha_k \leq \lambda_1. \) In particular, we need at most \( (n + 1)^k \) evaluations if \( r = 0. \)

Now let \( r = 1. \) We can replace by an odd element an arbitrary variable from \( Z_1 \) and get (up to the sign) the same value \( \varphi(f) \) since \( f \) is symmetric on \( Z_1. \) Suppose, say, that \( \varphi(z_i^1) = h \otimes e, e \in E_1, \) and \( \varphi(z_i^{1}) = g_1 \otimes a_1, \ldots, \varphi(z_i^{1}) = g_{\lambda_1 - 1} \otimes a_{\lambda_1 - 1}, \) where all \( a_j \) are even. If \( \alpha_1, \ldots, \alpha_k \) are the same integers as in the case \( r = 0, \) then the result of the substitution also depends only on \( \alpha_1, \ldots, \alpha_k. \) Hence for \( r = 1 \) we have at most
\[
\binom{l}{1} \lambda_i^1 \leq \binom{l}{1} (n + 1)^k
\]
variants for \( \varphi \) since \( 0 \leq \alpha_1, \ldots, \alpha_k \leq \lambda_1 - 1. \)

Similarly, for general \( 0 \leq r \leq l \) we have at most
\[
\binom{l}{r} (\lambda_1 + 1 - r)^k \leq \binom{l}{r} (n + 1)^k
\]
variants. Therefore, for evaluating all variables from \( Z_1 \) it suffices that
\[
\sum_{r=0}^{l} \binom{l}{r} (n + 1)^k = 2^l(n + 1)^k
\]
substitutions and for all symmetric variables we need at most
\[
(2^l(n + 1)^k)^k
\]
substitutions.

Now consider the alternating set \( Z_1' = \{ z_1^1, \ldots, z_{\mu_1}^1 \}. \) If \( \varphi(z_i^1) = g \otimes e, \varphi(z_j^1) = h \otimes e, \) for some \( i \neq j \) with the same \( e \in E_0, \) then \( \varphi(f) = 0. \) Hence we can choose
only $0 \leq r \leq k$ distinct basis elements $b_1, \ldots, b_r \in E_0$ for values of $\bar{z}_1^{i_1}, \ldots, \bar{z}_r^{i_r}$ of the type $g_i \otimes b_i$. Up to the sign, the result of the substitution does not depend on $i_1, \ldots, i_r$, and we have only $\binom{k}{r}$ options.

Suppose now that all $\varphi(\bar{z}_i^1), 1 \leq i \leq r$, are fixed even values. Let

$$\varphi(\bar{z}_r^{i+1}) = g_1 \otimes b_1, \ldots, \varphi(\bar{z})_{\mu_1} = g_{\mu_1-r} \otimes b_{\mu_1-r}, \ b_1, \ldots, b_{\mu_1-r} \in E_1.$$  

Then (up to the sign) the result of $\varphi$ depends only on the number of entries of $e_{k+1}, \ldots, e_{k+l}$ into the row $(b_1, \ldots, b_{\mu_1-r})$. Hence we have at most $(\mu_1 - r + 1)^l$ variants for the substitution of odd variables. As in the symmetric case we have the following upper bound:

$$\sum_{r=0}^{k} \binom{k}{r} \binom{n}{l} = 2^k(n + 1)^l$$

for one subset and $(2^k(n + 1)^l)^l$ for all skew variables.

We have proved that one can find $T \leq 2^{kl}(n + 1)^{2+k^2}$ evaluations $\varphi_1, \ldots, \varphi_T$ such that the relations

$$\varphi_1(f) = \ldots = \varphi_T(f) = 0$$

imply $\varphi(f) = 0$ for any evaluation $\varphi$; that is, $f$ is an identity of $\bar{L}$. Recall that $f = \delta_1 f_1 + \cdots + \delta_m f_m$. Therefore for any evaluation $\varphi$ the equality $\varphi(f) = 0$ can be viewed as a system of $k + l$ homogeneous linear equations in the algebra $\bar{L}$ on unknown coefficients $\delta_1, \ldots, \delta_m$. If (11) does not hold, then the system (13) has a non-trivial solution $\bar{\delta}_1, \ldots, \bar{\delta}_m$, and $f = \bar{\delta}_1 f_1 + \cdots + \bar{\delta}_m f_m$ is an identity of $\bar{L}$, a contradiction.

We have proved the inequality (11). From this inequality it follows that all multiplicities in (9) are bounded by $(k + l)^{2^{kl}}$. Finally, note that the number of partitions $\lambda \in H(k, l)$ is bounded by $n^{k+l}$. Hence

$$l_n(\bar{L}) < (k + l)^{2^{kl}} n^{k^2 + l^2 + kl},$$

and we have thus completed the proof. \hfill \qed

As a corollary of previous results we obtain the following:

**Proposition 1.** Let $L = L_0 \oplus L_1$ be a finite dimensional $\mathbb{Z}_2$-graded Lie algebra with $\dim L_0 = k, \dim L_1 = l$ and let $\bar{L} = G(L)$ be its Grassmann envelope. Then there exist constants $\alpha, \beta \in \mathbb{R}$ such that

$$c_n(\bar{L}) \leq \alpha n^\beta (k + l)^n.$$  

In particular,

$$\exp(\bar{L}) = \limsup_{n \to \infty} n^{\sqrt[n]{c_n(\bar{L})}} \leq k + l.$$

**Proof.** By [10] Lemma 6.2.5], there exist constants $C$ and $r$ such that

$$\sum_{\lambda \in H(k, l)} d_\lambda \leq C n^r (k + l)^n$$

for all $n = 1, 2, \ldots$. In particular,

$$\max\{d_\lambda | \lambda \vdash n, \lambda \in H(k, l)\} \leq C n^r (k + l)^n.$$  

Now Lemma 3 and the inequality (6) complete the proof. \hfill \qed
4. Existence of PI-exponents

**Proposition 2.** Let $L$ be a finite dimensional simple Lie algebra over an algebraically closed field of characteristic zero with some $\mathbb{Z}_2$-grading, $L = L_0 \oplus L_1$, \(\dim L_0 = k, \dim L_1 = l\). Also let $\tilde{L} = G(L)$ be its Grassmann envelope. Then there exist constants $\gamma > 0, \delta \in \mathbb{R}$ such that

\[ c_n(\tilde{L}) \geq \gamma n^\delta (k + l)^n. \]

In particular,

\[ \exp(\tilde{L}) = \liminf_{n \to \infty} \sqrt[n]{c_n(\tilde{L})} \geq k + l. \]

**Proof.** Denote $d = k + l = \dim L$. By \cite{19} Theorem 12.1, for the adjoint representation of $L$ there exists a multilinear assosciative polynomial $h = h(u_1^1, \ldots, u_d^1, \ldots, u_m^1, \ldots, u_d^m)$ alternating on each subset of indeterminates $\{u_1^i, \ldots, u_d^i\}, 1 \leq i \leq m$, such that under any evaluation $\varphi : u_j^i \to \text{ad} b_j^i, b_j^i \in L$, the value $\varphi(h)$ is a scalar linear transformation of $L$ and $\varphi(h) \neq 0$ for some $h$. It follows that for any integer $t \geq 1$ there exists a multilinear Lie polynomial

\[ f_t = f_t(u_1^1, \ldots, u_d^1, \ldots, u_m^1, \ldots, u_d^m, w) \]

alternating on each set $\{u_1^i, \ldots, u_d^i\}, 1 \leq i \leq mt$, such that $\varphi(f_t) \neq 0$ for some evaluation $\varphi : \{u_1^1, \ldots, u_m^1, w\} \to L_0 \cup L_1$. Since $f_t$ is multilinear and alternating on each set $\{u_1^i, \ldots, u_d^i\}$ and $d = \dim L_0 + \dim L_1$, it follows that for any $t \geq 1$ we get a graded multilinear polynomial

\[ f_t = f_t(x_1^1, \ldots, x_k^1, \ldots, x_1^{mt}, \ldots, x_k^{mt}, y_1^1, \ldots, y_1^{mt}, \ldots, y_l^{mt}, w) \]

which is not a graded identity of $L$ and is alternating on each subset $\{x_1^i, \ldots, x_k^i\}$ and on each subset $\{y_1^i, \ldots, y_l^i\}, 1 \leq i \leq mt$, where $x_j^i$’s are even and $y_j^i$’s are odd variables. The latter indeterminate $w$ can be taken of arbitrary parity; say, $w = x_0$ is even.

Consider an $S_p \times S_q$-action on

\[ P_{p+1,q} = P_{p+1,q}(x_0, x_1^1, \ldots, x_k^1, \ldots, x_1^{mt}, y_1^1, \ldots, y_l^{mt}), \]

where $p = mtk, q = mtl$ and $S_p, S_q$ act on $\{x_j^i\}, \{y_j^i\}$, respectively. It follows from Lemma\cite{3} that the $S_p \times S_q$-character of the submodule generated by $f$ in $P_{p+1,q}$ lies in the pair of strips $H(k, 0), H(l, 0)$, that is,

\[ \chi(F[S_p \times F_q]f) = \sum_{\lambda+p+q} m_{\lambda,\mu} \chi_{\lambda,\mu} \]

with $m_{\lambda,\mu} = 0$, unless $\lambda \in H(k, 0), \mu \in H(l, 0)$. Hence $\lambda$ is a partition of $mtk$ with at most $k$ rows. On the other hand, $f$ depends on $mt$ alternating subsets of even indeterminates of order $k$ each. It is well-known that in this case $m_{\lambda,\mu} = 0$ if $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda_1 \geq mt + 1$. It follows that only the rectangular partition

\[ \lambda = \underbrace{mt, \ldots, mt}_k \]

(14)

can appear in $F[S_p \times F_q]f$ with non-zero multiplicity. Similarly,

\[ \mu = \underbrace{mt, \ldots, mt}_l \]

(15)
if \( m_{\lambda,\mu} \neq 0 \). Hence we can assume that \( f \) has the form

\[
f = e_{T_x} e_{T_\mu} g(x_1^1, \ldots, y_l^{mT}, w)
\]

with \( \lambda \) and \( \mu \) of the types \([14], [15]\), respectively.

By Lemma \([2]\) the polynomial \( \tilde{f} \) is not an identity of the Lie superalgebra \( \tilde{L} = G(L) \), and by Lemma 4.8.6 from \([10]\), the graded polynomial \( \tilde{f} \) generates in \( P_{p+1,q}(\tilde{L}) \) an irreducible \( S_p \times S_q \) submodule with the character \((\chi_\lambda, \chi_\mu')\), where

\[
\mu' = (l, \ldots, l)_{mt}
\]

is conjugated to a \( \mu \) partition of \( mtl \).

First we apply the Littlewood-Richardson rule and induce this \( S_p \times S_q \) module up to an \( S_n \) module. Then we induce the obtained \( S_n \) module up to an \( S_{n+1} \) module, where \( n = p + q = mt(k + l) \). It follows from the Littlewood-Richardson rule that the induced \( S_{n+1} \) module can contain only a simple submodule corresponding to partitions \( \nu \vdash n + 1 \) such that the Young diagram \( D_\nu \) contains a subdiagram \( D_{\nu_0} \), where

\[
\nu_0 = h(k, l, t_0) = (l + t_0, \ldots, l + t_0, l, \ldots, l)_{k-t_0}
\]

is a finite hook with \( t_0 \geq l - k, mt - kl \). Since we are interested in an asymptotic of codimensions, we may assume that \( mt - kl > l - k \) and then \( t_0 = mt - kl \). In particular, \( \nu_0 \) is a partition of \( n_0 = (k + l)t_0 + kl \). Then \( n + 1 - n_0 = (k + l - 1)k + l + 1 \), and by \([10]\) Lemma 6.2.4

\[
d_{\nu_0} \leq d_\nu \leq nc_{\nu_0},
\]

where \( c = (k + l - 1)k + 1 \) and

\[
d_{h(k,l,t_0)} \simeq an_{n_0}^{b} (k + l)^{n_0} \quad \text{if} \ n_0 \to \infty
\]

for some constants \( a, b \) by Lemma 6.2.5 from \([10]\). Here the relation \( f(n) \simeq g(n) \) means that \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \). Since \( c_{n+1}(\tilde{L}) \geq d_\nu \) we get the inequality

\[
c_{n+1}(\tilde{L}) \geq \alpha (n + 1)^{\beta} (k + l)^{n+1}
\]

for all \( n = m(k + l)t \), \( t = 1, 2, \ldots \), for some constants \( \alpha > 0 \) and \( \beta \).

Since the Lie algebra \( L \) is simple, the Grassmann envelope \( \tilde{L} \) is a centerless Lie superalgebra. It is not difficult to see that in this case \( c_{r+1}(\tilde{L}) \geq c_r(\tilde{L}) \), for all \( r \geq 1 \). Hence by \([16]\) we have

\[
c_{n+j}(\tilde{L}) \geq \alpha (n + 1)^{\beta} (k + l)^{n+1}
\]

for any \( 1 \leq j \leq m(k + l) \). Since \( n = m(k + l)t \) one can find constants \( \gamma > 0 \) and \( \delta \) such that

\[
c_r(\tilde{L}) \geq \gamma r^{\delta} (k + l)^r
\]

for all positive integers \( r \), and we have completed the proof. \( \square \)

Theorem \([1]\) now easily follows from Propositions \([1, 2]\).
Proof of Theorem 2. First we obtain an upper bound for \(c_{\text{gr}}^r(\tilde{L})\):

\[
c_{\text{gr}}^r(\tilde{L}) = \sum_{q=0}^{n} \binom{n}{q} c_{q,n-q}(\tilde{L}),
\]

where

\[
c_{q,n-q}(\tilde{L}) = \sum_{\lambda \vdash q, \mu \vdash n-q} m_{\lambda,\mu} d_{\lambda,\mu}
\]

and \(d_{\lambda,\mu} = \deg \chi_{\lambda,\mu} = \deg \chi_{\lambda} \cdot \deg \chi_{\mu} = d_{\lambda} d_{\mu}\). Moreover, \(\lambda \in H(k,0), \mu \in H(0,l)\) by Lemma 4. Applying Lemma 6.2.5 from [10], we obtain

\[
\sum_{\lambda \in H(k,0), \lambda \vdash q} d_{\lambda} \leq C n^r k^q, \quad \sum_{\mu \in H(0,l), \mu \vdash n-q} d_{\mu} \leq C n^r l^{n-q}
\]

for some constants \(C, r\), and hence

\[
\sum_{\lambda \in H(k,0), \lambda \vdash q, \mu \in H(0,l), \mu \vdash n-q} d_{\lambda} d_{\mu} \leq C^2 n^{2r} k^q l^{n-q}.
\]

On the other hand, the graded colength

\[
l_{q,n-q}(\tilde{L}) = \sum_{\lambda \vdash q, \mu \vdash n-q} m_{\lambda,\mu}
\]

is not greater than the non-graded colength \(l_{n}(\tilde{L})\). Since \(l_{n}(\tilde{L})\) is polynomially bounded by Lemma 6 one can find a polynomial \(\varphi(n)\) such that

\[
m_{\lambda,\mu} \leq \varphi(n)
\]

for any \(m_{\lambda,\mu}\) in (17). It now follows from (17), (18) and (19) that for \(\psi(n) = C^2 n^{2r} \varphi(n)\) we have

\[
c_{\text{gr}}^r(\tilde{L}) \leq \psi(n) \sum_{q=1}^{n} \binom{n}{q} k^q l^{n-q} = \psi(n)(k+l)^n,
\]

and we have obtained an upper bound for \(c_{\text{gr}}^r(\tilde{L})\).

On the other hand, it was proved in [2, Lemma 3.1] that for any associative \(G\)-graded algebra \(A\), where \(G\) is a finite group, an ordinary \(n\)-th codimension is less than or equal to the graded \(n\)-th codimension, for any \(n\). Proof of this lemma does not use associativity. Hence

\[
c_{\text{gr}}^r(\tilde{L}) \geq c_{n}(\tilde{L}),
\]

and Theorem 2 now follows from (20), (21) and Proposition 2. \(\Box\)

References


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