INVERTIBLE WEIGHTED SHIFT OPERATORS WHICH ARE $m$-ISOMETRIES

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Abstract. For a bounded linear operator $T$ on a complex Hilbert space $H$, let

$$\Delta_{T,m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{m-k}T^{m-k}$$

for $m \in \mathbb{N}$. $T$ is called an $m$-isometry if $\Delta_{T,m} = 0$. In this paper, for every even number $m$, we give an example of invertible $(m+1)$-isometry which is not an $m$-isometry. Next we show that if $T$ is an $m$-isometry, then the operator $\Delta_{T,m-1}$ is not invertible.

1. Introduction

J. Agler and M. Stankus published excellent papers about $m$-isometric operators, [1], [2] and [3]. They showed that $m$-isometries have interesting spectral properties. For example, if $T$ is an $m$-isometry, then the approximate point spectrum of $T$ lies on the unit circle. In [10], Patel showed that Weyl’s theorem holds for a 2-isometry. Applying Uchiyama and Tanahashi’s result [11], in [8] we showed that if $T$ is an $m$-isometry, then $T$ has the single valued extension property. Let $H$ be a complex Hilbert space and $B(H)$ be a set of all bounded linear operators on $H$. Let $\binom{m}{k}$ be the binomial coefficient. For an operator $T \in B(H)$, let

$$\Delta_{T,m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{m-k}T^{m-k}.$$ 

$T$ is said to be an $m$-isometry if $\Delta_{T,m} = 0$. Agler and Stankus proved that if $T$ is an $m$-isometry, then $\Delta_{T,m-1} \geq 0$ (Proposition 1.5, [1]). It is easy to see that $T^\ast \Delta_{T,m}T - \Delta_{T,m} = \Delta_{T,m+1}$. Hence it holds that if $T$ is an $m$-isometry, then $T$ is an $(m+1)$-isometry. In [7], T. Bermudez, A. Martinon and E. Negrin studied characterizations of weighted shift operators which are $m$-isometries. In [1], Agler and Stankus proved that if $m$ is even and $T$ is an invertible $m$-isometry, then $T$ is an $(m-1)$-isometry. In [4] A. Athavale proved that if $S$ is a unilateral weighted shift
Se\(_n = w_ne_{n+1}\) with weights \(w_n = \sqrt{1 + \frac{m}{n}}\) for \(n = 1, 2, 3, \cdots\), then \(S\) is an \((m+1)\)-isometry which is not an \(m\)-isometry. The operator \(S\) is not invertible. We have not seen invertible \((m+1)\)-isometries which are weighted shifts for \(m\) even. We give an example of an invertible \((m+1)\)-isometry which is not an \(m\)-isometry for every even number \(m\). Next we prove that power bounded \(m\)-isometries are isometries, and if \(T\) is an \(m\)-isometry for a natural number \(m \geq 2\), then the operator \(\Delta_{T,m-1}\) is not invertible.

2. Invertible weighted shift of an \(m\)-isometry

Agler and Stankus showed the following result.

**Proposition A** (Proposition 1.23, [1]). *If \(T\) is an invertible \(m\)-isometry and \(m\) is even, then \(T\) is an \((m-1)\)-isometry.*

Their proof is fine. We give another proof.

**Proof.** Since \(T\) is an \(m\)-isometry, it holds that

\[
\Delta_{T,m-1} = \sum_{k=0}^{m-1} (-1)^k \begin{pmatrix} m-1 \\ k \end{pmatrix} T^{m-1-k} T^{m-1-k} \geq 0.
\]

It is easy to see that \(T^{-1}\) also is an \(m\)-isometry. Since \(T^{-1}\) is an \(m\)-isometry, it holds that

\[
\Delta_{T^{-1},m-1} = \sum_{k=0}^{m-1} (-1)^k \begin{pmatrix} m-1 \\ k \end{pmatrix} (T^{-1})^{m-1-k} (T^{-1})^{m-1-k} \geq 0.
\]

Since \(m-1\) is an odd number, by (2.2) we have

\[
-\Delta_{T,m-1} = \sum_{k=0}^{m-1} (-1)^k \begin{pmatrix} m-1 \\ k \end{pmatrix} T^k T^k
\]
\[
= \sum_{k=0}^{m-1} (-1)^k \begin{pmatrix} m-1 \\ k \end{pmatrix} T^{m-1} (T^{-1})^{m-1-k} T^{m-1-k} T^{m-1}
\]
\[
= T^{m-1} (\Delta_{T^{-1},m-1}) T^{m-1} \geq 0.
\]

Hence \(\Delta_{T,m-1} \leq 0\). By (2.1) we have \(\Delta_{T,m-1} = 0\), so the proof is complete. \(\square\)

For every even number \(m\), we give an example of an invertible \((m+1)\)-isometry \(T\) which is not an \(m\)-isometry.

**Theorem 1.** *For any even number \(m\), there exists an invertible \((m+1)\)-isometry \(T\) which is not an \(m\)-isometry.*

**Proof.** Let \(m\) be any even number and put \(\psi(x) = x(x+1) \cdots (x+m-1)\). Then \(\psi(x)\) is a polynomial of even degree \(m\). Hence \(\psi(n) \geq 0\) for \(n = 0, \pm 1, \pm 2, \cdots\). So, for any \(k > 0\), we have \(\psi(n) + k > 0\) for \(n = 0, \pm 1, \pm 2, \cdots\). Let

\[
w_n = \sqrt{\frac{\psi(n+1) + k}{\psi(n) + k}} > 0 \quad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots.
\]
Let \( \{e_n\}_{n=-\infty}^{\infty} \) be an orthonormal basis of \( \mathcal{H} = l^2 \) and \( T \) be a bilateral weighted shift such that \( T e_n = w_n e_{n+1} \). We define

\[
I_{m,n} = w_n^2 w_{n+1}^2 \cdots w_{n+m-1}^2 - \binom{m}{1} w_n^2 w_{n+1}^2 \cdots w_{n+m-2}^2 + \binom{m}{2} w_n^2 w_{n+1}^2 \cdots w_{n+m-3}^2 + \cdots + (-1)^{m-1} \binom{m}{m-1} w_n^2 + (-1)^m.
\]

Note that \( T \) is an \( m \)-isometry if and only if \( I_{m,n} = 0 \) for \( n = 0, \pm 1, \pm 2, \cdots \). To show \( T \) is an \((m+1)\)-isometry, we will show that \( I_{m+1,n} = 0 \) for \( n = 0, \pm 1, \pm 2, \cdots \).

We define a function \( f(x) \) as

\[
f(x) = x^{n+m-1}(1-x)^{m+1}
\]

\[
= x^{n+m-1} - \binom{m+1}{1} x^{n+m} + \binom{m+1}{2} x^{n+m+1}
\]

\[
+ \cdots + (-1)^m \binom{m+1}{m} x^{n+2m-1} + (-1)^m x^{n+2m}.
\]

Note that \( f^{(m)}(1) = 0 \). By differentiating \( m \) times, we have

\[
0 = f^{(m)}(1)
\]

\[
= (n + m - 1)(n + m - 2) \cdots n
\]

\[
- \binom{m+1}{1} (n + m)(n + m - 1) \cdots (n + 1)
\]

\[
+ \cdots + (-1)^m \binom{m+1}{m} (n + 2m - 1)(n + 2m - 2)(n + m) + (-1)^{m+1}(n + 2m)(n + 2m - 1) \cdots (n + m + 1).
\]

Hence

\[
(-1)^{m+1} I_{m+1,n} (\psi(n) + k)
\]

\[
= (n + m + 1)(n + m + 2) \cdots (n + 2m) + k
\]

\[
- \binom{m+1}{1} \{(n + m)(n + m + 1) \cdots (n + 2m - 1) + k\}
\]

\[
+ \cdots + (-1)^m \binom{m+1}{m} \{(n + 1)(n + 2) \cdots (n + m) + k\}
\]

\[
+ (-1)^{m+1} \{n(n + 1)(n + 2) \cdots (n + m - 1) + k\}
\]

\[
= f^{(m)}(1) + k(1 - 1)^{m+1} = 0.
\]

This implies that \( I_{m+1,n} = 0 \). Hence \( T \) is an \((m+1)\)-isometry.

To show that \( T \) is not an \( m \)-isometry, we will show that \( I_{m,n} \neq 0 \) for \( n = 0, \pm 1, \pm 2, \cdots \). We define a function \( g(x) \) as

\[
g(x) = x^{n+m-1}(1-x)^m
\]

\[
= x^{n+m-1} - \binom{m}{1} x^{n+m} + \binom{m}{2} x^{n+m+1}
\]

\[
+ \cdots + (-1)^{m-1} \binom{m}{m-1} x^{n+2m-2} + (-1)^m x^{n+2m-1}.
\]
Note that $g^{(m)}(1) = m!(-1)^m$. By differentiating $m$ times, we have
\begin{align*}
m!(-1)^m &= g^{(m)}(1) \\
&= (n + m - 1)(n + m - 2) \cdots n \\
&\quad - \binom{m}{1}(n + m)(n + m - 1) \cdots (n + 1) \\
&\quad + \cdots + (-1)^{m-1} \binom{m}{m-1}(n + 2m - 2)(n + 2m - 3) \cdots (n + m - 1) \\
&\quad + (-1)^m(n + 2m - 1)(n + 2m - 2) \cdots (n + m).
\end{align*}

Hence
\begin{align*}
(-1)^m I_{m,n} (\psi(n) + k) \\
= (n + m)(n + m + 1) \cdots (n + 2m - 1) + k \\
&\quad - \binom{m}{1}\{(n + m - 1)(n + m) \cdots (n + 2m - 2) + k\} \\
&\quad + \cdots + (-1)^{m-1} \binom{m}{m-1}\{(n + 1)(n + 2) \cdots (n + m) + k\} \\
&\quad + (-1)^m\{n(n + 1)(n + 2) \cdots (n + m - 1) + k\} \\
&= g^{(m)}(1) + k(1 - 1)^m = m!(-1)^m.
\end{align*}

This implies that $I_{m,n} = m!(\psi(n) + k)^{-1} \neq 0$. Hence $T$ is not an $m$-isometry.

Finally, since $w_n \to 1 (n \to \pm \infty)$, we have $\sigma(T) = \{\lambda : |\lambda| = 1\}$ by Proposition 2.6.8 (b) of [9]. Therefore $T$ is invertible. The proof is complete. \hfill \Box

Remark 1. This result provides an example of an $(m+1)$-isometry $T$ which is not an $m$-isometry if we take $Te_n = w_ne_{n+1}$ for $n = 1, 2, \cdots$ where
\begin{equation*}
w_n = \sqrt{\frac{(n + 1)(n + 2) \cdots (n + m) + k}{n(n + 1) \cdots (n + m - 1) + k}}
\end{equation*}
with $0 \leq k$. If $k = 0$, then
\begin{equation*}
w_n = \sqrt{\frac{(n + 1)(n + 2) \cdots (n + m) + 0}{n(n + 1) \cdots (n + m - 1) + 0}} = \sqrt{\frac{n + m}{n}}.
\end{equation*}

This is the result of A. Athavale, Proposition 8 of [4].

3. Some Properties of $m$-Isometries

An operator $T \in B(\mathcal{H})$ is said to be power bounded if there exists a positive number $M$ such that $\|T^n\| \leq M$ for every $n \in \mathbb{N}$.

Theorem 2. A power bounded $m$-isometry is an isometry.

To prove this result, we will use Berberian’s method (cf. [6], [12]).
**Proposition** (Lemma 2.7, [12]). Let $\mathcal{H}$ be a complex Hilbert space. Then there exist a Hilbert space $\mathcal{H}' \supset \mathcal{H}$ and a unital linear map $\circ : B(\mathcal{H}) \to B(\mathcal{H}')$ such that

(i) $(ST)^* = S^*T^*$, $(T^n)^* = (T^n)^*$, $\|T\| = \|T^*\|$,

(ii) $S \leq T \implies S^o \leq T^o$,

(iii) $\sigma(T) = \sigma(T^o), \sigma_a(T) = \sigma_a(T^o) = \sigma_p(T^o)$.

We prepare notation and a lemma. For a unit vector $x \in \mathcal{H}$, assume that $(T^*T - I)x = a_1x$. Let $a_n = (\Delta_{T,n}x, x)$ and $b_n = \|T^n x\|^2$ for $n = 1, 2, 3, \ldots$. Then we have the following lemma.

**Lemma.** For an operator $T \in B(\mathcal{H})$ with the above notation, it holds that

\begin{equation}
(3.1) \quad b_n = a_n + \left( \frac{n}{n - 1} \right) a_{n-1} + \left( \frac{n}{n - 2} \right) a_{n-2} + \cdots + na_1 + 1.
\end{equation}

**Proof.** Note that

$$b_1 = \|Tx\|^2 = \langle T^*Tx, x \rangle = \langle a_1x + x, x \rangle = a_1 + 1.$$  

Since

$$T^*T^2 = (T^*T^2 - 2T^*T + I) + 2(T^*T - I) + I = \Delta_{T,2} + 2\Delta_{T,1} + I,$$

we have

$$b_2 = \langle T^*T^2x, x \rangle = \langle \Delta_{T,2}x, x \rangle + 2\langle \Delta_{T,1}x, x \rangle + \langle x, x \rangle = a_2 + 2a_1 + 1.$$  

Hence (3.1) holds for $n = 1, 2$. Note that

$$t^n = (t - 1 + 1)^n = (t - 1)^n + \left( \frac{n}{1} \right) (t - 1)^{n-1} + \left( \frac{n}{2} \right) (t - 1)^{n-2} + \cdots + n(t - 1) + 1.$$  

Hence

$$T^{*n}T^n = \Delta_{T,n} + \left( \frac{n}{1} \right) \Delta_{T,n-1} + \left( \frac{n}{2} \right) \Delta_{T,n-2} + \cdots + n\Delta_{T,n-1} + I$$

and

$$b_n = \langle T^{*n}T^n x, x \rangle = a_n + \left( \frac{n}{1} \right) a_{n-1} + \left( \frac{n}{2} \right) a_{n-2} + \cdots + na_1 + 1 = a_n + \left( \frac{n}{n - 1} \right) a_{n-1} + \left( \frac{n}{n - 2} \right) a_{n-2} + \cdots + na_1 + 1.$$  

This completes the proof. \(\square\)

**Proof of Theorem.** Let $T$ be a power bounded $m$-isometry. Since $T^*T - I$ is self-adjoint, it suffices to show that

$$\sigma(T^*T - I) = \sigma_a(T^*T - I) = \{0\}.$$  

Assume that there exists a non-zero real number $a \in \sigma(T^*T - I)$. Since $a$ belongs to the approximate point spectrum of $T^*T - I$, by Berberian’s method we consider
an extension $\mathcal{H}^o$ of $\mathcal{H}$ and the mapping $S \to S^o$ of $B(\mathcal{H})$ into $B(\mathcal{H}^o)$. Then $a$ is an eigen-value of $T^*T^o - I^o$ and $T^o$ is also a power bounded $m$-isometry. For simplification, we denote $T^o$ by $T$. Since $a$ is an eigen-value of $T^*T - I$, there exists a unit vector $x$ such that $(T^*T - I)x = ax$. Hence

$$\langle (T^*T - I)x, x \rangle = \langle ax, x \rangle = a.$$ Let $a_1 = a$ and $a_n = \langle \Delta_{T,n}, x, x \rangle$. Then since $T$ is an $m$-isometry, we have $a_m = a_{m+1} = \cdots = a_n = 0$ for $n > m$. By the Lemma, we have

$$\|T^nx\|^2 = \left(\frac{n}{m-1}\right)a_{m-1} + \left(\frac{n}{m-2}\right)a_{m-2} + \cdots + na_1 + 1.$$ Hence

$$\|T^nx\|^2 = a_{m-1} + \left(\frac{1}{n}\right)\left\{\left(\frac{n}{m-2}\right)a_{m-2} + \cdots + na_1 + 1\right\}.$$ Since $T$ is power bounded, we have $a_{m-1} = 0$ by $n \to \infty$. Repeating this, we have

$$a_{m-1} = a_{m-2} = \cdots = a_1 = a = 0.$$ This is a contradiction, so the proof is complete. $\square$

Patel (Corollary 2.8, [10]) proved that if $T$ is a 2-isometry, then $1 \in \sigma(T^*T)$. That is, if $T$ is a 2-isometry, then $0 \in \sigma(\Delta_{T,1})$. We now generalize this result as follows.

**Theorem 3.** If $T$ is an $m$-isometry for $m \geq 2$, then $0 \in \sigma(\Delta_{T,m-1})$.

**Proof.** For simplification, we denote $\Delta_{T,m-1}$ by $\Delta$. Assume that $0 \notin \sigma(\Delta)$. Since $\Delta_{T,m} = T^*\Delta T - \Delta = 0$, it holds that $T^*\Delta T = \Delta$. Let $S = \Delta^{1/2}T\Delta^{-1/2}$. Then

$$S^*S = (\Delta^{-1/2}T^*\Delta^{1/2})(\Delta^{1/2}T\Delta^{-1/2}) = I.$$ Hence $S$ is an isometry and $T = \Delta^{-1/2}S\Delta^{1/2}$. So $T$ is similar to the isometry $S$. Since $S$ is power bounded, $T$ is an isometry by Theorem 3. Hence $\Delta_{T,1} = 0$, and so $\Delta_{T,m-1} = \Delta = 0$. This contradicts our assumption that $0 \notin \sigma(\Delta)$, so the proof is complete. $\square$

**Remark 2.** Since Weyl’s theorem holds for an isometry, Weyl’s theorem holds for a power bounded $m$-isometry by Theorem 3 (cf. [5], [10]).

Finally we show the following result. For a 2-isometry, Patel proved it (Theorem 2.1, [10]).

**Theorem 4.** A power of an $m$-isometry is again an $m$-isometry.

**Proof.** Let $T$ be an $m$-isometry, i.e.,

$$\Delta_{T,m} = T^*m^mTm - mT^*m^{-1}Tm^{-1} + \cdots + (-1)^m I = 0.$$ We prove $T^k$ is also an $m$-isometry. Write

$$(t^k - 1)^m = (t - 1)^m(t^{k-1} + t^{k-2} + \cdots + t + 1)^m$$

$$= (t^m - mt^{m-1} + \cdots + (-1)^m)(a_{m(k-1)}t^{m(k-1)} + \cdots + a_1 t + a_0).$$
Hence we have

\[ \Delta_{T^k,m} = \sum_{j=0}^{m(k-1)} a_j T^j \left( \Delta_{T^k,m} \right) T^j = 0. \]

So the proof is complete. \(\square\)

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