INVERTIBLE WEIGHTED SHIFT OPERATORS WHICH ARE $m$-ISOMETRIES

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Abstract. For a bounded linear operator $T$ on a complex Hilbert space $H$, let
$$
\Delta_{T,m} = \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} T^{*m-k}T^{m-k} \quad \text{for } m \in \mathbb{N}.
$$
$T$ is called an $m$-isometry if $\Delta_{T,m} = 0$. In this paper, for every even number $m$, we give an example of invertible $(m+1)$-isometry which is not an $m$-isometry. Next we show that if $T$ is an $m$-isometry, then the operator $\Delta_{T,m-1}$ is not invertible.

1. Introduction

J. Agler and M. Stankus published excellent papers about $m$-isometric operators, [1], [2] and [3]. They showed that $m$-isometries have interesting spectral properties. For example, if $T$ is an $m$-isometry, then the approximate point spectrum of $T$ lies on the unit circle. In [10], Patel showed that Weyl’s theorem holds for a 2-isometry. Applying Uchiyama and Tanahashi’s result [11], in [8] we showed that if $T$ is an $m$-isometry, then $T$ has the single valued extension property. Let $H$ be a complex Hilbert space and $B(H)$ be a set of all bounded linear operators on $H$. Let $\binom{m}{k}$ be the binomial coefficient. For an operator $T \in B(H)$, let
$$
\Delta_{T,m} = \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} T^{*m-k}T^{m-k}.
$$
$T$ is said to be an $m$-isometry if $\Delta_{T,m} = 0$. Agler and Stankus proved that if $T$ is an $m$-isometry, then $\Delta_{T,m-1} \geq 0$ (Proposition 1.5, [1]). It is easy to see that $T^{*}\Delta_{T,m}T - \Delta_{T,m} = \Delta_{T,m+1}$. Hence it holds that if $T$ is an $m$-isometry, then $T$ is an $(m+1)$-isometry. In [7], T. Bermudez, A. Martinon and E. Negrin studied characterizations of weighted shift operators which are $m$-isometries. In [1], Agler and Stankus proved that if $m$ is even and $T$ is an invertible $m$-isometry, then $T$ is an $(m-1)$-isometry. In [4] A. Athavale proved that if $S$ is a unilateral weighted shift

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$Se_n = w_n e_{n+1}$ with weights $w_n = \sqrt{1 + \frac{m}{n}}$ for $n = 1, 2, 3, \cdots$, then $S$ is an $(m+1)$-isometry which is not an $m$-isometry. The operator $S$ is not invertible. We have not seen invertible $(m+1)$-isometries which are weighted shifts for $m$ even. We give an example of an invertible $(m+1)$-isometry which is not an $m$-isometry for every even number $m$. Next we prove that power bounded $m$-isometries are isometries, and if $T$ is an $m$-isometry for a natural number $m \geq 2$, then the operator $\Delta_{T,m-1}$ is not invertible.

2. INVERTIBLE WEIGHTED SHIFT OF AN $m$-ISOMETRY

Agler and Stankus showed the following result.

**Proposition A** (Proposition 1.23, [1]). *If $T$ is an invertible $m$-isometry and $m$ is even, then $T$ is an $(m-1)$-isometry.*

Their proof is fine. We give another proof.

**Proof.** Since $T$ is an $m$-isometry, it holds that

$$\Delta_{T,m-1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} T^{m-1-k} T^{m-1-k} \geq 0. \quad (2.1)$$

It is easy to see that $T^{-1}$ also is an $m$-isometry. Since $T^{-1}$ is an $m$-isometry, it holds that

$$\Delta_{T^{-1},m-1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} (T^{-1})^{m-1-k} (T^{-1})^{m-1-k} \geq 0. \quad (2.2)$$

Since $m-1$ is an odd number, by (2.2) we have

$$-\Delta_{T,m-1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} T^{m-1} T^k = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} T^{m-1} (T^{-1})^{m-1-k} (T^{-1})^{m-1-k} T^{m-1}$$

$$= T^{m-1} (\Delta_{T^{-1},m-1}) T^{m-1} \geq 0.$$

Hence $\Delta_{T,m-1} \leq 0$. By (2.1) we have $\Delta_{T,m-1} = 0$, so the proof is complete. \[\□\]

For every even number $m$, we give an example of an invertible $(m+1)$-isometry $T$ which is not an $m$-isometry.

**Theorem 1.** *For any even number $m$, there exists an invertible $(m+1)$-isometry $T$ which is not an $m$-isometry.*

**Proof.** Let $m$ be any even number and put $\psi(x) = x(x+1) \cdots (x+m-1)$. Then $\psi(x)$ is a polynomial of even degree $m$. Hence $\psi(n) \geq 0$ for $n = 0, \pm 1, \pm 2, \cdots$. So, for any $k > 0$, we have $\psi(n) + k > 0$ for $n = 0, \pm 1, \pm 2, \cdots$. Let

$$w_n = \sqrt{\frac{\psi(n+1) + k}{\psi(n) + k}} > 0 \quad \text{for} \ n = 0, \pm 1, \pm 2, \cdots.$$
Let $\{e_n\}_{n=-\infty}^\infty$ be an orthonormal basis of $\mathcal{H} = \ell^2$ and $T$ be a bilateral weighted shift such that $T e_n = w_n e_{n+1}$. We define
\[
I_{m,n} = w_n^2 w_{n+1}^2 \cdots w_{n+m-1}^2 - \left( \frac{m}{1} \right) w_n^2 w_{n+1}^2 \cdots w_{n+m-2}^2 + \left( \frac{m}{2} \right) w_n^2 w_{n+1}^2 \cdots w_{n+m-3}^2 + \cdots + (-1)^{m-1} \left( \frac{m}{m-1} \right) w_n^2 + (-1)^m.
\]
Note that $T$ is an $m$-isometry if and only if $I_{m,n} = 0$ for $n = 0, \pm 1, \pm 2, \cdots$. To show $T$ is an $(m+1)$-isometry, we will show that $I_{m+1,n} = 0$ for $n = 0, \pm 1, \pm 2, \cdots$. We define a function $f(x)$ as
\[
f(x) = x^{n+m-1}(1 - x)^{m+1}
\]
\[
= x^{n+m-1} - \left( \frac{m+1}{1} \right) x^{n+m} + \left( \frac{m+1}{2} \right) x^{n+m+1}
\]
\[
+ \cdots + (-1)^m \left( \frac{m+1}{m} \right) x^{n+2m-1} + (-1)^{m+1}x^{n+2m}.
\]
Note that $f^{(m)}(1) = 0$. By differentiating $m$ times, we have
\[
0 = f^{(m)}(1)
\]
\[
= (n + m - 1)(n + m - 2) \cdots n
\]
\[
- \left( \frac{m+1}{1} \right) (n + m)(n + m - 1) \cdots (n + 1)
\]
\[
+ \cdots + (-1)^m \left( \frac{m+1}{m} \right) (n + 2m - 1)(n + 2m - 2) \cdots (n + m)
\]
\[
+ (-1)^{m+1}(n + 2m)(n + 2m - 1) \cdots (n + m + 1).
\]
Hence
\[
(-1)^{m+1}I_{m+1,n} \psi(n) + k
\]
\[
= (n + m + 1)(n + m + 2) \cdots (n + 2m) + k
\]
\[
- \left( \frac{m+1}{1} \right) \{ (n + m)(n + m + 1) \cdots (n + 2m - 1) + k \}
\]
\[
+ \cdots + (-1)^m \left( \frac{m+1}{m} \right) \{ (n + 1)(n + 2) \cdots (n + m) + k \}
\]
\[
+ (-1)^{m+1} \{ n(n + 1)(n + 2) \cdots (n + m - 1) + k \}
\]
\[
= f^{(m)}(1) + k(1 - 1)^{m+1} = 0.
\]
This implies that $I_{m+1,n} = 0$. Hence $T$ is an $(m+1)$-isometry.
To show that $T$ is not an $m$-isometry, we will show that $I_{m,n} \neq 0$ for $n = 0, \pm 1, \pm 2, \cdots$. We define a function $g(x)$ as
\[
g(x) = x^{n+m-1}(1 - x)^m
\]
\[
= x^{n+m-1} - \left( \frac{m}{1} \right) x^{n+m} + \left( \frac{m}{2} \right) x^{n+m+1}
\]
\[
+ \cdots + (-1)^{m-1} \left( \frac{m}{m-1} \right) x^{n+2m-2} + (-1)^m x^{n+2m-1}.
\]
Note that \( g^{(m)}(1) = m!(-1)^m \). By differentiating \( m \) times, we have
\[
m!(-1)^m = g^{(m)}(1)
= (n + m - 1)(n + m - 2) \cdots n
- \binom{m}{1} (n + m)(n + m - 1) \cdots (n + 1)
+ \cdots + (-1)^{m-1} \binom{m}{m-1} (n + 2m - 2)(n + 2m - 3) \cdots (n + m - 1)
+ (-1)^m (n + 2m - 1)(n + 2m - 2) \cdots (n + m).
\]

Hence
\[
(-1)^m I_{m,n} (\psi(n) + k)
= (n + m)(n + m + 1) \cdots (n + 2m - 1) + k
- \binom{m}{1} \{ (n + m - 1)(n + m) \cdots (n + 2m - 2) + k \}
+ \cdots + (-1)^{m-1} \binom{m}{m-1} \{ (n + 1)(n + 2) \cdots (n + m) + k \}
+ (-1)^m \{ n(n + 1)(n + 2) \cdots (n + m - 1) + k \}
= g^{(m)}(1) + k (1 - 1)^m = m!(-1)^m.
\]

This implies that \( I_{m,n} = m! (\psi(n) + k)^{-1} \neq 0 \). Hence \( T \) is not an \( m \)-isometry.

Finally, since \( \omega_n \to 1 \ (n \to \pm \infty) \), we have \( \sigma(T) = \{ \lambda : |\lambda| = 1 \} \) by Proposition 2.6.8 (b) of [9]. Therefore \( T \) is invertible. The proof is complete. \( \square \)

**Remark 1.** This result provides an example of an \((m+1)\)-isometry \( T \) which is not an \( m \)-isometry if we take \( T e_n = \omega_n e_{n+1} \) for \( n = 1, 2, \cdots \) where
\[
\omega_n = \sqrt{(n + 1)(n + 2) \cdots (n + m) + k}
\]
with \( 0 \leq k \). If \( k = 0 \), then
\[
\omega_n = \sqrt{(n + 1)(n + 2) \cdots (n + m) + 0} = \sqrt{n + m}.
\]

This is the result of A. Athavale, Proposition 8 of [4].

### 3. Some properties of \( m \)-isometries

An operator \( T \in B(\mathcal{H}) \) is said to be power bounded if there exists a positive number \( M \) such that \( \|T^n\| \leq M \) for every \( n \in \mathbb{N} \).

**Theorem 2.** A power bounded \( m \)-isometry is an isometry.

To prove this result, we will use Berberian’s method (cf. [6], [12]).
Proposition (Lemma 2.7, [12]). Let \( \mathcal{H} \) be a complex Hilbert space. Then there exist a Hilbert space \( \mathcal{H}' \supset \mathcal{H} \) and a unital linear map \( \circ : B(\mathcal{H}) \to B(\mathcal{H}') \) such that

(i) \( (ST)^\circ = S^oT^o, \) \( (T^*)^\circ = (T^o)^* \), \( \|T\| = \|T^o\| \),

(ii) \( S \leq T \implies S^o \leq T^o \),

(iii) \( \sigma(T) = \sigma(T^o), \sigma_a(T) = \sigma_a(T^o) = \sigma_p(T^o) \).

We prepare notation and a lemma. For a unit vector \( x \in \mathcal{H} \), assume that \( (T^* - I)x = a_1 x \). Let \( a_n = \langle \Delta T, n x \rangle \) and \( b_n = \| T^n x \|^2 \) for \( n = 1, 2, 3, \ldots \). Then we have the following lemma.

Lemma. For an operator \( T \in B(\mathcal{H}) \) with the above notation, it holds that

\[
(3.1) \quad b_n = a_n + \left( \frac{n}{n-1} \right) a_{n-1} + \left( \frac{n}{n-2} \right) a_{n-2} + \cdots + na_1 + 1.
\]

Proof. Note that

\[
b_1 = \|Tx\|^2 = \langle T^*Tx, x \rangle = \langle a_1 x + x, x \rangle = a_1 + 1.
\]

Since

\[
T^*T^2 = (T^*T^2 - 2T^*T + I) + 2(T^*T - I) + I = \Delta T, 2 + 2\Delta T, 1 + I,
\]

we have

\[
b_2 = \langle T^*T^2x, x \rangle = \langle \Delta T, 2x, x \rangle + 2\langle \Delta T, 1x, x \rangle + \langle x, x \rangle = a_2 + 2a_1 + 1.
\]

Hence (3.1) holds for \( n = 1, 2 \). Note that

\[
n^m = (t - 1 + 1)^n = (t - 1)^n + \left( \frac{n}{1} \right) (t - 1)^{n-1} + \left( \frac{n}{2} \right) (t - 1)^{n-2} + \cdots + n(t - 1) + 1.
\]

Hence

\[
T^{*m}T^n = \Delta T, n + \left( \frac{n}{1} \right) \Delta T, n-1 + \left( \frac{n}{2} \right) \Delta T, n-2 + \cdots + n\Delta T, n-1 + I
\]

and

\[
b_n = \langle T^{*m}T^n, x, x \rangle = a_n + \left( \frac{n}{1} \right) a_{n-1} + \left( \frac{n}{2} \right) a_{n-2} + \cdots + na_1 + 1
\]

This completes the proof. \( \square \)

Proof of Theorem \( \square \) Let \( T \) be a power bounded \( m \)-isometry. Since \( T^*T - I \) is self-adjoint, it suffices to show that

\[
\sigma(T^*T - I) = \sigma_a(T^*T - I) = \{0\}.
\]

Assume that there exists a non-zero real number \( a \in \sigma(T^*T - I) \). Since \( a \) belongs to the approximate point spectrum of \( T^*T - I \), by Berberian’s method we consider
an extension $\mathcal{H}^\circ$ of $\mathcal{H}$ and the mapping $S \to S^\circ$ of $B(\mathcal{H})$ into $B(\mathcal{H}^\circ)$. Then $a$ is an eigen-value of $T^*T^\circ - I^\circ$ and $T^\circ$ is also a power bounded $m$-isometry. For simplification, we denote $T^\circ$ by $T$. Since $a$ is an eigen-value of $T^*T - I$, there exists a unit vector $x$ such that $(T^*T - I)x = ax$. Hence
\[ \langle (T^*T - I)x, x \rangle = \langle ax, x \rangle = a. \]

Let $a_1 = a$ and $a_n = \langle \Delta T_n x, x \rangle$. Then since $T$ is an $m$-isometry, we have $a_m = a_{m+1} = \cdots = a_n = 0$ for $n > m$. By the Lemma, we have
\[ \|T^n x\|^2 = \left( \frac{n}{m-1} \right) a_{m-1} + \left( \frac{n}{m-2} \right) a_{m-2} + \cdots + na_1 + 1. \]

Hence
\[ \|T^n x\|^2 = a_{m-1} + \left( \frac{1}{n} \right) \left\{ \left( \frac{n}{m-2} \right) a_{m-2} + \cdots + na_1 + 1 \right\}. \]

Since $T$ is power bounded, we have $a_{m-1} = 0$ by $n \to \infty$. Repeating this, we have
\[ a_{m-1} = a_{m-2} = \cdots = a_1 = a = 0. \]

This is a contradiction, so the proof is complete. \hfill \Box

Patel (Corollary 2.8, [10]) proved that if $T$ is a 2-isometry, then $1 \in \sigma(T^*T)$. That is, if $T$ is a 2-isometry, then $0 \in \sigma(\Delta T, 1)$. We now generalize this result as follows.

**Theorem 3.** If $T$ is an $m$-isometry for $m \geq 2$, then $0 \in \sigma(\Delta T, m-1)$.

**Proof.** For simplification, we denote $\Delta T, m-1$ by $\Delta$. Assume that $0 \notin \sigma(\Delta)$. Since $\Delta T, m = T^* T \Delta, T - \Delta = 0$, it holds that $T^* T \Delta = \Delta$. Let $S = \Delta^{1/2} T \Delta^{-1/2}$. Then
\[ S^* S = (\Delta^{-1/2} T^* \Delta^{1/2})(\Delta^{1/2} T \Delta^{-1/2}) = I. \]

Hence $S$ is an isometry and $T = \Delta^{-1/2} S \Delta^{1/2}$. So $T$ is similar to the isometry $S$. Since $S$ is power bounded, $T$ is an isometry by Theorem [8]. Hence $\Delta T, 1 = 0$, and so $\Delta T, m-1 = \Delta = 0$. This contradicts our assumption that $0 \notin \sigma(\Delta)$, so the proof is complete. \hfill \Box

**Remark 2.** Since Weyl’s theorem holds for an isometry, Weyl’s theorem holds for a power bounded $m$-isometry by Theorem [9] (cf. [3], [10]).

Finally we show the following result. For a 2-isometry, Patel proved it (Theorem 2.1, [10]).

**Theorem 4.** A power of an $m$-isometry is again an $m$-isometry.

**Proof.** Let $T$ be an $m$-isometry, i.e.,
\[ \Delta T, m = T^* m T^m - m T^* m-1 T^{m-1} + \cdots + (-1)^m I = 0. \]

We prove $T^k$ is also an $m$-isometry. Write
\[ (t^k - 1)^m = (t - 1)^m (t^{k-1} + t^{k-2} + \cdots + t + 1)^m \]
\[ = (t^m - mt^{m-1} + \cdots + (-1)^m (a_{m(k-1)} t^{m(k-1)} + \cdots + a_1 t + a_0). \]
Hence we have

\[ \Delta_{T^k,m} = \sum_{j=0}^{m(k-1)} a_j T^j \left( \Delta_{T^k,m} \right) T^j = 0. \]

So the proof is complete. \( \square \)

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