Sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, bigraded resolutions, and coadjoint orbits of loop groups

ROGER BIELAWSKI AND LORENZ SCHWACHHÖFER

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Abstract. We construct a canonical linear resolution of acyclic 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ and discuss the resulting natural Poisson structure.

1. Introduction

The goal of this paper is to present a (yet another) variation on a theme developed by several authors, notably Moser, Adams, Harnad, Hurtubise, Previato [13], [1]–[5], and relating integrable systems, rank $r$ perturbations, spectral curves and their Jacobians, and coadjoint orbits of loop groups.

Let us briefly recall that, given matrices $A, Y, F, G$ of size, respectively, $N \times N$, $r \times r$, $N \times r$, and $r \times N$, one defines a $\mathfrak{gl}_r(\mathbb{C})$-valued rational map

\[ Y + G(A - \lambda)^{-1}F, \]

i.e. an element of the loop algebra $\mathfrak{gl}(r)^-$, consisting of loops extending holomorphically to the outside of some circle $S^1 \subset \mathbb{C}$. This determines a (shifted) reduced coadjoint orbit in $\mathfrak{gl}(r)^-$ (see Remark 4.5 for a definition). On the other hand, the polynomial (1.1) also determines (generically) a curve $S$ and a line bundle $L$ of degree $g + r - 1$: the curve is defined as the spectrum of (1.1), and $L$ is the dual of the eigenbundle of (1.1). This describes $S$ as an affine curve in $\mathbb{C}^2$, and the isospectral flows, corresponding to Hamiltonians on the space of rank $r$ perturbations, linearise on the Jacobian of the projective model of $S$.

In fact, as shown by Adams, Harnad, and Hurtubise [1,2], it is more convenient to compactify $S$ inside a Hirzebruch surface $F_d$, $d \geq 1$. This results in singularities, which may be partially resolved, but it gives a particularly nice description of $\text{Jac}^0(S)$, i.e. of the flow directions.

In this paper, we consider a different compactification of $S$, namely inside $\mathbb{P}^1 \times \mathbb{P}^1$ and defined as

\[ S = \left\{ (z, \lambda) \in \mathbb{P}^1 \times \mathbb{P}^1; \det \begin{pmatrix} Y - z & G \\ F & A - \lambda \end{pmatrix} = 0 \right\}. \]

This is a very natural thing to do, but we know of only one occurrence in the literature: the paper of Sanguinetti and Woodhouse [17] (we are grateful to Philip Boalch for this reference). In that paper, in addition to other results, the authors use the above compactification to give a nice picture of the duality phenomenon discussed in [3]. Our application is to another subtlety of the rank $r$ perturbation
Theorem 1.1. Let $S$ be a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$, defined by $[1,2]$ and corresponding to a (shifted) rank $r$ perturbation of the matrix $A$ ($r \leq N$). A line bundle $L \in \text{Jac}^{g-r+1}(S)$ corresponds to $(A, Y, F, G)$ with $\text{rank } F = \text{rank } G = r$ if and only if $L$ satisfies:

$$H^0(S, L(0, -1)) = H^1(S, L(0, -1)) = 0, \quad H^0(S, L(1, 0)) = 0, \quad H^1(S, L(1, -2)) = 0.$$  

We are interested in more than line bundles on smooth curves in $\mathbb{P}^1 \times \mathbb{P}^1$. The above approach generalises to acyclic (i.e. semistable) 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, with a fixed bigraded Hilbert polynomial. In §§2 and 3, we construct a natural linear resolution of such a sheaf, very much in the spirit of Beauville [6].

This gives us a linear polynomial matrix $M(z, \lambda)$ (up to a certain group action). If the support of the sheaf is a smooth curve of bidegree $(r, N)$, then the matrix has size $r \times N$. As long as the point $(\infty, \infty)$ does not belong to the support of the sheaf, then the matrices $M(z, \lambda)$ can be identified with the quadruples $(A, Y, F, G)$. The space $\mathcal{M}(k, l)$ of the $(A, Y, F, G)$ has a natural Poisson structure, obtained by identifying it with $\mathfrak{gl}_N(\mathbb{C})^* \oplus \mathfrak{gl}_r(\mathbb{C})^* \oplus T^* M_{N \times r}(\mathbb{C})$. Thus we obtain a Poisson structure on the quotient of an open subset of $\mathcal{M}(N, r)$ by $\text{GL}_N(\mathbb{C}) \times \text{GL}_r(\mathbb{C})$. The (generic) symplectic leaves are known, from [1,5], to be reduced coadjoint orbits of loop groups. Our aim is to describe these symplectic leaves directly in terms of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. We show that they correspond to symplectic leaves of a particular Mukai-Tyurin-Bottacin Poisson structure on the moduli space $M_Q(r, N)$ of simple sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with (bigraded) Hilbert polynomial $N x + r y$. The surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$ is an example of a Poisson surface, and consequently, for every choice of a Poisson structure on $Q$, i.e. a section $s$ of the anticanonical bundle $K_Q^* \simeq \mathcal{O}(2, 2)$, one obtains a Poisson structure on $M_Q(r, N)$ as a map

$$T_{[\mathcal{F}]} M_Q(r, N) \simeq \text{Ext}^1_Q(\mathcal{F}, \mathcal{F} \otimes K_Q) \xrightarrow{\cdot s} \text{Ext}^1_Q(\mathcal{F}, \mathcal{F}) \simeq T_{[\mathcal{F}]} M_Q(r, N).$$

We show that the (generic) symplectic leaves $\mathfrak{gl}_N(\mathbb{C})^* \oplus \mathfrak{gl}_r(\mathbb{C})^* \oplus T^* M_{N \times r}(\mathbb{C})$, i.e. reduced coadjoint orbits in $\mathfrak{gl}(r)^-$, are the symplectic leaves of the Mukai-Tyurin-Bottacin structure corresponding to $s(z, \lambda) = 1$, i.e. to the anticanonical divisor $2(\{\infty\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{\infty\}).$

2. Acyclic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ and their resolutions

Definition 2.1. Let $X$ be a complex manifold and let $\mathcal{F}$ be a coherent sheaf on $X$. Then:

(i) The support of $\mathcal{F}$ is the complex subspace $\text{supp } \mathcal{F}$ of $X$ defined as the zero-locus of the annihilator (in $\mathcal{O}_X$) of $\mathcal{F}$. The dimension $\text{dim } \mathcal{F}$ of $\mathcal{F}$ is the dimension of its support.

(ii) $\mathcal{F}$ is pure if $\text{dim } \mathcal{E} = \text{dim } \mathcal{F}$ for all nontrivial coherent subsheaves $\mathcal{E} \subset \mathcal{F}$.

(iii) $\mathcal{F}$ is acyclic if $H^*(\mathcal{F}) = 0$.

Remark 2.2. In the case of 1-dimensional sheaves on a smooth surface $X$, purity of $\mathcal{F}$ means that at every point $x \in \text{supp } \mathcal{F}$, the skyscraper sheaf $\mathcal{O}_x$ does not embed into $\mathcal{F}_x$. In addition, a 1-dimensional sheaf $\mathcal{F}$ on a smooth surface $X$ is pure if and
only if it is reflexive; i.e. after performing the duality $\mathcal{F} \mapsto \text{Ext}^1_X(\mathcal{F}, K_X)$ twice, we obtain $\mathcal{F}$ back (up to isomorphism) (see [3] §1.1).

In the remainder of the paper, all sheaves are coherent.

We shall now consider sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. For any $p, q \in \mathbb{Z}$ we denote by $\mathcal{O}(p, q)$ the line bundle $\pi_1^* \mathcal{O}(p) \otimes \pi_2^* \mathcal{O}(q)$, where $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ are the two projections. We shall also denote by $\zeta$ and $\eta$ the two affine coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. Associated to $\mathcal{F}$ is its bigraded Hilbert polynomial

$$P_{\mathcal{F}}(x, y) = \sum_{x, y \in \mathbb{Z}} \chi(\mathcal{F}(x, y)).$$

The sheaf $\mathcal{F}$ is 1-dimensional if and only if $P_{\mathcal{F}}$ is linear.

We begin by describing a canonical resolution of acyclic 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$.

**Theorem 2.3.** Let $\mathcal{F}$ be a 1-dimensional acyclic sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathcal{F}$ has a linear resolution by locally free sheaves of the form

$$0 \to \mathcal{O}(-2, -1)^{\oplus k} \oplus \mathcal{O}(-1, -2)^{\oplus l} \xrightarrow{M(\zeta, \eta)} \mathcal{O}(-1, -1)^{\oplus (k+l)} \to \mathcal{F} \to 0,$$

for some $k, l \geq 0$.

Conversely, any $\mathcal{F}$ defined as a cokernel of a map $M(\zeta, \eta)$ as above with $\det M(\zeta, \eta) \neq 0$ is acyclic and 1-dimensional.

**Remark 2.4.** Let $\mathcal{F}$ be a 1-dimensional acyclic sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with $P_{\mathcal{F}}(x, y) = lx + ky$. Then $\mathcal{F}$ is semistable with respect to $\mathcal{O}(1, 1)$.

**Remark 2.5.** This resolution is canonical, but not necessarily minimal, in the sense of being obtained from the minimal resolution of the bigraded module $\bigoplus_{i,j \in \mathbb{Z}} H^0(\mathcal{F}(i, j))$.

**Proof.** Let $h^0(\mathcal{F}(0, 1)) = k$ and $h^0(\mathcal{F}(1, 0)) = l$ so that $P_{\mathcal{F}} = lx + ky$. Let $\mathcal{E} = \mathcal{F}(1, 1)$, and let $\Gamma_*(\mathcal{E}) = \bigoplus_{i,j \geq 0} H^0(\mathcal{E}(i, j))$ be the associated bigraded module over the bigraded ring $\mathcal{S} = \bigoplus_{i,j \in \mathbb{Z}} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(i, j))$. Furthermore, let $\Gamma_*(\mathcal{E})|_{i=0} = \bigoplus_{i,j \geq 0} H^0(\mathcal{E}(i, j))$ be its truncation. Owing to [12] Lemma 6.8], the sheaf associated to $\Gamma_*(\mathcal{E})|_{i=0}$ is again $\mathcal{E}$. Moreover, [12] Theorem 6.9 implies, as $\mathcal{E}(-1, -1)$ is acyclic, that the natural map

$$H^0(\mathcal{E}) \otimes H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(p, q)) \to H^0(\mathcal{E}(p, q))$$

is surjective for any $p, q \geq 0$. Therefore, we have a surjective homomorphism

$$\mathcal{S}^{\oplus (k+l)} \to \Gamma_*(\mathcal{E})|_{i=0} \to 0$$

of bigraded $\mathcal{S}$-modules. Since $\mathcal{E}$ is of pure dimension 1, its projective dimension is 1, and, hence, the above homomorphism extends to a linear free resolution

$$0 \to \bigoplus_{i=1}^{k+l} \mathcal{S}(-p_i, -q_i) \to \bigoplus_{i=1}^{k+l} \mathcal{S} \to \Gamma_*(\mathcal{E})|_{i=0} \to 0,$$

where $p_i, q_i \geq 0$ and $p_i + q_i > 0$ for each $i$. The corresponding sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ give us a locally free resolution of $\mathcal{E}$:

$$0 \to \bigoplus_{i=1}^{k+l} \mathcal{O}(-p_i, -q_i) \to \bigoplus_{i=1}^{k+l} \mathcal{O} \to \mathcal{E} \to 0.$$
Since $H^*(E(-1,-1)) = 0$, either $p_i = 0$ or $q_i = 0$ for every $i$. Since $h^0(\mathcal{E}(-1,0)) = k$, we deduce, after tensoring (2.3) with $\mathcal{O}(-1,0)$, that $\sum p_i = k$. Similarly $\sum q_i = l$. Since $h^4(\mathcal{E}) = 0$, none of the $p_i$ or $q_i$ can be greater than 1, and so all nonzero $p_i$ and all nonzero $q_i$ are equal to 1. This proves the existence of resolution (2.2).

Conversely, if $\mathcal{F}$ admits a resolution of the form (2.2), then it is 1-dimensional. The long exact cohomology sequence implies that $\mathcal{F}$ is acyclic. 

Let us write $n = k + l$. The polynomial matrix $M(\zeta, \eta)$ in (2.3) has size $n \times n$ and is of the form

\[(2.4) \quad (A_0 + A_1 \zeta \ B_0 + B_1 \eta),\]

with $A_0, A_1 \in \text{Mat}_{n,k}(\mathbb{C})$, $B_0, B_1 \in \text{Mat}_{n,l}(\mathbb{C})$. Let us denote by $\mathcal{A}(k,l)$ the space of such matrices with nonzero determinant. The group $\text{GL}_n(\mathbb{C}) \times \text{GL}_k(\mathbb{C}) \times \text{GL}_l(\mathbb{C})$ acts on $\mathcal{M}(k,l)$ via

\[(2.5) \quad (g, h_1, h_2) \cdot (A(\zeta) \ B(\eta)) = g(A(\zeta) \ B(\eta)) \left( \begin{array}{cc} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{array} \right),\]

and we can restate Theorem 2.3 as follows:

**Corollary 2.6.** There exists a natural bijection between

- (a) isomorphism classes of 1-dimensional acyclic sheaves $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $h^0(\mathcal{F}(0,1)) = k$, $h^0(\mathcal{F}(1,0)) = l$

and

- (b) orbits of $\text{GL}_{k+l}(\mathbb{C}) \times \text{GL}_k(\mathbb{C}) \times \text{GL}_l(\mathbb{C})$ on $\mathcal{A}(k,l)$. 

For a sheaf defined by (2.2), we can describe its support as follows. As a set, the support of $\mathcal{F}$ is

\[S = \{(\zeta, \eta) \in \mathbb{P}^1 \times \mathbb{P}^1; \text{det } M(\zeta, \eta) = 0\}.\]

Let us write $\text{det } M(\zeta, \eta) = \prod_{i=1}^{s} q_i(\zeta, \eta)^{k_i}$, where $q_i$ are irreducible polynomials. We define the minimal polynomial $p_M(\zeta, \eta)$ of $M$ as $\prod_{i=1}^{s} q_i(\zeta, \eta)^{r_i}$, where

\[r_i = \max\{a_i b_i\}; \text{ at a generic point, } M(\zeta, \eta) \text{ has a Jordan block of size } a_i \text{ with eigenvalue } q_i(\zeta, \eta)^{b_i}\].

Then:

**Proposition 2.7.** The support of $\mathcal{F}$ is the curve $(S, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}/(p_M))$. 

Let us now fix the support $S$. For simplicity, we shall assume that it is an integral curve in the linear system $|\mathcal{O}(k,l)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$; i.e. $S$ is given by an irreducible polynomial $P(\zeta, \eta)$ of bidegree $(k,l)$, $k,l \geq 1$. This immediately implies that the rank of $\mathcal{F}$ is constant; i.e. $\mathcal{F}$ is locally free. Theorem 2.3 and Corollary 2.6 imply

**Corollary 2.8.** Let $P(\zeta, \eta)$ be an irreducible polynomial of bidegree $(k,l)$ and $S = \{(\zeta, \eta); P(\zeta, \eta) = 0\}$ be the corresponding integral curve of genus $g = (k-1)(l-1)$. There exists a canonical biholomorphism $\text{Jac}^{g-1}(S) - \Theta \simeq \{M \in \mathcal{A}(k,l); \text{det } M = P\}/\text{GL}_n(\mathbb{C}) \times \text{GL}_k(\mathbb{C}) \times \text{GL}_l(\mathbb{C})$.

Similarly, let $\mathcal{U}_S(r,d)$ be the moduli space of semistable vector bundles (locally free sheaves) on $S$. For $d = r(g-1)$ define the generalised theta divisor $\Theta$ as the set of bundles with nonzero section. Then we have:
Corollary 2.9. Let \( P(\zeta, \eta) \) be an irreducible polynomial of bidegree \((k, l)\) and \( S = \{(\zeta, \eta); P(\zeta, \eta) = 0\} \) be the corresponding integral curve of genus \( g = (k - 1)(l - 1)\). There exists a canonical biholomorphism

\[ U_S(r, r(g - 1)) - \Theta \simeq \{ M \in A(kr, lr); \det M = P^r \} / GL_{nr}(\mathbb{C}) \times GL_{kr}(\mathbb{C}) \times GL_{lr}(\mathbb{C}) \]

3. A geometric resolution

There is a much more geometric way of constructing resolution (2.2), which works under mild assumptions on the sheaf \( F \) (cf. [7] for the case of \( \sigma \)-sheaves).

Definition 3.1. Let \( F \) be a 1-dimensional sheaf on \( \mathbb{P}^1 \times \mathbb{P}^1 \). We say that \( F \) is bipure if \( F \) has no nontrivial coherent subsheaves supported on \( \{ z \} \times \mathbb{P}^1 \) or on \( \mathbb{P}^1 \times \{ z \} \) for any \( z \in \mathbb{P}^1 \).

Remark 3.2. Observe that bipure implies pure.

Now let \( F \) be an acyclic and bipure sheaf on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with Hilbert polynomial \( lx + ky \). As in the proof of Theorem 2.3, we consider the sheaf \( E = F(1, 1) \). Let \( D_\zeta \) and \( D_\eta \) denote the divisors \( \{ \zeta \} \times \mathbb{P}^1, \mathbb{P}^1 \times \{ \eta \} \). We set

\[ V_\zeta = \{ s \in H^0(E); s|_{D_\zeta} = 0 \}, \quad W_\eta = \{ s \in H^0(E); s|_{D_\eta} = 0 \} \]

For any \( \zeta \) and \( \eta \), consider the maps

\[ E(-1, 0) \to E, \quad E(0, -1) \to E \]

given by multiplication by global nonzero sections of \( O(1, 0) \) and \( O(0, 1) \), vanishing at \( \zeta \) and \( \eta \), respectively. Since \( E \) is bipure, these maps are injective, and therefore \( V_\zeta \simeq H^0(E(-1, 0)) \), \( W_\eta \simeq H^0(E(0, -1)) \) for any \( \zeta, \eta \). In particular,

\[ \dim V_\zeta = k, \quad \dim W_\eta = l, \quad \text{for any } \zeta \text{ and } \eta. \]

Therefore, \( \zeta \mapsto V_\zeta \) and \( \eta \mapsto W_\eta \) are subbundles of \( H^0(E) \otimes O \) on \( \mathbb{P}^1 \). They are isomorphic to \( H^0(E(-1, 0)) \otimes O(-1) \) and to \( H^0(E(0, -1)) \otimes O(-1) \). The isomorphism is realised explicitly via the map

\[ H^0(E(-1, 0)) \otimes O(-1) \to H^0(E) \otimes O, \]

defined as

\[ H^0(E(-1, 0)) \otimes O(-1) \ni (s, (a, b)) \mapsto (b\zeta - a)s \in H^0(E) \]

(here \((a, b) \in l, \) where \( l \) is the fibre of \( O(-1) \) over \([l]\)), and similarly for the subbundle \( W \). We now define a vector bundle \( U \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \), the fibre of which at \( \zeta, \eta \) is \( V_\zeta \oplus W_\eta \); i.e.

\[ U \simeq (H^0(E(-1, 0)) \otimes O(-1, 0)) \oplus (H^0(E(0, -1)) \otimes O(0, -1)). \]

We obtain an injective map of sheaves \( U \to H^0(E) \otimes O \). Let \( G \) be the cokernel, i.e.

\[ 0 \to U \to H^0(E) \otimes O \to G \to 0. \]

We claim that \( G \simeq E \), and so (3.2) is a natural resolution of \( E \). To prove this, tensor the resolution (2.2) by \( O(1, 1) \) to obtain

\[ 0 \to O(-1, 0) \oplus O(0, -1) \oplus M(\zeta, \eta) \otimes O(0, -1) \to E \to 0. \]

Clearly, the middle term is identified with \( H^0(E) \otimes O \). For any \( \zeta_0 \), consider the image of \( M(\zeta_0, \eta) \) restricted to \( O(-1, 0) \oplus O(0, -1) \). This image does not depend on \( \eta \), and since \( F \) is bipure, it is exactly \( V_{\zeta_0} \), defined in (3.1), i.e. sections vanishing on \( \zeta_0 \times \mathbb{P}^1 \). Similarly, for any \( \eta_0 \), the image of \( M(\zeta, \eta_0) \) restricted to \( O(0, -1) \) is precisely \( W_{\eta_0} \). Hence, there are canonical isomorphisms between both first and second terms in resolutions (3.2) and (3.3) which commute with the horizontal maps. Therefore \( G \simeq E \).
Proposition 4.2. Let \( \phi \) induces a bijection between

\[ k \text{hand, conjugacy classes of } GL_k(\mathbb{C}) \]

This can, of course, always be achieved via an automorphism of \( GL_k(\mathbb{C}) \). In terms of the matrix \( M(\zeta, \eta) \) corresponding to \( \mathcal{F} \), \((\infty, \infty) \notin \text{supp } \mathcal{F} \).

The residual group action is that of conjugation by the block-diagonal \( GL_k(\mathbb{C}) \times GL_l(\mathbb{C}) \). We denote this group by \( K \).

Remark 4.1. We are, essentially, in the situation of [5]. The only difference is that we do not fix \( X \) or \( Y \).

We denote by \( \mathcal{M}(k, l) \) the space of all matrices of the form \((4.2)\), which we identify with quadruples \((X, Y, F, G)\) as above. The action of \( K = GL_k(\mathbb{C}) \times GL_l(\mathbb{C}) \) on \( \mathcal{M}(k, l) \) is given by

\[
(g, h).((X, Y, F, G)) = (gXg^{-1}, hYh^{-1}, gFh^{-1}, hGg^{-1}).
\]

Let us also write \( \mathcal{S}(k, l) \) for the set of isomorphism classes of acyclic sheaves with Hilbert polynomial \( lx + ky \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) which satisfy \((4.1)\). The content of Corollary 2.6 is that there exists a natural bijection

\[
(4.4) \quad \mathcal{M}(k, l)/K \simeq \mathcal{S}(k, l).
\]

4.1. Poisson structure. The vector space \( \text{Mat}_{k,l} \times \text{Mat}_{l,k} \) has a natural \( K \)-invariant symplectic structure: \( \omega = \text{tr}(dF \wedge dG) \). On the other hand, \( \text{Mat}_{k,k} \simeq \mathfrak{gl}_k(\mathbb{C})^* \) and \( \text{Mat}_{l,l} \simeq \mathfrak{gl}_l(\mathbb{C})^* \) have canonical Poisson structures, and therefore \( \mathcal{M}(k, l) \) has a natural \( K \)-invariant Poisson structure. If \( \mathcal{M}(k, l)^0 \) is the subset of \( \mathcal{M}(k, l) \) on which the action of \( K \) is free and proper, then \( \mathcal{M}(k, l)^0/K \) is a Poisson manifold, and, consequently, we obtain a Poisson structure on the corresponding subset of acyclic sheaves with Hilbert polynomial \( lx + ky \) and satisfying \((4.1)\). We shall now describe symplectic leaves of \( \mathcal{M}(k, l)^0/K \) in terms of sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \).

First of all, let us describe sheaves corresponding to symplectic leaves in \( \mathcal{M}(k, l) \). Such a leaf is determined by fixing conjugacy classes of \( X \) and \( Y \). On the other hand, conjugacy classes of \( k \times k \) matrices correspond to isomorphism classes of torsion sheaves on \( \mathbb{P}^1 \), of length \( k \). This correspondence is given by associating to a matrix \( X \in \text{Mat}_{k,k}(\mathbb{C}) \) the sheaf \( \mathcal{G} \) via

\[
(4.5) \quad 0 \to \mathcal{O}(-1)^{\oplus k} \xrightarrow{X-\zeta} \mathcal{O}^{\oplus k} \to \mathcal{G} \to 0.
\]

If, for example, \( X \) is diagonalisable with distinct eigenvalues \( \zeta_1, \ldots, \zeta_r \) of multiplicities \( k_1, \ldots, k_r \), then \( \mathcal{G} \simeq \bigoplus_{i=1}^r \mathbb{C}^{k_i} |_{\zeta_i} \), i.e. \( \mathcal{G}|_{\zeta_i} \) is the skyscraper sheaf of rank \( k_i \).

Proposition 4.2. Let \( P \) be a conjugacy class of \( k \times k \) matrices. The bijection \((4.4)\) induces a bijection between

\[
\mathcal{M}(k, l)/K \simeq \mathcal{S}(k, l).
\]
(i) orbits of $GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\{(X,Y,F,G) \in \mathcal{M}(k,l); \, X \in P\}$ and
(ii) isomorphism classes of sheaves $\mathcal{F}$ in $\mathcal{S}(k,l)$ such that $\mathcal{F}|_{\gamma=\infty}$ is isomorphic to $\mathcal{G}$ defined by (1.5).

**Proof.** At $\eta = \infty$, the matrix (4.2) becomes
\[
\begin{pmatrix}
X - \zeta & 0 \\
G & -1
\end{pmatrix}.
\]
The statement follows from (4.5) and (2.2). \qed

Therefore symplectic leaves on $\mathcal{M}(k,l)$ correspond to fixing isomorphism classes of $\mathcal{F}|_{\gamma=\infty}$ and of $\mathcal{F}|_{\zeta=\infty}$. Symplectic leaves on $\mathcal{M}(k,l)^0/K$ are of course smaller than $K$-orbits of symplectic leaves on $\mathcal{M}(k,l)^0$. They are obtained by fixing $X$ and $Y$ and taking the symplectic quotient of $\text{Mat}_{k,I} \times \text{Mat}_{l,k}$ by $\text{Stab}(X) \times \text{Stab}(Y)$. We shall describe sheaves corresponding to a particular symplectic leaf in the case when $X$ and $Y$ are diagonalisable.

**4.2. Orbits of $GL_k(\mathbb{C})$ and matrix-valued rational maps.** We now consider only the action of $GL_k(\mathbb{C}) \simeq GL_k(\mathbb{C}) \times \{1\} \subset K$ on $\mathcal{M}(k,l)$. We fix a semisimple conjugacy class of $X$; i.e., we suppose that $X$ is diagonalisable, with distinct eigenvalues $\zeta_1, \ldots, \zeta_r$ of multiplicities $k_1, \ldots, k_r$. The stabiliser of $X$ is then isomorphic to $\prod_{i=1}^r GL_{k_i}(\mathbb{C})$. If the action of $GL_k(\mathbb{C})$ is to be free, we must have $k_i \leq l$, $i = 1, \ldots, r$. Let us diagonalise $X$ so that $X$ has the block-diagonal form $(\zeta_1 \cdot 1_{k_1 \times k_1}, \ldots, \zeta_r \cdot 1_{k_r \times k_r})$, and let $F_i, G_i$ denote the $k_i \times l$ and $l \times k_i$ submatrices of $F, G$ such that rows of $F$ and the columns of $G$ have the same coordinates as the block $\zeta_i \cdot 1_{k_i \times k_i}$. The action of $GL_k(\mathbb{C})$ is free and proper at $(X,Y,F,G)$ if and only if $\text{rank } F_i = \text{rank } G_i = k_i$ for $i = 1, \ldots, r$.

As in [15], we can associate to each element of $\mathcal{M}(k,l)$ a $\text{Mat}_{l,l}(\mathbb{C})$-valued rational map:
\[
R(\zeta) = Y + G(\zeta - X)^{-1} F.
\]
The mapping $(X,Y,F,G) \mapsto R(\zeta)$ is clearly $GL_k(\mathbb{C})$-invariant. If $X$ is diagonalisable, as above, i.e. $X = (\zeta_1 \cdot 1_{k_1 \times k_1}, \ldots, \zeta_r \cdot 1_{k_r \times k_r})$, then
\[
R(\zeta) = Y + \sum_{i=1}^r \frac{G_i F_i}{\zeta - \zeta_i}.
\]
We clearly have:

**Lemma 4.3.** Let $P$ be a semisimple conjugacy class of $k \times k$ matrices with eigenvalues $\zeta_1, \ldots, \zeta_r$ of multiplicities $k_1, \ldots, k_r$. The map $(X,Y,F,G) \mapsto R(\zeta)$ induces a bijection between
(i) $GL_k(\mathbb{C})$-orbits on $\{(X,Y,F,G) \in \mathcal{M}(k,l)^0; \, X \in P\}$ and
(ii) the set $\mathcal{R}_l(P)$ of all rational maps of the form
\[
R(\zeta) = Y + \sum_{i=1}^r \frac{R_i}{\zeta - \zeta_i},
\]
where $\text{rank } R_i = k_i$. \qed

**4.3. Orbits of loop groups.** A rational map of the form (4.6) may be viewed as an element of a loop Lie algebra $\tilde{gl}(l)^{-}$, consisting of maps from a circle $S^1$ in $\mathbb{C}$, containing the points $\zeta_i$ in its interior, which extend holomorphically outside $S^1$ (including $\infty$). The group $GL(l)^+$, consisting of smooth maps $g : S^1 \to GL_l(\mathbb{C})$,
extending holomorphically to the interior of $S^1$, acts on $\hat{\mathfrak{gl}}(l)$ by pointwise conjugation, followed by projection to $\mathfrak{gl}(l)$. In particular, if all eigenvalues of $X$ are distinct, then the action is

$$g(\zeta) \cdot \left(Y + \sum_{i=1}^{r} \frac{R_i}{\zeta - \zeta_i} \right) = Y + \sum_{i=1}^{r} \frac{g(\zeta_i) R_i g(\zeta_i)^{-1}}{\zeta - \zeta_i}.$$ 

Therefore, if we fix conjugacy classes of the $R_i$, we obtain an orbit of $\tilde{GL}(l)^+$ in $\hat{\mathfrak{gl}}(l)$. We shall now consider quotients of such orbits by $\text{Stab}(Y)$ and describe which sheaves correspond to elements of such an orbit. Let us give a name to such quotients:

**Definition 4.4.** The quotient of an orbit of $\tilde{GL}(l)^+$ in $\hat{\mathfrak{gl}}(l)$ by $GL_l(\mathbb{C})$ is called a semireduced orbit.

**Remark 4.5.** In the literature (see, e.g., [1]–[5]) a reduced orbit is the symplectic quotient of an orbit by $H_Y = \text{Stab}(Y)$. The $GL_l(\mathbb{C})$-moment map on $\hat{\mathfrak{gl}}(l)$ is identified with $Y + \sum_{i=1}^{r} R_i$ so that a reduced orbit is obtained by fixing the value of $a = \pi(\sum_{i=1}^{r} R_i)$, where $\pi$ is the projection $\mathfrak{gl}_l(\mathbb{C}) \to \mathfrak{gl}_l(\mathbb{C})/h_Y^+$ (with $\perp$ taken with respect to $\text{tr}$), and dividing by $\text{Stab}(a) \subset \text{Stab}(Y)$. Therefore, if $\text{Stab}(Y)$ fixes $a$, then a reduced orbit can be identified with a subset of a semireduced orbit.

Let us, therefore, fix a semireduced orbit of $\tilde{GL}(l)^+$. We choose $r$ distinct points $\zeta_1, \ldots, \zeta_r$ in $\mathbb{C}$. Furthermore, we choose $r + 1$ conjugacy classes $Q_0, Q_1, \ldots, Q_r$ of $l \times l$ matrices. This data determines a semireduced orbit $\Upsilon = \Upsilon(Q_0, \ldots, Q_r)$ of $\tilde{GL}(l)^+$ defined as

$$\Upsilon = \left\{ R(\zeta) = Y + \sum_{i=1}^{r} \frac{R_i}{\zeta - \zeta_i}; \ Y \in Q_0, \ \forall i \geq 1 R_i \in Q_i \right\} / GL_l(\mathbb{C}).$$

Let

$$k_i = \text{rank} Q_i, \ i = 1, \ldots, r, \ \ k = \sum_{i=1}^{r} k_i.$$

In the notation of Lemma [1.3], $\Upsilon \subset R_l(P)$, where $P$ is the semisimple conjugacy class of $k \times k$ matrices with eigenvalues $\zeta_i$ of multiplicities $k_i$.

Thanks to Proposition [4.2], the conjugacy class $P$ determines $\mathcal{F}|_{\eta = \infty}$, which, in the case at hand, is $\square_{i=1}^{r} \mathbb{C}^{k_i}|_{(\zeta_i, \infty)}$. Similarly, $Q_0$ determines the isomorphism class of $\mathcal{F}|_{\zeta = \infty}$. We now discuss the significance of the other conjugacy classes $Q_1, \ldots, Q_r$.

We claim that these classes determine the isomorphism class of $\mathcal{F}|_{p^2 = \infty}$, i.e., of $\mathcal{F}$ restricted to the first order neighbourhood of $\eta = \infty$. This is only to be expected if one thinks in terms of the Mukai-Tyurin-Bottacin Poisson structure; cf. [11].

We again consider the canonical resolution (2.2) of $\mathcal{F}$ with $M(\zeta, \eta)$ given by (4.2). Let $\tilde{\eta} = 1/\eta$ be a local coordinate near $\eta = \infty$ so that

$$M(\zeta, \tilde{\eta}) = \begin{pmatrix} X - \zeta & \tilde{\eta} F \\ G & \tilde{\eta} Y - 1 \end{pmatrix}.$$
Using action \((2.5)\), we can multiply \(M(\zeta, \tilde{\eta})\) on the right by \(\begin{pmatrix} 1 & 0 \\ 0 & (1 - \tilde{\eta}Y)^{-1} \end{pmatrix}\). On the scheme \(\tilde{\eta}^2 = 0\), we have \((1 - \tilde{\eta}Y)^{-1} = 1 + \tilde{\eta}Y\), and so \(M(\zeta, \tilde{\eta})\) becomes
\[
\begin{pmatrix} X - \zeta & \tilde{\eta}F \\ G & -1 \end{pmatrix}.
\]
To describe \(\mathcal{F}|_{\tilde{\eta}^2 = 0}\), it is enough to describe it near each \(\zeta_i\), i.e. to describe \(\mathcal{G}_i = \mathcal{F}|_{U_i \times \{\tilde{\eta}^2 = 0\}}\), where \(U_i\) is an open neighbourhood of \(\zeta_i\) (not containing the other \(\zeta_j\)). The resolution \((2.2)\) of \(\mathcal{F}\) restricted to \(U_i \times \{\tilde{\eta}^2 = 0\}\) becomes
\[
0 \to \mathcal{O}(-2, -1)^{\oplus k_i} \oplus \mathcal{O}(-1, -2)^{\oplus l} \xrightarrow{M_i(\zeta, \tilde{\eta})} \mathcal{O}(-1, 0)^{\oplus (k_i + l)} \to \mathcal{G}_i \to 0,
\]
where
\[
M_i(\zeta, \tilde{\eta}) = \begin{pmatrix} \zeta_i - \zeta & \tilde{\eta}F_i \\ G_i & -1 \end{pmatrix}.
\]
This implies that we have an exact sequence
\[
(4.10) \quad 0 \to \mathcal{O}(-2, -1)^{\oplus k_i} \oplus \mathcal{O}(-1, -2)^{\oplus l} \xrightarrow{M_i(\zeta, \tilde{\eta})} \mathcal{O}(-1, 0)^{\oplus (k_i + l)} \to \mathcal{G}_i \to 0
\]
on \(U_i \times \{\tilde{\eta}^2 = 0\}\). Therefore \(\mathcal{G}_i\) is determined by the \(GL_{k_i}(\mathbb{C})\)-conjugacy class of \(F_i G_i\), which is the same as the \(GL_l(\mathbb{C})\)-conjugacy class of \(G_i F_i\). Lemma \([4.3]\) and formula \((4.7)\) imply that the conjugacy class of \(G_i F_i\) is \(Q_i\). Thus, the conjugacy classes \(Q_1, \ldots, Q_r\), which determine the orbit \((4.8)\), correspond to the isomorphism class of \(\mathcal{G}_i\) restricted to \(U_i \times \{\tilde{\eta}^2 = 0\}\), i.e. to describe \(\mathcal{G}_i\) on \(U_i \times \{\tilde{\eta}^2 = 0\}\), it is enough to describe it near each \(\zeta_i\). Therefore the support of \(\mathcal{G}_i\) is given by \(\det((\zeta_i - \zeta) + \tilde{\eta}F_i G_i) = 0\). In other words, the eigenvalues of \(F_i G_i\) give \(\frac{\zeta - \zeta_i}{\tilde{\eta}}\) at \((\zeta, \tilde{\eta}) = (\zeta_i, 0)\), i.e. the first order neighbourhood of \(\text{supp} \mathcal{F}\) at \((\zeta_i, \infty)\).

Summing up, we have:

**Theorem 4.6.** There exists a natural bijection between elements of the semireduced rational orbit \((4.8)\) of \(GL(l)^+\) in \(\mathfrak{gl}(l)^-\) and isomorphism classes of 1-dimensional acyclic sheaves \(\mathcal{F}\) on \(\mathbb{P}^1 \times \mathbb{P}^1\) such that

(i) The Hilbert polynomial of \(\mathcal{F}\) is \(P_F(x, y) = lx + ky\).

(ii) \((\infty, \infty) \not\in \text{supp} S\), and \(\mathcal{F}|_{\tilde{\eta} = \infty} \simeq \bigoplus_{i=1}^r \mathbb{C}^{k_i}|_{\zeta_i, \infty}\).

(iii) The isomorphism class of \(\mathcal{F}|_{\tilde{\eta} = \infty}\) corresponds to \(Q_0\), as in Proposition \([4.2]\).

(iv) The isomorphism class of \(\mathcal{F}|_{\tilde{\eta} = \infty}\) corresponds to conjugacy classes \(Q_1, \ldots, Q_r\), as described above. \(\square\)

**Remark 4.7.** A variation of this result is probably well known to the integrable systems community (at least when \(\mathcal{F}\) is a line bundle supported on a smooth curve \(S\)). We think it useful, however, to state it in this language and in full generality.

4.4. **Symplectic leaves of \(\mathcal{L}(k,l)^0/K\).** We can finally describe symplectic leaves of \(\mathcal{L}(k,l)\), i.e. sheaves corresponding to a particular symplectic leaf \(L\) in \(\mathcal{L}(k,l)/K\), at least in the case when \(L \subset \mathcal{L}(k,l)^0/K\) and \(X\) and \(Y\) are semisimple. As we have already mentioned in \([4.1]\) a symplectic leaf in \(\mathcal{L}(k,l)^0/K\) is obtained by fixing \(X\) and \(Y\), as well as a coadjoint orbit \(\Lambda \subset \mathfrak{h}^*\) of \(H = \text{Stab}(X) \times \text{Stab}(Y)\). If \(\mu : \text{Mat}_{k,l} \times \text{Mat}_{l,k} \to \mathfrak{h}^*\) is the moment map for \(H\), then the symplectic leaf determined by these data is
\[
(4.11) \quad L = \{(X, Y, F, G) \in \mathcal{L}(k,l)^0; \text{ } X \text{ and } Y \text{ are given, } \mu(F, G) \in \Lambda\}/H.
\]
Let \(X\) be diagonal, written as in \([4.2]\) i.e. \(X = (\zeta_1 \cdot 1_{k_1} \times 1_{k_2}, \ldots, \zeta_r \cdot 1_{k_r} \times 1_{k_r})\), and let \(F_i, G_i, i = 1, \ldots, r\), be the corresponding submatrices of \(F\) and \(G\). Then \(\text{Stab}(X) \simeq \prod_{i=1}^r GL_{k_i}(\mathbb{C})\), and the moment map is the projection of the \(GL_{k_i}(\mathbb{C})\)-moment map,
i.e. \((F,G) \mapsto FG\), onto the Lie algebra of \(\text{Stab}(X)\). In other words, the \(\text{Stab}(X)\)-

moment map can be identified with \([5]\):

\[
\mu_X(F,G) = (F_1G_1, \ldots, F_rG_r).
\]

Similarly, if \(Y\) is diagonal with \(s\) distinct eigenvalues of multiplicities \(l_1, \ldots, l_s\), then we obtain \(l_i \times k\) and \(k \times l_i\) submatrices \(G^i, F^i\). The stabiliser of \(Y\) is isomorphic to \(\prod_{i=1}^s G L_{l_i}(\mathbb{C})\) and the moment map is

\[
\mu_Y(F,G) = (G^1F^1, \ldots, G^sF^s).
\]

Therefore, an orbit \(\Lambda\) corresponds to \(r + s\) conjugacy classes \(\pi_1, \ldots, \pi_r, \rho_1, \ldots, \rho_s\) of \(k_i \times k_i\) matrices for the \(\pi_i\) and \(l_j \times l_j\) matrices for the \(\rho_j\). The leaf \(L\) will be contained in \(\mathcal{M}(k,l)^0/K\) if and only if each conjugacy class consists of matrices of maximal rank \((k_i\) or \(l_j)\). From the discussion in the previous subsection, we immediately obtain:

**Proposition 4.8.** Let \(L\) be a symplectic leaf of the Poisson manifold \(\mathcal{M}(k,l)^0/K\), defined as in \([4,11]\), with semisimple \(X\) and \(Y\). Then the image of \(L\) under the bijection \([4.3]\) consists of isomorphism classes of sheaves \(F\) in \(\mathcal{S}(k,l)\) such that the isomorphism classes of \(F|_{\zeta^2 = \infty}\) and of \(F|_{\eta^2 = \infty}\) are fixed (and determined by \(L\)).

Spelling things out, \(X\) determines \(F|_{\eta^2 = \infty} \cong \bigoplus_{i=1}^r \mathbb{C}^{k_i}|_{(\zeta_i, \infty)}\), and each \(\pi_i, i = 1, \ldots, r\), determines \(F\) restricted to a neighbourhood of \((\zeta_i, \infty)\) in \(\eta^2 = \infty\) via \([4.10]\). Similarly, \(Y\) and the \(\rho_j\) determine \(F|_{\zeta^2 = \infty}\).

**Remark 4.9.** Symplectic leaves of \(\mathcal{M}(k,l)^0/K\) can also be identified with reduced orbits (cf. Remark \([4.5]\) of \(GL(l)^+\) in \(\mathfrak{gl}(l)^-\). Therefore, the last proposition describes sheaves corresponding to a reduced orbit with \(Y\) semisimple. Furthermore, if we view \(\mathcal{M}(k,l)^0/K\) as an open subset of the moduli space of semistable sheaves with Hilbert polynomial \(lx + ky\), then this map is a symplectomorphism between the Mukai-Tyurin-Bottacin symplectic structure, described in the introduction, and the Kostant-Kirillov form on a reduced orbit of a Lie group. For an open dense set where \(F\) is a line bundle on a smooth curve, this follows from results in \([24]\).

Since both symplectic structures extend everywhere, they must be isomorphic everywhere.

**Example 4.10.** If we want \(F\) to be a line bundle over its support, then we must require that all \(k_i\) and all \(l_j\) be equal to \(1\). A symplectic leaf in \(\mathcal{M}(k,l)^0/K\) is now given by fixing diagonal matrices \(X = \text{diag}(\zeta_1, \ldots, \zeta_k)\) and \(Y = \text{diag}(\eta_1, \ldots, \eta_l)\) with all \(\zeta_i\) and all \(\eta_j\) distinct, as well as the diagonal entries of \(FG\) and \(GF\), and quotienting by the group of \((k+l) \times (k+l)\) diagonal matrices (acting as in \([4.3]\)). If the diagonal entries of \(FG\) are fixed to be \(\alpha_1, \ldots, \alpha_k\) and the diagonal entries of \(GF\) are \(\beta_1, \ldots, \beta_l\), then the corresponding subset of \(\mathcal{S}(k,l)\) consists of sheaves \(F\) supported on a 1-dimensional scheme \(S\) such that

\[
S \cap \{\eta^2 = \infty\} = \bigcup_{i=1}^k \left\{ \zeta - \zeta_i = \frac{\alpha_i}{\eta} \right\}, \quad S \cap \{\zeta^2 = \infty\} = \bigcup_{j=1}^l \left\{ \eta - \eta_j = \frac{\beta_j}{\zeta} \right\}
\]

and the rank of \(F\) restricted to \(S \cap \{\eta^2 = \infty\}\) and \(S \cap \{\eta^2 = \infty\}\) is everywhere \(1\).

**Remark 4.11.** We expect that Proposition \([4.8]\) remains true if \(X\) or \(Y\) are not semisimple.
5. Rank \( k \) perturbations

Let us now assume that \( k \leq l \). In [1], the authors consider Hamiltonian flows on a subset \( \mathcal{M} \) of \( \mathcal{M}^0(k,l)/K \), where \( \operatorname{rank} F = \operatorname{rank} G = k \). It is clear from the previous section that a generic symplectic leaf of \( \mathcal{M}^0(k,l)/K \) is not contained in \( \mathcal{M} \). Therefore a flow may leave \( \mathcal{M} \) without becoming singular. Since such Hamiltonian flows on a particular symplectic leaf can be linearised on the Jacobian of a spectral curve, it is interesting to know which points of the (affine) Jacobian are outside \( \mathcal{M} \). We are going to give a very satisfactory answer to this in terms of cohomology of line bundles.

Let us therefore define the following set:

\[
\mathcal{M}(k,l)^1 = \{ M \in \mathcal{M}(k,l) \ ; \ \operatorname{rank} F = \operatorname{rank} G = k \}.
\]

Remark 5.1. The manifold of \( GL_k(\mathbb{C}) \)-orbits in \( \mathcal{M}(k,l)^1 \) with \( X = 0 \) and fixed \( Y \) can be identified with the set \( \{ Y + GF \} \), i.e. with the space of rank \( k \) perturbations of the matrix \( Y \), as considered first by Moser [13] \((k = 2)\) and then by many other authors, in particular Adams, Harnad, Hurtubise, Previato [1,5].

We now ask which acyclic sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) correspond to orbits of \( K = GL_k(\mathbb{C}) \times GL_l(\mathbb{C}) \) on \( \mathcal{M}(k,l)^1 \). We have:

**Proposition 5.2.** Let \( k \leq l \). The bijection of Corollary [2.6] induces a bijection between:

(i) orbits of \( GL_k(\mathbb{C}) \times GL_l(\mathbb{C}) \) on \( \mathcal{M}(k,l)^1 \) and

(ii) isomorphism classes of acyclic sheaves \( \mathcal{F} \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with Hilbert polynomial \( P_F(x,y) = lx + ky \), which satisfy, in addition, [1,1] and

\[
H^0(\mathcal{F}(-1,1)) = 0 \quad \text{and} \quad H^1(\mathcal{F}(1,-1)) = 0.
\]

**Proof.** Consider the short exact sequences

\[
0 \to \mathcal{O}(-1)^{\oplus k} \xrightarrow{(X,-\zeta,G)^T} \mathcal{O}^{\oplus (k+l)} \to W_1 \to 0,
\]

\[
0 \to \mathcal{O}(-1)^{\oplus l} \xrightarrow{(F,Y-Y)^T} \mathcal{O}^{\oplus (k+l)} \to W_2 \to 0.
\]

The condition that \( G \) has rank \( k \) is equivalent to \( W_1 \) being a vector bundle, isomorphic to \( \mathcal{O}(1)^{\oplus k} \oplus \mathcal{O}^{\oplus (l-k)} \). This is equivalent to \( H^0(W_1 \otimes \mathcal{O}(-2)) = 0 \). On the other hand, we claim that the condition that \( F \) has rank \( k \) is equivalent to \( H^1(W_2 \otimes \mathcal{O}(-2)) = 0 \). Indeed, any coherent sheaf on \( \mathbb{P}^1 \) splits into the sum of line bundles \( \mathcal{O}(i) \) and a torsion sheaf [16]. Since \( W_2 \) has a resolution as above, we know that all degrees \( i \) in the splitting are nonnegative and \( F \) has rank \( k \) if and only if all \( i \) are strictly positive, which is equivalent to \( H^1(W_2 \otimes \mathcal{O}(-2)) = 0 \).

We use the above exact sequences to obtain two further resolutions of \( \mathcal{E} = \mathcal{F}(1,1) \):

\[
0 \to \mathcal{O}(-1,0)^{\oplus k} \to \pi_2^* W_2 \to \mathcal{E} \to 0,
\]

\[
0 \to \mathcal{O}(0,-1)^{\oplus l} \to \pi_1^* W_1 \to \mathcal{E} \to 0,
\]

where the maps between the first two terms are given by the embedding in \( \mathcal{O}^{\oplus (k+l)} \) followed by the projection onto the quotients \( W_2, W_1 \). Tensoring \((5.2)\) with \( \mathcal{O}(0,-2) \) shows that \( H^1(W_2(-2)) = 0 \) if and only if \( H^1(\mathcal{E}(0,-2)) = 0 \), i.e. \( H^1(\mathcal{F}(1,-1)) = 0 \). Similarly, tensoring \((5.3)\) with \( \mathcal{O}(-2,0) \) shows that \( H^0(W_1(-2)) = 0 \) if and only if \( H^0(\mathcal{E}(-2,0)) = 0 \), i.e. \( H^0(\mathcal{F}(-1,1)) = 0 \). \(\square\)
Remark 5.3. In the case $k = l$, $H^0(\mathcal{E}(-2,0)) = 0$ implies that $\mathcal{E}(-2,0)$ is acyclic (and similarly, $H^1(\mathcal{E}(0,-2)) = 0$ implies that $\mathcal{E}(0,-2)$ is acyclic). In other words $\mathcal{G} = \mathcal{E}(-1,0)$ satisfies $H^*(\mathcal{G}(-1,0)) = 0$. Furthermore, the resolution \[ (5.3) \]
becomes the following resolution of $\mathcal{G}$:
\[ 0 \to O(-1,-1)^{\oplus k} \to O^k \to \mathcal{G} \to 0. \]
In the case when $S = \text{supp} \mathcal{G}$ is smooth and $\mathcal{G}$ is a line bundle, the corresponding part of $\text{Jac}^{g+k-1}(S)$ and the resolution \[ (5.4) \]
have been considered by Murray and Singer in [15].

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School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom

E-mail address: R.Bielawski@ed.ac.uk

Current address: Institut für Differentialgeometrie, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany

Fakultät für Mathematik, TU Dortmund, D-44221 Dortmund, Germany