A UNICITY THEOREM FOR MEROMORPHIC MAPS
OF A COMPLETE KÄHLER MANIFOLD
INTO \( \mathbb{P}^n(\mathbb{C}) \) SHARING HYPERSURFACES

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Abstract. In this paper, we give a unicity theorem for meromorphic maps of an \( m \)-dimensional complete Kähler manifold \( M \), whose universal covering is a ball in \( \mathbb{C}^m \), into \( \mathbb{P}^n(\mathbb{C}) \), sharing the hypersurfaces in general position in \( \mathbb{P}^n(\mathbb{C}) \), where the maps satisfy a certain growth condition.

1. Introduction

In 1926, R. Nevanlinna proved that two distinct nonconstant meromorphic functions \( f, g \) on the complex plane \( \mathbb{C} \) cannot have the same pre-images for more than four distinct values. The generalization of Nevanlinna’s result to the case of meromorphic maps of a complete Kähler manifold \( M \), whose universal covering is a ball in \( \mathbb{C}^m \), into \( \mathbb{P}^n(\mathbb{C}) \) satisfying a certain growth condition (see condition \( (C_\rho) \) below) and sharing hyperplanes is given by Fujimoto (see [5]). In this paper, by applying the results obtained in [11] by the authors, we extend Fujimoto’s result to the case where the meromorphic maps share hypersurfaces instead of hyperplanes.

Let \( M \) be an \( m \)-dimensional connected Kähler manifold with Kähler form \( \omega \) and let \( f \) be a meromorphic map of \( M \) into \( \mathbb{P}^n(\mathbb{C}) \). For \( \rho \geq 0 \), we say that \( f \) satisfies condition \( (C_\rho) \) if there exists a nonzero bounded continuous real-valued function \( h \) on \( M \) such that

\[
\rho \Omega_f + dd^c \log h^2 \geq \text{Ric}(\omega),
\]

where \( \Omega_f \) denotes the pull-back of the Fubini-Study metric form on \( \mathbb{P}^n(\mathbb{C}) \) by \( f \),

\[
d = \partial + \bar{\partial}, \quad d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial), \quad \text{and} \quad \text{Ric}(\omega) \text{ is the Ricci form of } \omega.
\]

The main theorem in this paper is as follows.

Main Theorem. Let \( M \) be a complete connected Kähler manifold whose universal covering is biholomorphic to either \( \mathbb{C}^m \) or the unit ball in \( \mathbb{C}^m \). Let \( f \) and \( g \) be algebraically nondegenerate meromorphic maps of \( M \) into \( \mathbb{P}^n(\mathbb{C}) \). Assume that \( f \) and \( g \) satisfy condition \( (C_\rho) \) and that there exist \( q \) hypersurfaces \( D_j, j = 1, \ldots, q \), of
degree $d_j$ located in general position in $\mathbb{P}^n(\mathbb{C})$ such that

\begin{enumerate}[(i)]
  \item $f = g$ on $\bigcup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j))$,
  \item $q > n + 1 + \frac{2n(L - 1) + 1}{d} + \frac{\rho L(L - 1)}{d}$,
\end{enumerate}

where $L = \left( \frac{N + n}{n} \right)$ with $N = 2d^2(d + 1)n(n + 1)(2^n - 1) + nd$, and $d = \text{lcm}\{d_1, d_2, ..., d_q\}$. Then $f \equiv g$.

From the proof we will easily see that if, in addition, we assume that $\dim(f^{-1}(D_i) \cap g^{-1}(D_j)) \leq m - 2$ for $i \neq j$, $1 \leq i, j \leq q$, we have the following result.

**Theorem 1.1.** Let $M$ be a complete connected Kähler manifold whose universal covering is biholomorphic to either $\mathbb{C}^m$ or the unit ball in $\mathbb{C}^m$. Let $f$ and $g$ be algebraically nondegenerate meromorphic maps of $M$ into $\mathbb{P}^n(\mathbb{C})$. Assume that $f$ and $g$ satisfy condition $(C_p)$ and that there exist $q$ hypersurfaces $D_j$, $j = 1, ..., q$, of degree $d_j$ located in general position in $\mathbb{P}^n(\mathbb{C})$ such that

\begin{enumerate}[(i)]
  \item $f = g$ on $\bigcup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j))$,
  \item $\dim(f^{-1}(D_i) \cap f^{-1}(D_j)) \leq m - 2$, for $i \neq j, 1 \leq i, j \leq q$,
  \item $q > n + 1 + \frac{2(L - 1) + 1}{d} + \frac{\rho L(L - 1)}{d}$,
\end{enumerate}

where $L = \left( \frac{N + n}{n} \right)$ with $N = 2d^2(d + 1)n(n + 1)(2^n - 1) + nd$, and $d = \text{lcm}\{d_1, d_2, ..., d_q\}$. Then $f \equiv g$.

In the case $M = \mathbb{C}^m$, we can take the flat metric whose Ricci form vanishes. Therefore all meromorphic maps of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ satisfy condition $(C_0)$, and we get the following version of the result of Dulock and Ru in \cite{12}.

**Corollary 1.2.** Let $f$ and $g$ be algebraically nondegenerate meromorphic maps of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Assume that there exist $q$ hypersurfaces $D_j$, $j = 1, ..., q$, of degree $d_j$ located in general position in $\mathbb{P}^n(\mathbb{C})$ such that

\begin{enumerate}[(i)]
  \item $f = g$ on $\bigcup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j))$,
  \item $q > n + 1 + \frac{2n(L - 1) + 1}{d}$,
\end{enumerate}

where $L = \left( \frac{N + n}{n} \right)$ with $N = 2d^2(d + 1)n(n + 1)(2^n - 1) + nd$, and $d = \text{lcm}\{d_1, d_2, ..., d_q\}$. Then $f \equiv g$.

In the case when $M$ is a ball in $\mathbb{C}^m$ but without the growth assumption for $f$ and $g$ (i.e. we do not assume that both $f$ and $g$ satisfy condition $(C_p)$), we also have the following result.
**Theorem 1.3.** Let \( f \) and \( g \) be algebraically nondegenerate meromorphic maps of the unit ball \( B(1) \subset \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \). Assume that, for a fixed \( r_0 \) with \( 0 < r_0 < 1 \),
\[
\lambda := \lim_{r \to 1} \sup \frac{\log(1/(1-r))}{T_f(r, r_0) + T_g(r, r_0)} < \infty.
\]
Assume that there exist \( q \) hypersurfaces \( D_j, j = 1, \ldots, q \), of degree \( d_j \) located in general position in \( \mathbb{P}^n(\mathbb{C}) \) such that
\[
(i) \quad f = g \quad \text{on} \quad \bigcup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j)),
\]
\[
(ii) \quad q > n + 1 + \lambda L(L-1) + \frac{2n(L-1)+1}{d},
\]
where \( L = \left( \frac{N+n}{n} \right) \) with \( N = 2d^2(d+1)n(n+1)(2^n-1) + nd \), and \( d = \text{lcm}\{d_1, d_2, \ldots, d_q\} \). Then \( f \equiv g \).

2. **Preliminaries**

Let \( h \) be a nonconstant holomorphic function on an open domain \( G \subset \mathbb{C}^m \). For a set \( \alpha = (\alpha_1, \ldots, \alpha_m) \) of integers \( \alpha_i \geq 0 \), we set \( |\alpha| = \alpha_1 + \cdots + \alpha_m \) and \( D^\alpha h = D_1^{\alpha_1} \cdots D_m^{\alpha_m} h \), where \( D_1 h = (\partial/\partial z_1)h \), for \( i = 1, \ldots, m \). We define \( \nu^0_h : G \to \mathbb{Z} \) by
\[
\nu^0_h(z) := \max\{k : D^\alpha h(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < k\} \quad (z \in G).
\]
By a divisor on a domain \( G \) in \( \mathbb{C}^m \) we mean a map \( \nu \) of \( G \) into \( \mathbb{Z} \) such that, for each \( z_0 \in G \), there are nonzero holomorphic functions \( h \) and \( g \) on a connected neighborhood \( U(\subset G) \) of \( z_0 \) so that \( \nu(z) = \nu^0_h(z) - \nu^0_g(z) \) for each \( z \in U \) outside an analytic set of dimension \( \leq m - 2 \). Two divisors are regarded as the same if they are identical outside an analytic set of dimension \( \leq m - 2 \). Take a nonzero meromorphic function \( \varphi \) on a domain \( G \) in \( \mathbb{C}^m \). For each \( z_0 \in G \), we choose nonzero meromorphic functions \( g \) and \( h \) on a neighborhood \( U(\subset G) \) of \( z_0 \) such that \( \varphi = \frac{g}{h} \) on \( U \) and \( \dim(g^{-1}(0) \cup h^{-1}(0)) \leq m - 2 \). We define \( \nu^\infty_{\varphi} := \nu^0_h, \quad \nu^\varphi_{\varphi} := \nu^0_{g-ah} \) for \( a \in \mathbb{C} \) and \( \nu^\varphi_{\varphi} = \nu^\varphi_{\varphi} - \nu^\infty_{\varphi} \), which are independent of the choices of \( h \) and \( g \).

Let \( f \) be a meromorphic map of \( B(0) \subset \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \), \( 0 < R_0 \leq \infty \). We take holomorphic functions \( f_0, f_1, \ldots, f_n \) such that \( I_f := \{z \in B(0) : f_0(z) = \cdots = f_n(z) = 0\} \) is of dimension at most \( m - 2 \) and \( f(z) = (f_0(z) : \cdots : f_n(z)) \) on \( B(0) - I_f \) in terms of homogeneous coordinates \((w_0 : \cdots : w_m)\) on \( \mathbb{P}^n(\mathbb{C}) \). We call such representation \( f(z) = (f_0(z) : \cdots : f_n(z)) \) a reduced representation of \( f \). For \( z = (z_1, \ldots, z_m) \in \mathbb{C}^m \) we set \( ||z|| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2} \) and define \( B(r) = \{z \in \mathbb{C}^m : ||z|| < r\} \), \( S(r) = \{z \in \mathbb{C}^m : ||z|| = r\} \) for \( 0 < r \leq +\infty \), where we mean \( B(\infty) = \mathbb{C}^m \) and \( S(\infty) = \emptyset \). Define
\[
\sigma_m := d^m \log ||z||^2 \wedge (dd^c \log ||z||^2)^{m-1} \quad \text{on } \mathbb{C}^m - \{0\} \quad \text{and} \quad v_l := (dd^c ||z||^2)^l, \quad 0 \leq l \leq m.
\]
The pullback of the normalized Fubini-Study metric form \( \Omega \) on \( \mathbb{P}^n(\mathbb{C}) \) by \( f \) is given by
\[
\Omega_f = dd^c \log ||f||^2,
\]
where \( \|f\|^2 := |f_0|^2 + \cdots + |f_n|^2 \). Fix \( r_0 < R_0 \); the characteristic function of \( f \) is defined by

\[
T_f(r, r_0) = \int_{r_0}^{r} \frac{dt}{t^{2m-1}} \int_{B(t)} \Omega_f \wedge v_{m-1} \quad (0 < r_0 < r < R_0).
\]

We then have (see [13])

\[
(2.1) \quad T_f(r, r_0) = \int_{r_0(r)} \log \|f\|\sigma_m - \int_{S(r_0)} \log \|f\|\sigma_m.
\]

Let \( \mu_0 \) be a positive integer or \( \infty \) and \( \nu \) be divisors on a domain \( B(R_0) \subset \mathbb{C}^m \). Set \( |\nu| = \{ z \in B(R_0): \nu(z) \neq 0 \} \). We define the counting function of \( \nu \) truncated by \( \mu_0 \) by

\[
N_\nu^{[\mu_0]}(r, r_0) = \int_{r_0}^{r} \frac{n^{[\mu_0]}(t)}{t^{2m-1}} dt,
\]

where

\[
n^{[\mu_0]}(t) = \int_{|\nu| \cap B(t)} \min\{ \nu, \mu_0 \} v_{m-1} \quad \text{for } m \geq 2 \quad \text{and} \quad n^{[\mu_0]}(t) = \sum_{|z| \leq t} \min\{ \nu(z), \mu_0 \} \quad \text{for } m = 1.
\]

For a meromorphic function \( \phi \) on \( B(R_0) \), we may regard \( \phi \) as a meromorphic map into \( \mathbb{P}^1(\mathbb{C}) \). We define

\[
N_\phi^0(r, r_0) = \int_{r_0}^{r} \frac{n^0_\phi(t)}{t^{2m-1}} dt,
\]

where

\[
n^0_\phi(t) = \int_{|\nu| \cap B(t)} \nu^0_\phi v_{m-1} \quad \text{for } m \geq 2 \quad \text{and} \quad n^0_\phi(t) = \sum_{|z| \leq t} \nu^0_\phi(z) \quad \text{for } m = 1.
\]

Let \( f \) and \( g \) be algebraically nondegenerate meromorphic maps from \( B(R_0) \subset \mathbb{C}^m \) \((0 < R_0 \leq \infty)\) into \( \mathbb{P}^n(\mathbb{C}) \). For a positive integer \( N \), let \( V_N \) denote the space of homogeneous polynomials of degree \( N \) in \( \mathbb{C}[X_0, \ldots, X_n] \) and fix a (arbitrary) basis \( \phi_1, \ldots, \phi_l \), where \( l = \dim V_N \). Let

\[
(2.2) \quad F = [\phi_1(f) : \cdots : \phi_l(f)] \quad \text{and} \quad G = [\phi_1(g) : \cdots : \phi_l(g)].
\]

Then \( F, G: B(R_0) \to \mathbb{P}^{l-1}(\mathbb{C}) \) are linearly nondegenerate. Hence (see Proposition 3.2 in [11]) there exist \( \alpha_j = (\alpha_{j1}, \ldots, \alpha_{jm}) \) (resp. \( \beta_j = (\beta_{j1}, \ldots, \beta_{jm}) \)) with \( \alpha_{ji} \geq 0 \) being integers (resp. \( \beta_{ji} \geq 0 \) being integers), \( |\alpha| := \alpha_{j1} + \cdots + \alpha_{jm} \leq l-1 \) for \( 1 \leq j \leq l \) (resp. \( |\beta| \leq l-1 \) for \( 1 \leq j \leq l \)), and \( |\alpha| + \cdots + |\alpha| \leq l(l-1)/2 \) (resp. \( |\beta| + \cdots + |\beta| \leq l(l-1)/2 \)) such that both

\[
(2.3) \quad W_{\alpha_1, \ldots, \alpha_l}(F) := \det \left( D^{\alpha_1} F, \ldots, D^{\alpha_l} F \right), \quad W_{\beta_1, \ldots, \beta_l}(G) := \det \left( D^{\beta_1} G, \ldots, D^{\beta_l} G \right)
\]

are not identically zero on \( B(R_0) \). Let \( D_1, \ldots, D_q \) be hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) of degree \( d_1, \ldots, d_q \), located in general position. Let \( Q_j, 1 \leq j \leq q \), be the homogeneous polynomials defining \( D_j \). Replacing \( Q_j \) by \( Q_j^{d/d_j} \) if necessary, where \( d = \text{lcm}\{d_1, \ldots, d_q\} \), we can assume that \( Q_1, \ldots, Q_q \) have the same degree \( d \).
Choose distinct $\gamma_{i_1}, \ldots, \gamma_{i_n} \in \{Q_1, \ldots, Q_q\}$. Arrange the \(n\)-tuples by lexicographic order and consider the \(n\)-tuples \(i = (i_1, \ldots, i_n)\) of nonnegative integers with \(\sigma(i) := i_1 + \cdots + i_n \leq N/d\). Define the spaces
\[
W_i = W_{N,i} := \sum_{\{\epsilon\} \geq i} \gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_n}^{\epsilon_n} V_{N-d\sigma(i)}.
\]
Clearly, \(W_{(0,\ldots,0)} = V_N, W_i \supset W_{i'}\) if \(i' \geq i\), and \(\{W_i\}\) defines a filtration of \(V_N\). Let
\[
\Delta_i := \dim \frac{W_i}{W_{i'}},
\]
where \(i' > i\) are consecutive \(n\)-tuples with \(W_{i'} \subset W_i\). Then it is known (see Lemma 4.1 in [11]) that each \(\Delta_i\) is independent of the choice of \(\gamma_{i_1}, \ldots, \gamma_{i_n}\). Hence
\[
\Delta := \frac{1}{n} \sum_i \Delta_i \sigma(i)
\]
is a positive integer depending only on \(D_1, \ldots, D_q\). The following lemma gives the estimate of \(\Delta\).

**Lemma 2.1.** With \(N = 2d^2(d + 1)n(n + 1)(2^n - 1) + nd\) and \(L = \binom{N}{n}\), we have
\[
\frac{LN}{\Delta} \leq d(n + 1) + \frac{1}{2d}.
\]

The proof of the above lemma is given by the same argument by setting \(\epsilon = 1/d\) as in the proof of [11] Lemma 4.3.

**Proposition 2.2 ([11] Proposition 4.4).** Set
\[
\psi = \frac{W_{\alpha^1 \cdots \alpha^l}(F)}{Q^\Delta_1(f) \cdots Q^\Delta_q(f)}.
\]
Then
\[
\nu^\infty_{\psi} \leq \sum_{j=1}^q \Delta \min\{\nu^\infty_{Q_j(f)}, l - 1\}
\]
outside an analytic set of codimension at least two.

**Proposition 2.3** (See (4.21) on page 1158 in [11]). Set \(l_0 = |\alpha^1| + \cdots + |\alpha^l|\) and take \(t, p'\) with \(0 < t_0 < p' < 1\). Then, for fixed \(r_0\) with \(0 < r_0 < R_0\) there exists a positive constant \(K\) such that for \(r_0 < r < R < R_0\),
\[
\int_{S(r)} \left| z^{\alpha^1 + \cdots + \alpha^l} \frac{W_{\alpha^1 \cdots \alpha^l}(f)}{Q^\Delta_1(f) \cdots Q^\Delta_q(f)} \right|^t \|f\|^{(dq\Delta - lN)} \sigma_m \leq K \left( \frac{R^{2m-1}}{R - r} T_F(R, r_0) \right)^{p'}.
\]

**Theorem 2.4.** Let \(f : B(R_0) \rightarrow \mathbb{P}^n(\mathbb{C}), 0 < R_0 \leq \infty\), be an algebraically nondegenerate meromorphic map and \(D_1, \ldots, D_q\) be hypersurfaces of degree \(d_j, 1 \leq j \leq q\), in \(\mathbb{P}^n(\mathbb{C})\) located in general position. Then, with \(N = 2d^2(d + 1)n(n + 1)(2^n - 1) + nd\) and \(L = \binom{N}{n}\),
\[
(q - (n + 1 + (1/d))) T_f(r, r_0) \leq \sum_{j=1}^q d_j^{-1} N_j^{[L-1]} (r, D_j) + S_f(r),
\]
where
\[
\Delta_j = \frac{N_j}{n}.
\]
where $S_f(r)$ is evaluated as follows:

1. In the case $R_0 < \infty$,
   \[
   S_f(r) \leq \frac{L(L-1)}{2} \log^+ \frac{1}{R_0 - r} + K \log^+ T_f(r, r_0)
   \]
   for every $r \in [0, R_0)$ excluding a set $E$ with $\int_E \frac{1}{R_0 - t} dt < \infty$, where $K$ is a positive constant.

2. In the case $R_0 = \infty$,
   \[
   S_f(r) \leq K(\log^+ T_f(r, r_0) + \log r)
   \]
   for every $r \in [0, \infty)$ excluding a set $E'$ with $\int_{E'} dt < \infty$, where $K$ is a positive constant.

**Remark 2.5.** If $R_0 = \infty$ and $\lim_{r \to \infty} T_f(r, r_0)/\log r < \infty$, then we can choose $S_f(r)$ to be bounded.

**Proof.** The proof is given by the same argument by taking $\epsilon = 1/d$ as in the proof of [11, Theorem 4.5] except for the following modifications in the case $R_0 > +\infty$:

From Proposition 2.3, for $0 < tl_0 < p' < 1$, we have

\begin{equation}
\int_{S(r)} z^{\alpha_1 + \cdots + \alpha_L} \frac{W_{\alpha_1, \ldots, \alpha_L}(F)}{Q_1^\alpha(f) \cdots Q_q^\Delta(f)} t^\ell \|f\|^{\ell(dq - LN)} \sigma_m \leq K \left( \frac{R^{2m-1}}{R - r} T_F(R, r_0) \right)^{p'},
\end{equation}

for $r_0 < r < R < R_0$. Hence, by virtue of the concavity of the logarithm, the above inequality implies that

\[
\int_{S(r)} \log |z^{\alpha_1 + \cdots + \alpha_L}| \sigma_m + \int_{S(r)} \log \left| \frac{W_{\alpha_1, \ldots, \alpha_L}(F)}{Q_1^\alpha(f) \cdots Q_q^\Delta(f)} \right| \sigma_m \\
+ (dq \triangle - NL) \int_{S(r)} \log \|f\| \sigma_m + \leq \frac{p'}{t} \left( \log \frac{1}{R - r} + \log^+ T_F(R, r_0) \right) + O(1),
\]

for $r_0 < r < R < R_0$. Since $l_0 = |\alpha_1| + \cdots + |\alpha_L| \leq L(L-1)/2$, by letting $(p'/t) > l_0$ approach $l_0$, we get

\[
\int_{S(r)} \log |z^{\alpha_1 + \cdots + \alpha_L}| \sigma_m + \int_{S(r)} \log \left| \frac{W_{\alpha_1, \ldots, \alpha_L}(F)}{Q_1^\alpha(f) \cdots Q_q^\Delta(f)} \right| \sigma_m \\
+ (dq \triangle - NL) \int_{S(r)} \log \|f\| \sigma_m \leq \frac{L(L-1)}{2} \left( \log \frac{1}{R - r} + \log^+ T_F(R, r_0) \right) + O(1).
\]

The rest of the arguments follow in the proof of [11, Theorem 4.5].

\[\square\]

### 3. The proof of the Main Theorem for particular cases

In this section, we prove Corollary 1.2 and Theorem 1.3 stated in section 1.

**Proof of Corollary 1.2** Assume that $f \not\equiv g$. Fix a reduced representation $(f_0, f_1, \ldots, f_n)$ of $f$ and a reduced representation $(g_0, g_1, \ldots, g_n)$ of $g$. We may choose distinct indices $i_0$ and $j_0$ such that

\begin{equation}
\chi := f_{i_0} g_{j_0} - f_{j_0} g_{i_0}
\end{equation}
is not identically zero. Applying Theorem 2.4 to the maps \( f \) and \( g \) (without loss of generality, we assume that \( d_1 = \cdots = d_q = d \)), we get

\[
\left( q - (n + 1 + (1/d)) \right) T_f(r, r_0) \leq \sum_{j=1}^{q} d^{-1} N_f^{[L-1]}(r, D_j) + S_f(r),
\]

\[
\left( q - (n + 1 + (1/d)) \right) T_g(r, r_0) \leq \sum_{j=1}^{q} d^{-1} N_g^{[L-1]}(r, D_j) + S_g(r),
\]

where \( S_f(r) \) and \( S_g(r) \) satisfy

\[
S_f(r) \leq_{exc} K(\log^+ T_f(r, r_0) + \log r) \quad \text{and} \quad S_g(r) \leq_{exc} K(\log^+ T_g(r, r_0) + \log r),
\]

where \( \leq_{exc} \) means the inequality holds for all \( r \in [0, \infty) \) excluding a set \( E' \) with \( \int_{E'} dt < \infty \). Adding these two inequalities gives

\[
\left( q - (n + 1 + (1/d)) \right) \left( T_f(r, r_0) + T_g(r, r_0) \right) \leq \frac{1}{d} \sum_{j=1}^{q} (N_f^{[L-1]}(r, D_j) + N_g^{[L-1]}(r, D_j)) + S_g(r) + S_f(r).
\]

Using the fact that the \( D_j \)'s are in general position, from assumption (i) in Corollary 1.2 we have trivially \( \sum_{j=1}^{q} (N_f^{[L-1]}(r, D_j) + N_g^{[L-1]}(r, D_j)) \leq 2n(L-1)N(r, A) \), where \( A = \bigcup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j)) \). Hence,

\[
(3.2) \quad \left( q - (n + 1 + (1/d)) \right) \left( T_f(r, r_0) + T_g(r, r_0) \right) \leq \frac{2n(L-1)}{d} N(r, A) + S_g(r) + S_f(r).
\]

We now claim that

\[
N(r, A) \leq T_f(r_0, r) + T_g(r_0, r) + O(1).
\]

Indeed, consider the map \( \chi := f_i g_{j_0} - f_{j_0} g_i \) defined above. If \( z \in A \), then \( f(z) = g(z) \) from assumption (i) in Corollary 1.2 and so \( \chi(z) = 0 \). It then follows that \( N(r, A) \leq N^0_\chi(r, r_0) \). By the First Main Theorem of Nevanlinna,

\[
N^0_\chi(r, r_0) \leq T_\chi(r, r_0) + O(1) \leq T_f(r, r_0) + T_g(r, r_0) + O(1).
\]

The claim then follows.

Therefore (3.2) gives

\[
\left( q - (n + 1 + (1/d)) \right) \left( T_f(r, r_0) + T_g(r, r_0) \right) \leq \frac{2n(L-1)}{d} \left( T_f(r, r_0) + T_g(r, r_0) \right) \]

or

\[
(3.3) \quad \left( q - (n + 1 + (1/d)) - \frac{2n(L-1)}{d} \right) \leq \frac{S_g(r) + S_f(r)}{T_f(r, r_0) + T_g(r, r_0)}.
\]

In the case that \( f, g \) are rational, then, as in Remark 2.5 \( S_f(r) \) and \( S_g(r) \) can be taken to be bounded. Thus

\[
\lim_{r \to \infty} \frac{S_f(r) + S_g(r)}{T_f(r, r_0) + T_g(r, r_0)} = 0,
\]
and hence \( q \leq n + 1 + \frac{2n(L-1)+1}{d} \), which leads to a contradiction. If either \( f \) or \( g \) is transcendental, then
\[
\lim_{r \to \infty} \frac{\log r}{T_f(r, r_0) + T_g(r, r_0)} = 0.
\]
On the other hand,
\[
S_f(r) \leq_{exc} K(\log^+ T_f(r, r_0) + \log r) \quad \text{and} \quad S_g(r) \leq_{exc} K(\log^+ T_g(r, r_0) + \log r),
\]
so
\[
\lim_{r \to \infty} \inf \frac{S_f(r) + S_g(r)}{T_f(r, r_0) + T_g(r, r_0)} = 0.
\]
This again implies that \( q \leq n + 1 + \frac{2n(L-1)+1}{d} \), which leads to a contradiction. This completes the proof of Corollary 1.2.

**Proof of Theorem 1.3** The proof is similar to the above argument. Assume that \( f \not\equiv g \). Conclusion (1) of Theorem 2.4 implies that there exists a subset \( E \) of \([0,1)\) such that
\[
\int_E (1-r)^{-1} dr < \infty \quad \text{and, for every } r \not\in E,
\]
\[
S_f(r) + S_g(r) \leq L(L-1) \log(1/(1-r)) + K \log^+(T_f(r, r_0) + T_g(r, r_0)).
\]
From this and (3.3), we conclude that
\[
(q - (n + 1 + (1/d)) - \frac{2n(L-1)}{d}) \leq L(L-1) \liminf_{r \to 1, r \not\in E} \frac{\log(1/(1-r)) + K \log^+(T_f(r, r_0) + T_g(r, r_0))}{T_f(r, r_0) + T_g(r, r_0)}
\]
\[
\leq L(L-1) \limsup_{r \to 1} \frac{\log(1/(1-r))}{T_f(r, r_0) + T_g(r, r_0)} \leq \lambda L(L-1).
\]
Thus
\[
q \leq n + 1 + \frac{2n(L-1)+1}{d} + \lambda L(L-1),
\]
which contradicts assumption (ii) of Theorem 1.3. This completes the proof.

**Remark 3.1.** As is easily seen from the above proof, the quantity \( \lambda \) in the conclusion of Theorem 1.3 can be replaced by the least upper bound of the quantities \( \bar{\lambda} \) such that
\[
\bar{\lambda} = \liminf_{r \to 1, r \not\in E} \frac{\log(1/(1-r))}{T_f(r, r_0) + T_g(r, r_0)}
\]
for some subset \( E \) of \([0,1)\) with \( \int_E (1-r)^{-1} dr < \infty \).

4. **Proof of the Main Theorem**

**Proof.** We now proceed to prove the Main Theorem. By lifting \( f \) and \( g \) to the universal covering of \( M \) if necessary, we may assume that \( M = B(R_0)(\subset \mathbb{C}^m) \) with \( 0 < R_0 \leq \infty \). The case when \( R_0 = \infty \) (i.e. \( M = \mathbb{C}^m \)) is nothing but Corollary 1.2 and so it suffices to study the case \( M = B(1) \). Moreover, by virtue
of Remark 3.1 the Main Theorem is true unless there exists a subset $E$ of $[0,1)$ with $\int_{E} (1 - r)^{-1} dr < \infty$ and

$$\limsup_{r \to 1, r \notin E} \frac{T_f(r, r_0) + T_g(r, r_0)}{\log(1/(1 - r))} < \infty.$$  

We note that by the same argument as in the proof of [4, Proposition 5.5], (4.1) indeed implies that there exists a positive constant $K$ such that

$$T_f(r, r_0) + T_g(r, r_0) \leq K \log \frac{1}{1 - r} \quad (0 < r_0 \leq r < 1).$$

Assuming (4.2), we shall show that it will lead to a contradiction if $f \not\equiv g$.

Let $F, G$ be the meromorphic maps of $B(1) \subset \mathbb{C}^m$ into $\mathbb{P}^{L-1}(\mathbb{C})$ defined in (2.2). Let $\alpha^j$ and $\beta^j, j = 1, \ldots, L$, be $m$-tuples of nonnegative integers with $|\alpha^j| \leq L - 1$, $|\beta^j| \leq L - 1$ for $1 \leq j \leq L$, and $|\alpha^1| + \cdots + |\alpha^L| \leq L(L - 1)/2$, $|\beta^1| + \cdots + |\beta^L| \leq L(L - 1)/2$ such that $W_{\alpha^1, \ldots, \alpha^L}(F) \neq 0$ and $W_{\beta^1, \ldots, \beta^L}(G) \neq 0$, where $W_{\alpha^1, \ldots, \alpha^L}(F)$ and $W_{\beta^1, \ldots, \beta^L}(G)$ are defined in (2.3). Set $\phi := z^{\alpha^1+\cdots+\alpha^L} \Phi$ and $\psi := z^{\beta^1+\cdots+\beta^L} \Psi$, where

$$\Phi = \frac{W_{\alpha^1, \ldots, \alpha^L}(F)}{Q_1^\Delta(f) \cdots Q_q^\Delta(f)} \quad \text{and} \quad \Psi = \frac{W_{\beta^1, \ldots, \beta^L}(G)}{Q_1^\Delta(g) \cdots Q_q^\Delta(g)}.$$  

We fix a reduced representation $(f_0, f_1, \ldots, f_n)$ of $f$ and a reduced representation $(g_0, g_1, \ldots, g_n)$ of $g$. Since $f \not\equiv g$, we may choose distinct indices $i_0$ and $j_0$ such that

$$\chi := f_{i_0}g_{j_0} - f_{j_0}g_{i_0} \neq 0.$$  

Notice that for a point $p \in f^{-1}(D_j) \subset \bigcup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j))$, $f(p) = g(p)$. Thus $\chi(p) = 0$. Furthermore, from Proposition 2.2 by noticing that at $p \in B(1)$ there are at most $n Q_j$’s with $Q_j(f)(p) = 0$ due to the assumption that $D_1, \ldots, D_q$ are in general position, we have $\nu_{\phi}^\Delta \leq \Delta n(L - 1)\nu_{\phi}^0$ and $\nu_{\psi}^\Delta \leq \Delta n(L - 1)\nu_{\psi}^0$, outside an analytic set of codimension $\geq 2$. It then follows that, outside an analytic set of codimension $\geq 2$, the functions $\phi^\Delta \chi^{\Delta n(L-1)}$ and $\psi^\Delta \chi^{\Delta n(L-1)}$ are both holomorphic on $B(1)$. Therefore if we let

$$t := \frac{\rho}{dq \Delta - LN - 2 \Delta n(L - 1)}$$  

and

$$u := t \log |\phi^\Delta \psi^\Delta|^2 \Delta n(L-1),$$

then $u$ is plurisubharmonic on $B(1)$ outside an analytic set of codimension $\geq 2$. From the assumption that both $f$ and $g$ satisfy condition $(C_p)$, there are continuous plurisubharmonic functions $u_1, u_2$ on $B(1)$ such that

$$e^{u_1} \det(h_{ij})^{1/2} \leq ||f||^\rho,$$

$$e^{u_2} \det(h_{ij})^{1/2} \leq ||g||^\rho.$$  

Since $\rho + 2 \Delta t n(L - 1) = t(dq \Delta - LN)$ and noticing that $|\chi| \leq 2 ||f|| ||g||$, we obtain

$$\det(h_{ij})e^{u_1 + u_1 + u_2} \leq ||\phi^t \psi^t||^2 \chi^{2\Delta t n(L-1)} ||f||^\rho ||g||^\rho$$

$$\leq K ||\phi^t \psi^t||^\rho ||f||^{\rho + 2n t n(L - 1)} ||g||^{\rho + 2n t n(L - 1)}$$

$$= K ||\phi^t \psi^t||^\rho ||f||^{t(dq \Delta - LN)} ||g||^{t(dq \Delta - LN)}$$
for some constant $K$. Note that the volume form on $B(1)$ is given by

$$dV := c_m \det(h_{ij}) v_m;$$

therefore,

$$\int_{B(1)} e^{u+u_1+u_2} dV \leq K \int_{B(1)} \|\tilde{\phi}\|^{t(dq\triangle - LN)} \|f\|^{t(dq\triangle - LN)} v_m.$$

Thus, by the Hölder inequality and by noticing that

$$v_m = (dd^c\|z\|^2)^m = 2m\|z\|^{2m-1} \sigma_m \land d\|z\|,$$

we obtain

$$\int_{B(1)} e^{u+u_1+u_2} dV \leq K \left( \int_0^1 r^{2m-1} \left( \int_{S(r)} \|\tilde{\phi}\|^{2t(dq\triangle - LN)} \|f\|^{2t(dq\triangle - LN)} \sigma_m \right) dr \right)^{1/2} \times \left( \int_0^1 r^{2m-1} \left( \int_{S(r)} \|\tilde{\psi}\|^{2t(dq\triangle - LN)} \|g\|^{2t(dq\triangle - LN)} \sigma_m \right) dr \right)^{1/2}.$$  

From Lemma 2.1

$$-\frac{LN}{\Delta} \geq -d(n+1) - \frac{1}{2d}.$$  

Therefore, together with assumption (ii) in the Main Theorem, we have

$$dq \triangle - LN - 2 \Delta n(L - 1) = \Delta \left( dq - \frac{LN}{\Delta} - 2n(L - 1) \right) \geq \Delta \left( dq - d(n+1) - \frac{1}{2d} - 2n(L - 1) \right) = d \Delta \left( q - (n+1) - \frac{1}{2d^2} - \frac{2n(L - 1)}{d} \right) > d \Delta \left( \frac{pL(L - 1)}{d} \right) = \Delta \rho L(L - 1),$$

and thus

$$(2t) \cdot L(L - 1)/2 = tL(L - 1) = \frac{L(L - 1) \rho}{dq \triangle - LN - 2 \Delta n(L - 1)} < \frac{1}{\Delta} < 1.$$  

Take some $p'$ with $0 < tL(L - 1) < p' < 1$. Then, since $0 < 2t(|a^1| + \cdots + |a^L|) \leq tL(L - 1) < p' < 1$, it follows from Proposition 2.3 that for $r_0 < r < R < 1$,

$$\int_{S(r)} \|\tilde{\phi}\|^{2t(dq\triangle - LN)} \sigma_m = \int_{S(r)} \left| z^{a^1 + \cdots + a^L} W_{a^1 \cdots a^L}(F) \|f\|^{(dq\triangle - LN)} Q^\Delta_1(f) \cdots Q^\Delta_r(f) \right|^{2t} \sigma_m \leq K_3 \left( \frac{1}{R-r} T_F(R, r_0) \right)^{p'} \leq K_3 \left( \frac{d}{R-r} T_F(R, r_0) \right)^{p'}.$$  

Using (4.2) and by taking $R = \frac{1+x_r}{2}$, (4.5) becomes

$$\int_{S(r)} \|\tilde{\phi}\|^{2t(dq\triangle - LN)} \sigma_m \leq \frac{K}{(1-r)^{p'}} \left( \log \frac{1}{1-r} \right)^{p'}.$$  

Likewise,

$$\int_{S(r)} \|\tilde{\psi}\|^{2t(dq\triangle - LN)} \sigma_m \leq \frac{K}{(1-r)^{p'}} \left( \log \frac{1}{1-r} \right)^{p'}.$$
Therefore, by combining (4.4), (4.6), and (4.7), we have
\[
\int_{B(1)} e^{u+u_1+u_2} dV \leq K \int_0^1 r^{2m-1} \left( \log \frac{1}{1-r} \right)^{p'} dr < \infty,
\]
since \( p' < 1 \). On the other hand, by the result of S.T. Yau (14) and L. Karp (7), we necessarily have
\[
\int_{B(1)} e^{u+u_1+u_2} dV = \infty
\]
because \( u + u_1 + u_2 \) is plurisubharmonic. This is a contradiction. Thus, the Main Theorem is proved. □

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