PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 12, December 2013, Pages 4229–4239
S 0002-9939(2013)11718-1
Article electronically published on August 1, 2013

A UNICITY THEOREM FOR MEROMORPHIC MAPS OF A COMPLETE KÄHLER MANIFOLD INTO $\mathbb{P}^n(\mathbb{C})$ SHARING HYPERSURFACES

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(Communicated by Mei-Chi Shaw)

Abstract. In this paper, we give a unicity theorem for meromorphic maps of an $m-$dimensional complete Kähler manifold M, whose universal covering is a ball in $\mathbb{C}^m$, into $\mathbb{P}^n(\mathbb{C})$, sharing the hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$, where the maps satisfy a certain growth condition.

1. Introduction

In 1926, R. Nevanlinna proved that two distinct nonconstant meromorphic functions f, g on the complex plane $\mathbb{C}$ cannot have the same pre-images for more than four distinct values. The generalization of Nevanlinna's result to the case of meromorphic maps of a complete Kähler manifold M, whose universal covering is a ball in $\mathbb{C}^m$, into $\mathbb{P}^n(\mathbb{C})$ satisfying a certain growth condition (see condition (C_ρ) below) and sharing hyperplanes is given by Fujimoto (see [5]). In this paper, by applying the results obtained in [11] by the authors, we extend Fujimoto's result to the case where the meromorphic maps share hypersurfaces instead of hyperplanes.

Let M be an m-dimensional connected Kähler manifold with Kähler form ω and let f be a meromorphic map of M into $\mathbb{P}^n(\mathbb{C})$. For $\rho \geq 0$, we say that f satisfies condition (C_ρ) if there exists a nonzero bounded continuous real-valued function h on M such that

$$\rho\Omega_f + dd^c \log h^2 \geq \mathrm{Ric}(\omega),$$

where Ω_f denotes the pull-back of the Fubini-Study metric form on $\mathbb{P}^n(\mathbb{C})$ by f, $d = \partial + \bar{\partial}$, $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$, and $\mathrm{Ric}(\omega)$ is the Ricci form of ω. The main theorem in this paper is as follows.

Main Theorem. *Let M be a complete connected Kähler manifold whose universal covering is biholomorphic to either $\mathbb{C}^m$ or the unit ball in $\mathbb{C}^m$. Let f and g be algebraically nondegenerate meromorphic maps of M into $\mathbb{P}^n(\mathbb{C})$. Assume that f and g satisfy condition (C_ρ) and that there exist q hypersurfaces $D_j, j = 1, ..., q$, of*

Received by the editors October 17, 2011 and, in revised form, January 30, 2012.
2010 *Mathematics Subject Classification.* Primary 32H30; Secondary 53A10.
The first author was supported in part by NSA H98230-11-1-0201.

degree d_j located in general position in $\mathbb{P}^n(\mathbb{C})$ such that

$$(i)\ f = g \quad on \quad \bigcup_{j=1}^{q}(f^{-1}(D_j) \cup g^{-1}(D_j)),$$

$$(ii)\ q > n + 1 + \frac{2n(L-1)+1}{d} + \frac{\rho L(L-1)}{d},$$

where $L = \begin{pmatrix} N+n \\ n \end{pmatrix}$ *with* $N = 2d^2(d+1)n(n+1)(2^n-1) + nd$, *and* $d = lcm\{d_1, d_2, ..., d_q\}$. *Then* $f \equiv g$.

From the proof we will easily see that if, in addition, we assume that $\dim(f^{-1}(D_i) \cap f^{-1}(D_j)) \leq m-2$ for $i \neq j$, $1 \leq i, j \leq q$, we have the following result.

Theorem 1.1. *Let M be a complete connected Kähler manifold whose universal covering is biholomorphic to either $\mathbb{C}^m$ or the unit ball in $\mathbb{C}^m$. Let f and g be algebraically nondegenerate meromorphic maps of M into $\mathbb{P}^n(\mathbb{C})$. Assume that f and g satisfy condition (C_ρ) and that there exist q hypersurfaces $D_j, j = 1, ..., q$, of degree d_j located in general position in $\mathbb{P}^n(\mathbb{C})$ such that*

$$(i)\ f = g \quad on \quad \bigcup_{j=1}^{q}(f^{-1}(D_j) \cup g^{-1}(D_j)),$$

$$(ii)\ \dim(f^{-1}(D_i) \cap f^{-1}(D_j)) \leq m-2,\ for\ i \neq j, 1 \leq i, j \leq q,$$

$$(iii)\ q > n + 1 + \frac{2(L-1)+1}{d} + \frac{\rho L(L-1)}{d},$$

where $L = \begin{pmatrix} N+n \\ n \end{pmatrix}$ *with* $N = 2d^2(d+1)n(n+1)(2^n-1) + nd$, *and* $d = lcm\{d_1, d_2, ..., d_q\}$. *Then* $f \equiv g$.

In the case $M = \mathbb{C}^m$, we can take the flat metric whose Ricci form vanishes. Therefore all meromorphic maps of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ satisfy condition (C_0), and we get the following version of the result of Dulock and Ru in [12].

Corollary 1.2. *Let f and g be algebraically nondegenerate meromorphic maps of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Assume that there exist q hypersurfaces $D_j, j = 1, ..., q$, of degree d_j located in general position in $\mathbb{P}^n(\mathbb{C})$ such that*

$$(i)\ f = g \quad on \quad \bigcup_{j=1}^{q}(f^{-1}(D_j) \cup g^{-1}(D_j)),$$

$$(ii)\ q > n + 1 + \frac{2n(L-1)+1}{d},$$

where $L = \begin{pmatrix} N+n \\ n \end{pmatrix}$ *with* $N = 2d^2(d+1)n(n+1)(2^n-1) + nd$, *and* $d = lcm\{d_1, d_2, ..., d_q\}$. *Then* $f \equiv g$.

In the case when M is a ball in $\mathbb{C}^m$ but without the growth assumption for f and g (i.e. we do not assume that both f and g satisfy condition (C_ρ)), we also have the following result.

Theorem 1.3. *Let f and g be algebraically nondegenerate meromorphic maps of the unit ball $B(1) \subset \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Assume that, for a fixed r_0 with $0 < r_0 < 1$,*

$$\lambda := \limsup_{r \to 1} \frac{\log(1/(1-r))}{T_f(r, r_0) + T_g(r, r_0)} < \infty.$$

Assume that there exist q hypersurfaces $D_j, j = 1, ..., q$, of degree d_j located in general position in $\mathbb{P}^n(\mathbb{C})$ such that

$$(i)\ f = g \quad on \quad \bigcup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j)),$$

$$(ii)\ q > n + 1 + \lambda L(L-1) + \frac{2n(L-1)+1}{d},$$

where $L = \begin{pmatrix} N+n \\ n \end{pmatrix}$ with $N = 2d^2(d+1)n(n+1)(2^n - 1) + nd$, and $d = lcm\{d_1, d_2, ..., d_q\}$. Then $f \equiv g$.

2. Preliminaries

Let h be a nonconstant holomorphic function on an open domain $G \subset \mathbb{C}^m$. For a set $\alpha = (\alpha_1, ..., \alpha_m)$ of integers $\alpha_i \geq 0$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $D^\alpha h = D_1^{\alpha_1} \cdots D_m^{\alpha_m} h$, where $D_i h = (\partial/\partial z_i)h$, for $i = 1, \ldots, m$. We define $\nu_h^0 : G \to \mathbb{Z}$ by

$$\nu_h^0(z) := \max\{k : D^\alpha h(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < k\}\ (z \in G).$$

By a divisor on a domain G in $\mathbb{C}^m$ we mean a map ν of G into $\mathbb{Z}$ such that, for each $z_0 \in G$, there are nonzero holomorphic functions h and g on a connected neighborhood $U(\subset G)$ of z_0 so that $\nu(z) = \nu_h^0(z) - \nu_g^0(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m-2$. Take a nonzero meromorphic function φ on a domain G in $\mathbb{C}^m$. For each $z_0 \in G$, we choose nonzero holomorphic functions g and h on a neighborhood $U(\subset G)$ of z_0 such that $\varphi = \frac{g}{h}$ on U and $\dim(g^{-1}(0) \cup h^{-1}(0)) \leq m-2$. We define $\nu_\varphi^\infty := \nu_h^0$, $\nu_\varphi^a := \nu_{g-ah}^0$ for $a \in \mathbb{C}$ and $\nu_\varphi = \nu_\varphi^0 - \nu_\varphi^\infty$, which are independent of the choices of h and g.

Let f be a meromorphic map of $B(R_0) \subset \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, $0 < R_0 \leq \infty$. We take holomorphic functions $f_0, f_1, \ldots, f_n$ such that $I_f := \{z \in B(R_0), f_0(z) = \cdots = f_n(z) = 0\}$ is of dimension at most $m-2$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ on $B(R_0) - I_f$ in terms of homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{P}^n(\mathbb{C})$. We call such representation $f(z) = (f_0(z) : \cdots : f_n(z))$ a reduced representation of f. For $z = (z_1, ..., z_m) \in \mathbb{C}^m$ we set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ and define $B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}$, $S(r) = \{z \in \mathbb{C}^m : \|z\| = r\}$ for $0 < r \leq +\infty$, where we mean $B(\infty) = \mathbb{C}^m$ and $S(\infty) = \emptyset$. Define

$$\sigma_m := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1} \text{ on } \mathbb{C}^m - \{0\} \text{ and } v_l := (dd^c\|z\|^2)^l,\ 0 \leq l \leq m.$$

The pullback of the normalized Fubini-Study metric form Ω on $\mathbb{P}^n(\mathbb{C})$ by f is given by

$$\Omega_f = dd^c \log\|f\|^2,$$

where $\|f\|^2 := |f_0|^2 + \cdots + |f_n|^2$. Fix $r_0 < R_0$; the characteristic function of f is defined by

$$T_f(r, r_0) = \int_{r_0}^{r} \frac{dt}{t^{2m-1}} \int_{B(t)} \Omega_f \wedge v_{m-1} \qquad (0 < r_0 < r < R_0).$$

We then have (see [13])

$$T_f(r, r_0) = \int_{S(r)} \log\|f\|\sigma_m - \int_{S(r_0)} \log\|f\|\sigma_m. \tag{2.1}$$

Let μ_0 be a positive integer or ∞ and ν be divisors on a domain $B(R_0) \subset \mathbb{C}^m$. Set $|\nu| = \overline{\{z \in B(R_0) : \nu(z) \neq 0\}}$. We define the counting function of ν truncated by μ_0 by

$$N_\nu^{[\mu_0]}(r, r_0) = \int_{r_0}^{r} \frac{n^{[\mu_0]}(t)}{t^{2m-1}} dt,$$

where

$$n^{[\mu_0]}(t) = \int_{|\nu| \cap B(t)} \min\{\nu, \mu_0\} v_{m-1} \quad \text{for } m \geq 2 \quad \text{and}$$
$$n^{[\mu_0]}(t) = \sum_{|z| \leq t} \min\{\nu(z), \mu_0\} \quad \text{for } m = 1.$$

For a meromorphic function ϕ on $B(R_0)$, we may regard ϕ as a meromorphic map into $\mathbb{P}^1(\mathbb{C})$. We define

$$N_\phi^0(r, r_0) = \int_{r_0}^{r} \frac{n_\phi^0(t)}{t^{2m-1}} dt,$$

where

$$n_\phi^0(t) = \int_{|\nu| \cap B(t)} \nu_\phi^0 v_{m-1} \quad \text{for } m \geq 2 \quad \text{and}$$
$$n_\phi^0(t) = \sum_{|z| \leq t} \nu_\phi^0(z) \quad \text{for } m = 1.$$

Let f and g be algebraically nondegenerate meromorphic maps from $B(R_0) \subset \mathbb{C}^m$ $(0 < R_0 \leq \infty)$ into $\mathbb{P}^n(\mathbb{C})$. For a positive integer N, let V_N denote the space of homogeneous polynomials of degree N in $\mathbb{C}[X_0, ..., X_n]$ and fix a (arbitrary) basis $\phi_1, ..., \phi_l$, where $l = \dim V_N$. Let

$$F = [\phi_1(f) : \cdots : \phi_l(f)] \text{ and } G = [\phi_1(g) : \cdots : \phi_l(g)]. \tag{2.2}$$

Then $F, G : B(R_0) \to \mathbb{P}^{l-1}(\mathbb{C})$ are linearly nondegenerate. Hence (see Proposition 3.2 in [11]) there exist $\alpha^j = (\alpha_{j1}, \ldots, \alpha_{jm})$ (resp. $\beta^j = (\beta_{j1}, \ldots, \beta_{jm})$) with $\alpha_{ji} \geq 0$ being integers (resp. $\beta_{ji} \geq 0$ being integers), $|\alpha^j| := \alpha_{j1} + \cdots + \alpha_{jm} \leq l - 1$ for $1 \leq j \leq l$ (resp. $|\beta^j| \leq l - 1$ for $1 \leq j \leq l$), and $|\alpha^1| + \cdots + |\alpha^l| \leq l(l-1)/2$ (resp. $|\beta^1| + \cdots + |\beta^l| \leq l(l-1)/2$) such that both

$$W_{\alpha^1 \ldots \alpha^l}(F) := \det\left(D^{\alpha^1} F, \ldots, D^{\alpha^l} F\right), \; W_{\beta^1 \ldots \beta^l}(G) := \det\left(D^{\beta^1} G, \ldots, D^{\beta^l} G\right) \tag{2.3}$$

are not identically zero on $B(R_0)$. Let $D_1, ..., D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree $d_1, \ldots, d_q$, located in general position. Let Q_j, $1 \leq j \leq q$, be the homogeneous polynomials defining D_j. Replacing Q_j by Q_j^{d/d_j} if necessary, where $d = lcm\{d_1, \ldots, d_q\}$, we can assume that $Q_1, \ldots, Q_q$ have the same degree d.

Choose distinct $\gamma_{i_1}, ..., \gamma_{i_n} \in \{Q_1, ..., Q_q\}$. Arrange the n-tuples by lexicographic order and consider the n-tuples $\mathbf{i} = (i_1, ..., i_n)$ of nonnegative integers with $\sigma(\mathbf{i}) := i_1 + \cdots + i_n \leq N/d$. Define the spaces

$$W_{\mathbf{i}} = W_{N,\mathbf{i}} := \sum_{(e)\geq(i)} \gamma_1^{e_1} ... \gamma_n^{e_n} V_{N-d\sigma(e)}.$$

Clearly, $W_{(0,...,0)} = V_N$, $W_{\mathbf{i}} \supset W_{\mathbf{i}'}$ if $\mathbf{i}' \geq \mathbf{i}$, and $\{W_{\mathbf{i}}\}$ defines a filtration of V_N. Let

$$\triangle_{\mathbf{i}} := \dim \frac{W_{\mathbf{i}}}{W_{\mathbf{i}'}},$$

where $\mathbf{i}' > \mathbf{i}$ are consecutive n-tuples with $W_{\mathbf{i}'} \subset W_{\mathbf{i}}$. Then it is known (see Lemma 4.1 in [11]) that each $\triangle_{\mathbf{i}}$ is independent of the choice of $\gamma_{i_1}, ..., \gamma_{i_n}$. Hence

$$\triangle := \frac{1}{n} \sum_{\mathbf{i}} \triangle_{\mathbf{i}} \sigma(\mathbf{i})$$

is a positive integer depending only on $D_1, \ldots, D_q$. The following lemma gives the estimate of $\triangle$.

Lemma 2.1. *With* $N = 2d^2(d+1)n(n+1)(2^n-1) + nd$ *and* $L = \begin{pmatrix} N+n \\ n \end{pmatrix}$, *we have*

$$\frac{LN}{\triangle} \leq d(n+1) + \frac{1}{2d}. \tag{2.4}$$

The proof of the above lemma is given by the same argument by setting $\epsilon = 1/d$ as in the proof of [11, Lemma 4.3].

Proposition 2.2 ([11, Proposition 4.4])**.** *Set*

$$\psi = \frac{W_{\alpha^1 \cdots \alpha^l}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)}.$$

Then

$$\nu_{\psi}^{\infty} \leq \sum_{j=1}^{q} \triangle \min\{\nu^0_{Q_j(f)}, l-1\}$$

outside an analytic set of codimension at least two.

Proposition 2.3 (See (4.21) on page 1158 in [11])**.** *Set* $l_0 = |\alpha^1| + \cdots + |\alpha^l|$ *and take* t, p' *with* $0 < tl_0 < p' < 1$. *Then, for fixed* r_0 *with* $0 < r_0 < R_0$ *there exists a positive constant* K *such that for* $r_0 < r < R < R_0$,

$$\int_{S(r)} \left| z^{\alpha^1 + \cdots + \alpha^l} \frac{W_{\alpha^1 \ldots \alpha^l}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \right|^t \|f\|^{t(dq\triangle - lN)} \sigma_m \leq K \Big(\frac{R^{2m-1}}{R-r} T_F(R, r_0) \Big)^{p'}.$$

Theorem 2.4. *Let* $f : B(R_0) \longrightarrow \mathbb{P}^n(\mathbb{C})$, $0 < R_0 \leq \infty$, *be an algebraically nondegenerate meromorphic map and* $D_1, ..., D_q$ *be hypersurfaces of degree* $d_j, 1 \leq j \leq q$, *in* $\mathbb{P}^n(\mathbb{C})$ *located in general position. Then, with* $N = 2d^2(d+1)n(n+1)(2^n-1)+nd$ *and* $L = \begin{pmatrix} N+n \\ n \end{pmatrix}$,

$$\Big(q - (n+1+(1/d))\Big) T_f(r, r_0) \leq \sum_{j=1}^{q} d_j^{-1} N_f^{[L-1]}(r, D_j) + S_f(r),$$

where $S_f(r)$ is evaluated as follows:

(1) In the case $R_0 < \infty$,

$$S_f(r) \leq \frac{L(L-1)}{2} \log^+ \frac{1}{R_0 - r} + K \log^+ T_f(r, r_0)$$

for every $r \in [0, R_0)$ excluding a set E with $\int_E \frac{1}{R_0 - t} dt < \infty$, where K is a positive constant.

(2) In the case $R_0 = \infty$,

$$S_f(r) \leq K(\log^+ T_f(r, r_0) + \log r)$$

for every $r \in [0, \infty)$ excluding a set E' with $\int_{E'} dt < \infty$, where K is a positive constant.

Remark 2.5. If $R_0 = \infty$ and $\lim_{r \to \infty} T_f(r, r_0)/\log r < \infty$, then we can choose $S_f(r)$ to be bounded.

Proof. The proof is given by the same argument by taking $\epsilon = 1/d$ as in the proof of [11, Theorem 4.5] except for the following modifications in the case $R_0 < +\infty$: From Proposition 2.3, for $0 < tl_0 < p' < 1$, we have

(2.5)
$$\int_{S(r)} \Big| z^{\alpha^1 + \cdots + \alpha^L} \frac{W_{\alpha^1 \ldots \alpha^L}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \Big|^t \|f\|^{t(dq \triangle - LN)} \sigma_m \leq K \Big(\frac{R^{2m-1}}{R - r} T_F(R, r_0) \Big)^{p'},$$

for $r_0 < r < R < R_0$. Hence, by virtue of the concavity of the logarithm, the above inequality implies that

$$\int_{S(r)} \log |z^{\alpha^1 + \cdots + \alpha^L}| \sigma_m + \int_{S(r)} \log \Big| \frac{W_{\alpha^1 \ldots \alpha^L}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \Big| \sigma_m$$
$$+ (dq \triangle - NL) \int_{S(r)} \log \|f\| \sigma_m + \leq \frac{p'}{t} \Big(\log \frac{1}{R - r} + \log^+ T_F(R, r_0) \Big) + O(1),$$

for $r_0 < r < R < R_0$. Since $l_0 = |\alpha^1| + \cdots + |\alpha^L| \leq L(L-1)/2$, by letting $(p'/t) > l_0$ approach l_0, we get

$$\int_{S(r)} \log |z^{\alpha^1 + \cdots + \alpha^L}| \sigma_m + \int_{S(r)} \log \Big| \frac{W_{\alpha^1 \ldots \alpha^L}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \Big| \sigma_m$$
$$+ (dq \triangle - NL) \int_{S(r)} \log \|f\| \sigma_m \leq \frac{L(L-1)}{2} \Big(\log \frac{1}{R - r} + \log^+ T_F(R, r_0) \Big) + O(1).$$

The rest of the arguments follow in the proof of [11, Theorem 4.5]. □

3. The proof of the Main Theorem for particular cases

In this section, we prove Corollary 1.2 and Theorem 1.3 stated in section 1.

Proof of Corollary 1.2. Assume that $f \not\equiv g$. Fix a reduced representation $(f_0, f_1, ..., f_n)$ of f and a reduced representation $(g_0, g_1, ..., g_n)$ of g. We may choose distinct indices i_0 and j_0 such that

$$\chi := f_{i_0} g_{j_0} - f_{j_0} g_{i_0} \tag{3.1}$$

is not identically zero. Applying Theorem 2.4 to the maps f and g (without loss of generality, we assume that $d_1 = \cdots = d_q = d$), we get

$$\Big(q - (n+1+(1/d))\Big)T_f(r,r_0) \le \sum_{j=1}^{q} d^{-1}N_f^{[L-1]}(r,D_j) + S_f(r),$$

$$\Big(q - (n+1+(1/d))\Big)T_g(r,r_0) \le \sum_{j=1}^{q} d^{-1}N_g^{[L-1]}(r,D_j) + S_g(r),$$

where $S_f(r)$ and $S_g(r)$ satisfy

$$S_f(r) \le_{exc} K(\log^+ T_f(r,r_0) + \log r) \text{ and } S_g(r) \le_{exc} K(\log^+ T_g(r,r_0) + \log r),$$

where $\le_{exc}$ means the inequality holds for all $r \in [0,\infty)$ excluding a set E' with $\int_{E'} dt < \infty$. Adding these two inequalities gives

$$\Big(q - (n+1+(1/d))\Big)\Big(T_f(r,r_0) + T_g(r,r_0)\Big) \le \frac{1}{d}\sum_{j=1}^{q}(N_f^{[L-1]}(r,D_j) + N_g^{[L-1]}(r,D_j)) + S_g(r) + S_f(r).$$

Using the fact that the D_j's are in general position, from assumption (i) in Corollary 1.2, we have trivially $\sum_{j=1}^{q}(N_f^{[L-1]}(r,D_j) + N_g^{[L-1]}(r,D_j)) \le 2n(L-1)N(r,A)$, where $A = \bigcup_{j=1}^{q}(f^{-1}(D_j) \cup g^{-1}(D_j))$. Hence,

(3.2)
$$\Big(q - (n+1+(1/d))\Big)\Big(T_f(r,r_0) + T_g(r,r_0)\Big) \le \frac{2n(L-1)}{d}N(r,A) + S_g(r) + S_f(r).$$

We now claim that

$$N(r,A) \le T_f(r_0,r) + T_g(r_0,r) + O(1).$$

Indeed, consider the map $\chi := f_{i_0}g_{j_0} - f_{j_0}g_{i_0}$ defined above. If $z \in A$, then $f(z) = g(z)$ from assumption (i) in Corollary 1.2, and so $\chi(z) = 0$. It then follows that $N(r,A) \le N_\chi^0(r,r_0)$. By the First Main Theorem of Nevanlinna,

$$N_\chi^0(r,r_0) \le T_\chi(r,r_0) + O(1) \le T_f(r,r_0) + T_g(r,r_0) + O(1).$$

The claim then follows.

Therefore (3.2) gives

$$\Big(q - (n+1+(1/d))\Big)\Big(T_f(r,r_0) + T_g(r,r_0)\Big) \le \frac{2n(L-1)}{d}\Big(T_f(r,r_0) + T_g(r,r_0)\Big) + S_g(r) + S_f(r)$$

or

$$\Big(q - (n+1+(1/d)) - \frac{2n(L-1)}{d}\Big) \le \frac{S_g(r) + S_f(r)}{T_f(r,r_0) + T_g(r,r_0)}. \tag{3.3}$$

In the case that f, g are rational, then, as in Remark 2.5, $S_f(r)$ and $S_g(r)$ can be taken to be bounded. Thus

$$\lim_{r\longrightarrow\infty} \frac{S_f(r) + S_g(r)}{T_f(r,r_0) + T_g(r,r_0)} = 0,$$

and hence $q \leq n+1+\frac{2n(L-1)+1}{d}$, which leads to a contradiction. If either f or g is transcendental, then

$$\lim_{r\longrightarrow\infty} \frac{\log r}{T_f(r,r_0)+T_g(r,r_0)} = 0.$$

On the other hand,

$$S_f(r) \leq_{exc} K(\log^+ T_f(r,r_0)+\log r) \text{ and } S_g(r) \leq_{exc} K(\log^+ T_g(r,r_0)+\log r),$$

so

$$\lim_{r\to\infty} \inf \frac{S_f(r)+S_g(r)}{T_f(r,r_0)+T_g(r,r_0)} = 0.$$

This again implies that $q \leq n+1+\frac{2n(L-1)+1}{d}$, which leads to a contradiction. This completes the proof of Corollary 1.2.

Proof of Theorem 1.3. The proof is similar to the above argument. Assume that $f \not\equiv g$. Conclusion (1) of Theorem 2.4 implies that there exists a subset E of $[0,1)$ such that $\int_E (1-r)^{-1}dr < \infty$ and, for every $r \notin E$,

$$\frac{S_f(r)+S_g(r)}{T_f(r,r_0)+T_g(r,r_0)} \leq \frac{L(L-1)\log(1/(1-r))+K\log^+(T_f(r,r_0)+T_g(r,r_0))}{T_f(r,r_0)+T_g(r,r_0)}.$$

From this and (3.3), we conclude that

$$\begin{aligned}
&\Big(q-(n+1+(1/d))-\frac{2n(L-1)}{d}\Big)\\
&\leq L(L-1)\liminf_{r\to 1, r\notin E} \frac{\log(1/(1-r))+K\log^+(T_f(r,r_0)+T_g(r,r_0))}{T_f(r,r_0)+T_g(r,r_0)}\\
&\leq L(L-1)\limsup_{r\to 1} \frac{\log(1/(1-r))}{T_f(r,r_0)+T_g(r,r_0)} \leq \lambda L(L-1).
\end{aligned}$$

Thus

$$q \leq n+1+\frac{2n(L-1)+1}{d}+\lambda L(L-1),$$

which contradicts assumption (ii) of Theorem 1.3. This completes the proof.

Remark 3.1. As is easily seen from the above proof, the quantity λ in the conclusion of Theorem 1.3 can be replaced by the least upper bound of the quantities $\tilde{\lambda}$ such that

$$\tilde{\lambda} = \liminf_{r\to 1, r\notin E} \frac{\log(1/(1-r))}{T_f(r,r_0)+T_g(r,r_0)}$$

for some subset E of $[0,1)$ with $\int_E (1-r)^{-1}dr < \infty$.

4. Proof of the Main Theorem

Proof. We now proceed to prove the Main Theorem. By lifting f and g to the universal covering of M if necessary, we may assume that $M = B(R_0)(\subset \mathbb{C}^m)$ with $0 < R_0 \leq \infty$. The case when $R_0 = \infty$ (i.e. $M = \mathbb{C}^m$) is nothing but Corollary 1.2, and so it suffices to study the case $M = B(1)$. Moreover, by virtue

of Remark 3.1, the Main Theorem is true unless there exists a subset E of $[0,1)$ with $\int_E (1-r)^{-1}dr < \infty$ and

$$\limsup_{r\to 1, r\notin E} \frac{T_f(r,r_0)+T_g(r,r_0)}{\log(1/(1-r))} < \infty. \tag{4.1}$$

We note that by the same argument as in the proof of [4, Proposition 5.5], (4.1) indeed implies that there exists a positive constant K such that

$$T_f(r,r_0)+T_g(r,r_0) \le K \log \frac{1}{1-r} \ (0 < r_0 \le r < 1). \tag{4.2}$$

Assuming (4.2), we shall show that it will lead to a contradiction if $f \not\equiv g$.

Let F, G be the meromorphic maps of $B(1) \subset \mathbb{C}^m$ into $\mathbb{P}^{L-1}(\mathbb{C})$ defined in (2.2). Let α^j and $\beta^j, j = 1, \dots, L$, be m-tuples of nonnegative integers with $|\alpha^j| \le L-1$, $|\beta^j| \le L-1$ for $1 \le j \le L$, and $|\alpha^1| + \cdots + |\alpha^L| \le L(L-1)/2$, $|\beta^1| + \cdots + |\beta^L| \le L(L-1)/2$ such that $W_{\alpha^1\dots\alpha^L}(F) \not\equiv 0$ and $W_{\beta^1\dots\beta^L}(G) \not\equiv 0$, where $W_{\alpha^1\dots\alpha^L}(F)$ and $W_{\beta^1\dots\beta^L}(G)$ are defined in (2.3). Set $\tilde{\phi} := z^{\alpha^1+\cdots+\alpha^L}\Phi$ and $\tilde{\psi} := z^{\beta^1+\cdots+\beta^L}\Psi$, where

$$\Phi = \frac{W_{\alpha^1\dots\alpha^L}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)} \text{ and } \Psi = \frac{W_{\beta^1\dots\beta^L}(G)}{Q_1^{\triangle}(g)\cdots Q_q^{\triangle}(g)}.$$

We fix a reduced representation $(f_0, f_1, ..., f_n)$ of f and a reduced representation $(g_0, g_1, ..., g_n)$ of g. Since $f \not\equiv g$, we may choose distinct indices i_0 and j_0 such that

$$\chi := f_{i_0}g_{j_0} - f_{j_0}g_{i_0} \not\equiv 0. \tag{4.3}$$

Notice that for a point $p \in f^{-1}(D_j) \subset \bigcup_{j=1}^q (f^{-1}(D_j) \cup g^{-1}(D_j))$, $f(p) = g(p)$. Thus $\chi(p) = 0$. Furthermore, from Proposition 2.2, by noticing that at $p \in B(1)$ there are at most n Q_j's with $Q_j(f)(p) = 0$ due to the assumption that $D_1, \dots, D_q$ are in general position, we have $\nu_\Phi^\infty \le \triangle n(L-1)\nu_\chi^0$ and $\nu_\Psi^\infty \le \triangle n(L-1)\nu_\chi^0$, outside an analytic set of codimension ≥ 2. It then follows that, outside an analytic set of codimension ≥ 2, the functions $\tilde{\phi}\chi^{\triangle n(L-1)}$ and $\tilde{\psi}\chi^{\triangle n(L-1)}$ are both holomorphic on $B(1)$. Therefore if we let

$$t := \frac{\rho}{dq \triangle -LN - 2 \triangle n(L-1)}$$

and

$$u := t \log |\tilde{\phi}\tilde{\psi}\chi^{2\triangle n(L-1)}|,$$

then u is plurisubharmonic on $B(1)$ outside an analytic set of codimension ≥ 2. From the assumption that both f and g satisfy condition (C_ρ), there are continuous plurisubharmonic functions u_1, u_2 on $B(1)$ such that

$$e^{u_1}\det(h_{i\bar{j}})^{1/2} \le \|f\|^\rho,$$
$$e^{u_2}\det(h_{i\bar{j}})^{1/2} \le \|g\|^\rho.$$

Since $\rho + 2\triangle tn(L-1) = t(dq \triangle -LN)$ and noticing that $|\chi| \le 2\|f\|\|g\|$, we obtain

$$\begin{aligned}\det(h_{i\bar{j}})e^{u+u_1+u_2} &\le |\tilde{\phi}|^t|\tilde{\psi}|^t|\chi|^{2\triangle tn(L-1)}\|f\|^\rho\|g\|^\rho \\ &\le K|\tilde{\phi}|^t|\tilde{\psi}|^t\|f\|^{\rho+2\triangle tn(L-1)}\|g\|^{\rho+2\triangle tn(L-1)} \\ &= K|\tilde{\phi}|^t|\tilde{\psi}|^t\|f\|^{t(dq\triangle -LN)}\|g\|^{t(dq\triangle -LN)}\end{aligned}$$

for some constant K. Note that the volume form on $B(1)$ is given by

$$dV := c_m \det(h_{i\bar{j}})v_m;$$

therefore,

$$\int_{B(1)} e^{u+u_1+u_2} dV \leq K \int_{B(1)} |\tilde{\phi}|^t \|f\|^{t(dq\triangle - LN)} |\tilde{\psi}|^t \|g\|^{t(dq\triangle - LN)} v_m.$$

Thus, by the Hölder inequality and by noticing that

$$v_m = (dd^c\|z\|^2)^m = 2m\|z\|^{2m-1}\sigma_m \wedge d\|z\|,$$

we obtain

$$\begin{aligned}\int_{B(1)} e^{u+u_1+u_2} dV \leq K\Big(\int_0^1 r^{2m-1}\big(\int_{S(r)} |\tilde{\phi}|^{2t}\|f\|^{2t(dq\triangle - LN)}\sigma_m\big)dr\Big)^{1/2} \\ \times \Big(\int_0^1 r^{2m-1}\big(\int_{S(r)} |\tilde{\psi}|^{2t}\|g\|^{2t(dq\triangle - LN)}\sigma_m\big)dr\Big)^{1/2}.\end{aligned} \tag{4.4}$$

From Lemma 2.1,

$$-\frac{LN}{\triangle} \geq -d(n+1) - \frac{1}{2d}.$$

Therefore, together with assumption (ii) in the Main Theorem, we have

$$\begin{aligned} dq \triangle -LN - 2 \triangle n(L-1) &= \triangle\Big(dq - \frac{LN}{\triangle} - 2n(L-1)\Big) \\ &\geq \triangle\Big(dq - d(n+1) - \frac{1}{2d} - 2n(L-1)\Big) \\ &= d \triangle \Big(q - (n+1) - \frac{1}{2d^2} - \frac{2n(L-1)}{d}\Big) \\ &> d \triangle \big(\frac{\rho L(L-1)}{d}\Big) = \triangle \rho L(L-1), \end{aligned}$$

and thus

$$(2t)\cdot L(L-1)/2 = tL(L-1) = \frac{L(L-1)\rho}{dq \triangle -LN - 2 \triangle n(L-1)} < \frac{1}{\triangle} < 1.$$

Take some p' with $0 < tL(L-1) < p' < 1$. Then, since $0 < 2t(|\alpha^1| + \cdots + |\alpha^L|) \leq tL(L-1) < p' < 1$, it follows from Proposition 2.3 that for $r_0 < r < R < 1$,

$$\begin{aligned}\int_{S(r)} |\tilde{\phi}|^{2t}\|f\|^{2t(dq\triangle - LN)}\sigma_m = \int_{S(r)} \Big| z^{\alpha^1+\cdots+\alpha^L} \frac{W_{\alpha^1\ldots\alpha^L}(F)\|f\|^{(dq\triangle - LN)}}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}\Big|^{2t}\sigma_m \\ \leq K_3\Big(\frac{1}{R-r}T_F(R,r_0)\Big)^{p'} \leq K_3\Big(\frac{d}{R-r}T_f(R,r_0)\Big)^{p'}.\end{aligned} \tag{4.5}$$

Using (4.2) and by taking $R = \frac{1+r}{2}$, (4.5) becomes

$$\int_{S(r)} |\tilde{\phi}|^{2t}\|f\|^{2t(dq\triangle - LN)}\sigma_m \leq \frac{K}{(1-r)^{p'}}\Big(\log\frac{1}{1-r}\Big)^{p'}. \tag{4.6}$$

Likewise,

$$\int_{S(r)} |\tilde{\psi}|^{2t}\|g\|^{2t(dq\triangle - LN)}\sigma_m \leq \frac{K}{(1-r)^{p'}}\Big(\log\frac{1}{1-r}\Big)^{p'}. \tag{4.7}$$

Therefore, by combining (4.4), (4.6), and (4.7), we have

$$\int_{B(1)} e^{u+u_1+u_2} dV \leq K \int_0^1 \frac{r^{2m-1}}{(1-r)^{p'}} \Big(\log \frac{1}{1-r} \Big)^{p'} dr < \infty,$$

since $p' < 1$. On the other hand, by the result of S.T. Yau ([14]) and L. Karp ([7]), we necessarily have

$$\int_{B(1)} e^{u+u_1+u_2} dV = \infty$$

because $u + u_1 + u_2$ is plurisubharmonic. This is a contradiction. Thus, the Main Theorem is proved. □

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