A NOTE ON DIMENSION OF TRIANGULATED CATEGORIES

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Abstract. In this note we study the behavior of the dimension of the perfect derived category Perf($A$) of a dg-algebra $A$ over a field $k$ under a base field extension $K/k$. In particular, we show that the dimension of a perfect derived category is invariant under a separable algebraic extension $K/k$. As an application we prove the following statement: Let $A$ be a self-injective algebra over a perfect field $k$. If the dimension of the stable category mod$A$ is 0, then $A$ is of finite representation type. This theorem is proved by M. Yoshiwaki in the case when $k$ is an algebraically closed field. Our proof depends on his result.

1. Introduction

In [3] R. Rouquier introduced the dimension of triangulated categories and showed that it gives an upper bound or a lower bound of other dimensions in algebraic geometry or in representation theory (see also [4]). The dimension of triangulated categories is studied by many researchers.

In this note we study the behavior of the dimension of the perfect derived category Perf($A$) of a dg-algebra $A$ over a field $k$ under a base field extension $K/k$. For a field extension $K/k$, we denote $A \otimes_k K$ by $A_K$.

Theorem 1.1.

(1) For an algebraic extension $K/k$, we have

$$ \text{tridim} \text{ Perf}(A) \leq \text{tridim} \text{ Perf}(A_K). $$

(2) If moreover $K/k$ is separable, then equality holds.

As an application we prove the following theorem, which gives evidence that dimension of triangulated categories captures some representation theoretic properties.

The stable category mod$A$ plays an important role in the study of a self-injective algebra $A$ (cf. [2][4]). If a self-injective algebra $A$ is of finite representation type, then the dimension of the stable category mod$A$ is zero. Then a natural question arises as to whether the converse should also hold.

Theorem 1.2. Let $A$ be a self-injective finite dimensional algebra over a perfect field $k$. If tridim mod$A = 0$, then $A$ is of finite representation type.

In the case when $k$ is an algebraically closed field, this theorem is proved by M. Yoshiwaki in [5]. Our proof depends on his result.
2. Proof of Theorem 1.1

We recall the definition of the dimension of triangulated categories. Let $\mathcal{T}$ be a triangulated category. For a full subcategory $\mathcal{I}$ of $\mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{I}$ which is closed under taking shifts, finite direct sums, direct summands and isomorphisms. For full subcategories $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{T}$ we denote by $\mathcal{I} \ast \mathcal{J}$ the full subcategory of $\mathcal{T}$ consisting of those objects $M \in \mathcal{T}$ such that there exists an exact triangle $I \to M \to J \overset{[1]}{\to}$ with $I \in \mathcal{I}$ and $J \in \mathcal{J}$. Set $\mathcal{I} \circ \mathcal{J} := \langle \mathcal{I} \ast \mathcal{J} \rangle$. For $n \geq 1$ we define inductively

$$\langle \mathcal{I} \rangle_n := \begin{cases} \langle \mathcal{I} \rangle & \text{for } n = 1; \\ \langle \mathcal{I} \rangle \circ \langle \mathcal{I} \rangle_{n-1} & \text{for } n \geq 2. \end{cases}$$

Now we define the dimension of a triangulated category $\mathcal{T}$ to be

$$\text{tridim } \mathcal{T} := \min \{ n \mid \langle E \rangle_{n+1} = \mathcal{T} \text{ for some } E \in \mathcal{T} \}.$$

To prove Theorem 1.1 we prepare some notation. We fix a field $k$. In the sequel the term dg-$A$-modules means right dg-$A$-modules. We denote by $C(A)$ the category of dg-$A$-modules. There is an adjoint pair

$$- \bigotimes_{k} A_{K} = - \bigotimes_{k} K : C(A) \rightleftarrows C(A_{K}) : \text{Hom}_{A_{K}}(A_{K}, -) =: U_{K/k}.$$

We denote by $\mathcal{D}(A)$ the derived category of dg-$A$-modules. The functors $- \bigotimes_{k} K$ and $U_{K/k}$ preserve quasi-isomorphisms. Hence we obtain an adjoint pair

$$- \bigotimes_{k} K : \mathcal{D}(A) \rightleftarrows \mathcal{D}(A_{K}) : U_{K/k}.$$

If there seems to be no ambiguity, we denote $U_{K/k}$ by $U$.

We denote by $\text{Perf}(A)$ the perfect derived category of $A$. This is the full triangulated subcategory of $\mathcal{D}(A)$ consisting of objects $M \in \mathcal{D}(A)$ obtained from $A \in \mathcal{D}(A)$ by taking shifts, finite direct sums, direct summands, isomorphisms and cones: $\text{Perf}(A) := \bigcup_{n \geq 1} \langle A \rangle_n$. For $M \in \mathcal{D}(A)$ we denote by $\mathcal{M}$ the smallest full subcategory of $\mathcal{D}(A)$ containing $M$ which is closed under taking finite direct sums and direct summands.

**Lemma 2.1.** If $K/k$ is a finite dimensional extension, then for a dg-$A$-module $M$, we have $\text{add } M = \text{add } U(M \otimes_{k} K)$ in $\mathcal{D}(A)$. Therefore we have $\langle M \rangle = \langle U(M \otimes_{k} K) \rangle$. Hence we have $\langle M \rangle_n = \langle U(M \otimes_{k} K) \rangle_n$ for $n \geq 1$.

**Proof.** The functor $U$ is a forgetful functor. Hence we have a isomorphism $U(M \otimes_{k} K) \cong M^{\oplus \dim_{k} K}$ in $\mathcal{D}(A)$. □

**Lemma 2.2.** Let $K/k$ be an algebraic extension and $E$ an object of $\mathcal{D}(A)$.

1. If an object $\mathcal{G}$ of $\mathcal{D}(A_{K})$ belongs to $\langle E \otimes_{k} K \rangle_n$, then there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over $k$ such that there exists an object $\mathcal{G}'$ of $\langle E \otimes_{k} K_0 \rangle_n$, such that $\mathcal{G}' \otimes_{K_0} K \cong \mathcal{G}$ in $\mathcal{D}(A_{K})$.

2. Let $G$ be an object of $\mathcal{D}(A)$. If $G \otimes_{k} K$ belongs to $\langle E \otimes_{k} K \rangle_n$, then there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over $k$ such that $G \otimes_{K} K_0$ belongs to $\langle E \otimes_{k} K_0 \rangle_n$.

In the sequel overlined objects and morphisms are those of $\mathcal{D}(A_{K})$, and objects and morphisms with dashes are those of $\mathcal{D}(A_{K_0})$. 

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Proof. (1) First note that for dg-$A$-modules $M, N$ we have a natural isomorphism of Hom-spaces

$$\text{Hom}_{\mathcal{D}(A_K)}(M \otimes_k K, N \otimes_k K) \cong \text{Hom}_{\mathcal{D}(A)}(M, N) \otimes_k K.$$ 

It is easy to see that the problem is reduced to the following two cases, (a), (b), by induction.

(a) Let $E_1, E_2$ be objects of $\mathcal{D}(A)$ such that we have an exact triangle

$$E_1 \otimes_k K \xrightarrow{f} E_2 \otimes_k K \rightarrow G[1]$$

in $\mathcal{D}(A_K)$. By the above remark there are $f_1, \ldots, f_n \in \text{Hom}_{\mathcal{D}(A)}(E_1, E_2)$ and $\alpha_1, \ldots, \alpha_n \in K$ such that we have $f = \sum_{i=1}^n f_i \alpha_i$. Set $K_0 := k(\alpha_1, \ldots, \alpha_n)$ and $f' := \sum_{i=1}^n f_i = \text{Hom}_{\mathcal{D}(A_K)}(E_1 \otimes_k K_0, E_2 \otimes_k K_0)$. Then $f' \otimes_{K_0} K = f$. Therefore we have an isomorphism $G' \otimes_{K_0} K \cong G$ in $\mathcal{D}(A_K)$ where $G' := c(f')$ is the cone of $f'$.

(b) Let $E_1$ be an object of $\mathcal{D}(A)$. Assume that $G$ is a direct summand of $E_1 \otimes_k K$. Let $\tau$ be an idempotent endomorphism of $E_1 \otimes_k K$ which corresponds to $G$, i.e., $\tau : E_1 \otimes_k K \rightarrow G \rightarrow E_1 \otimes_k K$, where the left arrow is the canonical projection and the right arrow is the canonical injection. Note that we can obtain the direct summand $G$ from the corresponding idempotent endomorphism $\tau$ in the following way: We define a morphism $\overline{X} : (E_1 \otimes_k K)^{\oplus n} \rightarrow (E_1 \otimes_k K)^{\oplus n}$ to be the morphism which is represented by the matrix given by

$$\overline{X} = \begin{pmatrix} 1 - \tau & 0 & 0 & 0 & \cdots \\ \tau & 1 - \tau & 0 & 0 & \cdots \\ 0 & \tau & 1 - \tau & 0 & \cdots \\ 0 & 0 & \tau & 1 - \tau & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} : (E_1 \otimes_k K)^{\oplus n} \rightarrow (E_1 \otimes_k K)^{\oplus n}.$$

Then the cone $c(\overline{X})$ of $\overline{X}$ is isomorphic to $G$ in $\mathcal{D}(A_K)$. By the same method as above, we see that there is an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over $k$ such that there is an idempotent endomorphism $e'$ of $E_1 \otimes_k K_0$ such that we have $e' \otimes_{K_0} K = \tau$. Hence we obtain an endomorphism $X'$ of $(E_1 \otimes_k K_0)^{\oplus n}$ such that $X' \otimes_{K_0} K_0 = \overline{X}$. Then $G$ is isomorphic to $G' \otimes_{K_0} K_0$ in $\mathcal{D}(A_K)$, where $G' := c(X')$ is a cone of $X'$. We see that $G'$ is a direct summand of $E_1 \otimes_k K_0$ by [1] Proposition 3.2.

(2) By (1) there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over $k$ such that $G' \otimes_{K_0} K \cong G \otimes_k K$ in $\mathcal{D}(A_K)$ for some object $G'$ of $\langle E \otimes_k K_0 \rangle_n$. Let $\overline{f} : G' \otimes_{K_0} K \rightarrow (G \otimes_k K_0) \otimes_{K_0} K$ be an isomorphism in $\mathcal{D}(A_K)$. By the same method as in the proof of (1), we see that there exists an intermediate field $K_0 \subset K_1 \subset K$ which is finite dimensional over $k$ such that there exists an isomorphism $f' : G' \otimes_{K_0} K_1 \cong (G \otimes_k K_0) \otimes_{K_0} K_1 \cong G \otimes_k K_1$. Hence $G \otimes_k K_1$ belongs to $\langle E \otimes_k K_1 \rangle_n$. \hfill \square

Proof of Theorem 1.1. (1) First we prove the case when $K/k$ is a finite dimensional extension. Assume that there exists an object $\overline{E}$ of $\text{Perf}(A_K)$ such that $\langle \overline{E} \rangle_n = \text{Perf}(A_K)$. Since $K/k$ is a finite $k$ dimensional extension, $U(\overline{E})$ belongs to $\text{Perf}(A)$. It is enough to prove that $\langle U(\overline{E}) \rangle_n = \text{Perf}(A)$. Let $G$ be an object of $\text{Perf}(A)$. Then $G \otimes_k K$ belongs to $\text{Perf}(A_K)$. Therefore $G \otimes_k K$ belongs to $\langle \overline{E} \rangle_n$. Since $U$ is exact, $U(G \otimes_k K)$ belongs to $\langle U(\overline{E}) \rangle_n$. By Lemma 2.1 $G$ belongs to $\langle U(\overline{E}) \rangle_n$.
Next we prove the general case. Assume that there exists an object $E$ of Perf$(A_K)$ such that $(E)_n = Perf(A_K)$. By Lemma 2.2, there exists an intermediate field $k \subseteq K_0 \subseteq K$ which is finite dimensional over $k$ such that there exists an object $E'$ of Perf$(A_{K_0})$ such that $E' \otimes_{K_0} K \cong E$. By the first step, it is enough to prove that $(E')_n = Perf(A_{K_0})$. Let $G'$ be an object of Perf$(A_{K_0})$. Since $G' \otimes_{K_0} K$ belongs to Perf$(A_K)$, by Lemma 2.2 there exists an intermediate field $K_0 \subseteq K_1 \subseteq K$ which is finite dimensional over $k$ such that $G' \otimes_{K_0} K_1$ belongs to $(E' \otimes_{K_0} K_1)_n$. Therefore $U_{K_1/K_0}(G' \otimes_{K_0} K_1)$ belongs to $(U_{K_1/K_0}(E' \otimes_{K_0} K_1))_n$. Hence by Lemma 2.1 $G'$ belongs to $(E')_n$.

(2) We assume that $K/k$ is a separable algebraic extension. We prove that for a perfect dg-$A$-module $E \in Perf(A)$ such that $(E)_n = Perf(A)$ for some $n \in \mathbb{N}$ we have $(E \otimes_k K')_n = Perf(A_K)$.

First we assume that the separable extension $K/k$ is finite dimensional. Let $G$ be a perfect dg-$A_{K}$-module. Since the extension $K/k$ is finite dimensional, $U(G) \subseteq Perf(A)$. Hence $U(G) \otimes_k K \subseteq Perf(A)$, and $U(G) \otimes_k K \subseteq Perf(A)$.

Theorem 2.3. (1) If an algebraic extension $K/k$ is not separable, then the dimension tridim Perf$(A_K)$ is possibly larger than the dimension tridim Perf$(A)$.

Here is an example. Let $F$ be a field of characteristic $p > 0$. Let $K := F(t)$ be a rational function field in one variable and define $k := F(t^p) \subseteq K = F(t)$. Set $A := K$. Then it is easy to see that $A_K \cong K[x]/(x^p)$. Since $	ext{gldim} A_K = \infty$, we see that tridim Perf$(A_K) = \infty$ by [3] Proposition 7.26]. However since $A = K$ is a field, we have tridim Perf$(A) = 0$.

(2) When the extension $K/k$ is not algebraic, the dimension tridim Perf$(A_K)$ is possibly larger than tridim Perf$(A)$ even if an extension $K/k$ is separable.

Here is an example. Assume that for simplicity $k$ is algebraically closed. Let $K = k(y)$ and $A = k(x)$ be rational function fields in one variable over $k$. We claim that tridim Perf$(A_K) = 1$. First note that $A_K$ is the localization $S^{-1}k[x,y]$ of the polynomial algebra $k[x,y]$ in two variables by the multiplicative set $S = \{f(x)g(y) \mid f(x) \in k[x] \setminus 0, g(y) \in k[y] \setminus 0\}$. Hence it is easy to see that $A_K$ is a regular algebra essentially of finite type over $k$ of dimension 1. Therefore we see that Perf$(A_K) \cong D^b(\text{mod} A_K)$ and tridim Perf$(A_K) \leq 1$ by [3] Proposition 7.4]. We denote by $W$ the image of the canonical embedding $i : \text{Spec} A_K \hookrightarrow \text{Spec} k[x,y]$. It is easy to see that for every maximal ideal $m$ of $k[x,y]$ there exists a prime ideal $p$ of height 1 contained in $m$ which belongs to $W$ (for a maximal ideal $m = (x - a, y - b)$ where $a, b \in k$, it is enough to set $p := (x + y - a - b)$). Therefore we see that for
every nonempty open set $U$ of $\text{Spec} \ k[x, y]$, there exists a prime ideal $p$ of height 1 which belongs to $U \cap W$. Since the derived pullback functor $\mathbb{R}i^* : D^b(\text{mod} \ k[x, y]) \to D^b(\text{mod} \ A_K)$ is essentially surjective, by the method of proof of [3, Theorem 7.17] we conclude that $\text{tridim} \ \text{Perf}(A_K) = 1$. However, since $A = k(x)$ is a field, we see that $\text{tridim} \ \text{Perf}(A) = 0$.

3. Proof of Theorem 1.2

Proof of Theorem 1.2 We denote the algebraic closure of $k$ by $\overline{k}$. By Lemma 3.1 below it is enough to show that $A_{\overline{k}} := A \otimes_k \overline{k}$ has finite representation type. By [5] it is enough to show that $\text{tridim} \ \text{mod} A_{\overline{k}} = 0$. Since we assume that $\text{tridim} \ \text{mod} A = 0,$ there exists a finite $A$-module $E$ such that $\langle E \rangle_1 = \text{mod} A$. By the same method of the proof of Theorem 1.1 (2), we see that $\langle E \otimes \overline{k} \rangle_1 = \text{mod} A_{\overline{k}}$. This completes the proof of Theorem 1.2.

Lemma 3.1. Let $A$ be a finite dimensional $k$-algebra. If $A_{\overline{k}}$ is of finite representation type, then $A$ is of finite representation type.

For a finitely generated $A$-module $M$, we denote by $\text{add}(M)$ the smallest full subcategory of $\text{mod} A$ which is closed under taking finite direct sums, direct summands and isomorphisms.

Proof. Assume that $A_{\overline{k}}$ is of finite representation type. There exists a finitely generated $A_{\overline{k}}$-module $\mathcal{N}$ such that $\text{add} \mathcal{N} = \text{mod} A_{\overline{k}}$. Since $\mathcal{N}$ has a finite presentation $A_{\overline{k}}^{\oplus n} \to A_{\overline{k}}^{\oplus m} \to \mathcal{N} \to 0$, by the same method as in the proof of Lemma 2.2 we see that there exists an intermediate field $k \subset K_0 \subset k$ which is finite dimensional over $k$ such that there exists a finitely generated $A_{K_0}$-module $N'$ such that $N' \otimes_{K_0} k \cong \mathcal{N}$ as $A_{\overline{k}}$-modules. We prove that $\text{add}(U(N')) = \text{mod} A$ where $U = U_{K_0/k} : \text{mod} A_{K_0} \to \text{mod} A$ is the forgetful functor. Let $M$ be a finitely generated $A$-module. Since $M \otimes_k k$ belongs to $\text{add}(N' \otimes_{K_0} k)$, by the same method as in the proof of Lemma 2.2 we see that there exists an intermediate field $K_0 \subset K_1 \subset k$ which is finite dimensional over $k$ such that $M \otimes_k K_1$ belongs to $\text{add}(N' \otimes_{K_0} K_1)$. Therefore we have $\text{add}(M) \subset \text{add}(N' \otimes_{K_0} K_1)$. Let $U' := U_{K_1/k} : \text{mod} A_{K_1} \to \text{mod} A$ be the forgetful functor. By the same method as in the proof of Lemma 2.2 we see that $\text{add}(M) = \text{add}(U'(M \otimes_k K_1))$ and that $\text{add}(U(N')) = \text{add}(U'(N' \otimes_{K_0} K_1))$. Hence $M$ belongs to $\text{add}(U(N'))$.

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