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A NOTE ON DIMENSION OF TRIANGULATED CATEGORIES

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Abstract. In this note we study the behavior of the dimension of the perfect derived category $\mathrm{Perf}(A)$ of a dg-algebra A over a field k under a base field extension K/k. In particular, we show that the dimension of a perfect derived category is invariant under a separable algebraic extension K/k. As an application we prove the following statement: Let A be a self-injective algebra over a perfect field k. If the dimension of the stable category $\underline{\mathrm{mod}}A$ is 0, then A is of finite representation type. This theorem is proved by M. Yoshiwaki in the case when k is an algebraically closed field. Our proof depends on his result.

1. Introduction

In [3] R. Rouquier introduced the dimension of triangulated categories and showed that it gives an upper bound or a lower bound of other dimensions in algebraic geometry or in representation theory (see also [4]). The dimension of triangulated categories is studied by many researchers.

In this note we study the behavior of the dimension of the perfect derived category $\mathrm{Perf}(A)$ of a dg-algebra A over a field k under a base field extension K/k. For a field extension K/k, we denote $A \otimes_k K$ by A_K.

Theorem 1.1.

(1) *For an algebraic extension K/k, we have*

$$\mathrm{tridim}\ \mathrm{Perf}(A) \leq \mathrm{tridim}\ \mathrm{Perf}(A_K).$$

(2) *If moreover K/k is separable, then equality holds.*

As an application we prove the following theorem, which gives evidence that dimension of triangulated categories captures some representation theoretic properties.

The stable category $\underline{\mathrm{mod}}A$ plays an important role in the study of a self-injective algebra A (cf. [2, 4]). If a self-injective algebra A is of finite representation type, then the dimension of the stable category $\underline{\mathrm{mod}}A$ is zero. Then a natural question arises as to whether the converse should also hold.

Theorem 1.2. *Let A be a self-injective finite dimensional algebra over a perfect field k. If* $\mathrm{tridim}\ \underline{\mathrm{mod}}A = 0$*, then A is of finite representation type.*

In the case when k is an algebraically closed field, this theorem is proved by M. Yoshiwaki in [5]. Our proof depends on his result.

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2. Proof of Theorem 1.1

We recall the definition of the dimension of triangulated categories. Let $\mathcal{T}$ be a triangulated category. For a full subcategory $\mathcal{I}$ of $\mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{I}$ which is closed under taking shifts, finite direct sums, direct summands and isomorphisms. For full subcategories $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{T}$ we denote by $\mathcal{I} * \mathcal{J}$ the full subcategory of $\mathcal{T}$ consisting of those objects $M \in \mathcal{T}$ such that there exists an exact triangle $I \to M \to J \xrightarrow{[1]}$ with $I \in \mathcal{I}$ and $J \in \mathcal{J}$. Set $\mathcal{I} \diamond \mathcal{J} := \langle \mathcal{I} * \mathcal{J} \rangle$. For $n \geq 1$ we define inductively

$$\langle \mathcal{I} \rangle_n := \begin{cases} \langle \mathcal{I} \rangle & \text{for } n = 1; \\ \langle \mathcal{I} \rangle \diamond \langle \mathcal{I} \rangle_{n-1} & \text{for } n \geq 2. \end{cases}$$

Now we define the dimension of a triangulated category $\mathcal{T}$ to be

$$\operatorname{tridim} \mathcal{T} := \min\{n \mid \langle E \rangle_{n+1} = \mathcal{T} \text{ for some } E \in \mathcal{T}\}.$$

To prove Theorem 1.1 we prepare some notation. We fix a field k. In the sequel the term dg-A-modules means right dg-A-modules. We denote by $C(A)$ the category of dg-A-modules. There is an adjoint pair

$$-\bigotimes_A A_K = -\bigotimes_k K : C(A) \rightleftarrows C(A_K) : \operatorname{Hom}_{A_K}(A_K, -) =: U_{K/k}.$$

We denote by $\mathcal{D}(A)$ the derived category of dg-A-modules. The functors $-\bigotimes_k K$ and $U_{K/k}$ preserve quasi-isomorphisms. Hence we obtain an adjoint pair

$$-\bigotimes_k K : \mathcal{D}(A) \rightleftarrows \mathcal{D}(A_K) : U_{K/k}.$$

If there seems to be no ambiguity, we denote $U_{K/k}$ by U.

We denote by $\operatorname{Perf}(A)$ the perfect derived category of A. This is the full triangulated subcategory of $\mathcal{D}(A)$ consisting of objects $M \in \mathcal{D}(A)$ obtained from $A \in \mathcal{D}(A)$ by taking shifts, finite direct sums, direct summands, isomorphisms and cones: $\operatorname{Perf}(A) := \bigcup_{n \geq 1} \langle A \rangle_n$. For $M \in \mathcal{D}(A)$ we denote by $\operatorname{add} M$ the smallest full subcategory of $\mathcal{D}(A)$ containing M which is closed under taking finite direct sums and direct summands.

Lemma 2.1. *If K/k is a finite dimensional extension, then for a dg-A-module M, we have* $\operatorname{add} M = \operatorname{add} U(M \otimes_k K)$ *in $\mathcal{D}(A)$. Therefore we have $\langle M \rangle = \langle U(M \otimes_k K) \rangle$. Hence we have $\langle M \rangle_n = \langle U(M \otimes_k K) \rangle_n$ for $n \geq 1$.*

Proof. The functor U is a forgetful functor. Hence we have a isomorphism $U(M \otimes_k K) \cong M^{\oplus \dim_k K}$ in $\mathcal{D}(A)$. □

Lemma 2.2. *Let K/k be an algebraic extension and E an object of $\mathcal{D}(A)$.*

(1) *If an object $\overline{G}$ of $\mathcal{D}(A_K)$ belongs to $\langle E \otimes_k K \rangle_n$, then there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that there exists an object G' of $\langle E \otimes_k K_0 \rangle_n$, such that $G' \otimes_{K_0} K \cong \overline{G}$ in $\mathcal{D}(A_K)$.*

(2) *Let G be an object of $\mathcal{D}(A)$. If $G \otimes_k K$ belongs to $\langle E \otimes_k K \rangle_n$, then there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that $G \otimes_k K_0$ belongs to $\langle E \otimes_k K_0 \rangle_n$.*

In the sequel overlined objects and morphisms are those of $\mathcal{D}(A_K)$, and objects and morphisms with dashes are those of $\mathcal{D}(A_{K_0})$.

Proof. (1) First note that for dg-A-modules M, N we have a natural isomorphism of Hom-spaces

$$\mathrm{Hom}_{\mathcal{D}(A_K)}(M \otimes_k K, N \otimes_k K) \cong \mathrm{Hom}_{\mathcal{D}(A)}(M, N) \otimes_k K.$$

It is easy to see that the problem is reduced to the following two cases, $(a), (b)$, by induction.

(a) Let E_1, E_2 be objects of $\mathcal{D}(A)$ such that we have an exact triangle

$$E_1 \otimes_k K \xrightarrow{\overline{f}} E_2 \otimes_k K \to \overline{G} \xrightarrow{[1]}$$

in $\mathcal{D}(A_K)$. By the above remark there are $f_1, \ldots, f_n \in \mathrm{Hom}_{\mathcal{D}(A)}(E_1, E_2)$ and $\alpha_1, \ldots, \alpha_n \in K$ such that we have $\overline{f} = \sum_{i=1}^n f_i \alpha_i$. Set $K_0 := k(\alpha_1, \ldots, \alpha_n)$ and $f' := \sum_{i=1}^n f_i \alpha_i \in \mathrm{Hom}_{\mathcal{D}(A_{K_0})}(E_1 \otimes_k K_0, E_2 \otimes_k K_0)$. Then $f' \otimes_{K_0} K = \overline{f}$. Therefore we have an isomorphism $G' \otimes_{K_0} K \cong \overline{G}$ in $\mathcal{D}(A_K)$ where $G' := \mathrm{c}(f')$ is the cone of f'.

(b) Let E_1 be an object of $\mathcal{D}(A)$. Assume that $\overline{G}$ is a direct summand of $E_1 \otimes_k K$. Let $\overline{e}$ be an idempotent endomorphism of $E_1 \otimes_k K$ which corresponds to $\overline{G}$, i.e., $\overline{e} : E_1 \otimes_k K \twoheadrightarrow \overline{G} \hookrightarrow E_1 \otimes_k K$, where the left arrow is the canonical projection and the right arrow is the canonical injection. Note that we can obtain the direct summand $\overline{G}$ from the corresponding idempotent endomorphism $\overline{e}$ in the following way: We define a morphism $\overline{X} : (E_1 \otimes_k K)^{\oplus \mathbb{N}} \to (E_1 \otimes_k K)^{\oplus \mathbb{N}}$ to be the morphism which is represented by the matrix given by

$$\overline{X} = \begin{pmatrix} 1-\overline{e} & 0 & 0 & 0 & \cdots \\ \overline{e} & 1-\overline{e} & 0 & 0 & \cdots \\ 0 & \overline{e} & 1-\overline{e} & 0 & \cdots \\ 0 & 0 & \overline{e} & 1-\overline{e} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} : (E_1 \otimes_k K)^{\oplus \mathbb{N}} \to (E_1 \otimes_k K)^{\oplus \mathbb{N}}.$$

Then the cone $\mathrm{c}(\overline{X})$ of $\overline{X}$ is isomorphic to $\overline{G}$ in $\mathcal{D}(A_K)$. By the same method as above, we see that there is an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that there is an idempotent endomorphism e' of $E_1 \otimes_k K_0$ such that we have $e' \otimes_{K_0} K = \overline{e}$. Hence we obtain an endomorphism X' of $(E_1 \otimes_k K_0)^{\oplus \mathbb{N}}$ such that $X' \otimes_k K_0 = \overline{X}$. Then $\overline{G}$ is isomorphic to $G' \otimes_k K_0$ in $\mathcal{D}(A_K)$, where $G' := c(X')$ is a cone of X'. We see that G' is a direct summand of $E_1 \otimes_k K_0$ by [1, Proposition 3.2.].

(2) By (1) there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that $G' \otimes_{K_0} K \cong G \otimes_k K$ in $\mathcal{D}(A_K)$ for some object G' of $\langle E \otimes_k K_0 \rangle_n$. Let $\overline{f} : G' \otimes_{K_0} K \xrightarrow{\cong} (G \otimes_k K_0) \otimes_{K_0} K$ be an isomorphism in $\mathcal{D}(A_K)$. By the same method as in the proof of (1), we see that there exists an intermediate field $K_0 \subset K_1 \subset K$ which is finite dimensional over k such that there exists an isomorphism $f' : G' \otimes_{K_0} K_1 \xrightarrow{\cong} (G \otimes_k K_0) \otimes_{K_0} K_1 \cong G \otimes_k K_1$. Hence $G \otimes_k K_1$ belongs to $\langle E \otimes_k K_1 \rangle_n$. □

Proof of Theorem 1.1. (1) First we prove the case when K/k is a finite dimensional extension. Assume that there exists an object $\overline{E}$ of $\mathrm{Perf}(A_K)$ such that $\langle \overline{E} \rangle_n = \mathrm{Perf}(A_K)$. Since K/k is a finite k dimensional extension, $U(\overline{E})$ belongs to $\mathrm{Perf}(A)$. It is enough to prove that $\langle U(\overline{E}) \rangle_n = \mathrm{Perf}(A)$. Let G be an object of $\mathrm{Perf}(A)$. Then $G \otimes_k K$ belongs to $\mathrm{Perf}(A_K)$. Therefore $G \otimes_k K$ belongs to $\langle \overline{E} \rangle_n$. Since U is exact, $U(G \otimes_k K)$ belongs to $\langle U(\overline{E}) \rangle_n$. By Lemma 2.1, G belongs to $\langle U(\overline{E}) \rangle_n$.

Next we prove the general case. Assume that there exists an object $\overline{E}$ of $\operatorname{Perf}(A_K)$ such that $\langle \overline{E} \rangle_n = \operatorname{Perf}(A_K)$. By Lemma 2.2, there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that there exists an object E' of $\operatorname{Perf}(A_{K_0})$ such that $E' \otimes_{K_0} K \cong \overline{E}$. By the first step, it is enough to prove that $\langle E' \rangle_n = \operatorname{Perf}(A_{K_0})$. Let G' be an object of $\operatorname{Perf}(A_{K_0})$. Since $G' \otimes_{K_0} K$ belongs to $\operatorname{Perf}(A_K) = \langle E' \otimes_{K_0} K \rangle_n$, by Lemma 2.2 there exists an intermediate field $K_0 \subset K_1 \subset K$ which is finite dimensional over k such that $G' \otimes_{K_0} K_1$ belongs to $\langle E' \otimes_{K_0} K_1 \rangle_n$. Therefore $U_{K_1/K_0}(G' \otimes_{K_0} K_1)$ belongs to $\langle U_{K_1/K_0}(E' \otimes_{K_0} K_1) \rangle_n$. Hence by Lemma 2.1, G' belongs to $\langle E' \rangle_n$.

(2) We assume that K/k is a separable algebraic extension. We prove that for a perfect dg-A-module $E \in \operatorname{Perf}(A)$ such that $\langle E \rangle_n = \operatorname{Perf}(A)$ for some $n \in \mathbb{N}$ we have $\langle E \otimes_k K \rangle_n = \operatorname{Perf}(A_K)$.

First we assume that the separable extension K/k is finite dimensional. Let $\overline{G}$ be a perfect dg-A_K-module. Since the extension K/k is finite dimensional, $U(\overline{G})$ belongs to $\langle E \rangle_n = \operatorname{Perf}(A)$. Hence $U(\overline{G}) \otimes_k K$ belongs to $\langle E \otimes_k K \rangle_n$. Since the extension K/k is separable, the morphism $\mu : K \otimes_k K \to K$, $x \otimes y \mapsto xy$ of $K \otimes_k K$-modules has a section. Hence the morphism $\overline{G} \otimes_K \mu$ of dg-A_K-modules has a section

$$U(\overline{G}) \otimes_k K \cong \overline{G} \otimes_K K \otimes_k K \xrightarrow{\overline{G} \otimes_K \mu} \overline{G} \otimes_K K \cong \overline{G}.$$

Therefore $\overline{G}$ is a direct summand of $U(\overline{G}) \otimes_k K$. Hence by Lemma 2.1, $\overline{G}$ belongs to $\langle E \otimes_k K \rangle_n$.

Finally we prove the general case. Let $\overline{G}$ be a perfect dg-A_K-module. By Lemma 2.2 there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that there exists a perfect A_{K_0}-module G' such that $G' \otimes_{K_0} K \cong \overline{G}$ in $\mathcal{D}(A_K)$. Since G' belongs to $\langle E \otimes_k K_0 \rangle_n$ by the first step, we conclude that $\overline{G}$ belongs to $\langle E \otimes_k K \rangle_n$. $\square$

Remark 2.3. (1) If an algebraic extension K/k is not separable, then the dimension tridim $\operatorname{Perf}(A_K)$ is possibly larger than the dimension tridim $\operatorname{Perf}(A)$.

Here is an example. Let F be a field of characteristic $p > 0$. Let $K := F(t)$ be a rational function field in one variable and define $k := F(t^p) \subset K = F(t)$. Set $A := K$. Then it is easy to see that $A_K \cong K[x]/(x^p)$. Since $\operatorname{gldim} A_K = \infty$, we see that tridim $\operatorname{Perf}(A_K) = \infty$ by [3, Proposition 7.26]. However since $A = K$ is a field, we have tridim $\operatorname{Perf}(A) = 0$.

(2) When the extension K/k is not algebraic, the dimension tridim $\operatorname{Perf}(A_K)$ is possibly larger than tridim $\operatorname{Perf}(A)$ even if an extension K/k is separable.

Here is an example. Assume that for simplicity k is algebraically closed. Let $K = k(y)$ and $A = k(x)$ be rational function fields in one variable over k. We claim that tridim $\operatorname{Perf}(A_K) = 1$. First note that A_K is the localization $S^{-1}k[x,y]$ of the polynomial algebra $k[x,y]$ in two variables by the multiplicative set $S = \{f(x)g(y) \mid f(x) \in k[x] \setminus 0, g(y) \in k[y] \setminus 0\}$. Hence it is easy to see that A_K is a regular algebra essentially of finite type over k of dimension 1. Therefore we see that $\operatorname{Perf}(A_K) \simeq \mathcal{D}^b(\operatorname{mod} A_K)$ and tridim $\operatorname{Perf}(A_K) \leq 1$ by [3, Proposition 7.4]. We denote by W the image of the canonical embedding $i : \operatorname{Spec} A_K \hookrightarrow \operatorname{Spec} k[x,y]$. It is easy to see that for every maximal ideal $\mathfrak{m}$ of $k[x,y]$ there exists a prime ideal $\mathfrak{p}$ of height 1 contained in $\mathfrak{m}$ which belongs to W (for a maximal ideal $\mathfrak{m} = (x-a, y-b)$ where $a, b \in k$, it is enough to set $\mathfrak{p} := (x+y-a-b)$). Therefore we see that for

every nonempty open set U of $\operatorname{Spec} k[x,y]$, there exists a prime ideal $\mathfrak{p}$ of height 1 which belongs to $U \cap W$. Since the derived pullback functor $\mathbb{R}i^* : \mathcal{D}^b(\operatorname{mod} k[x,y]) \to \mathcal{D}^b(\operatorname{mod} A_K)$ is essentially surjective, by the method of proof of [3, Theorem 7.17] we conclude that $\operatorname{tridim} \operatorname{Perf}(A_K) = 1$. However, since $A = k(x)$ is a field, we see that $\operatorname{tridim} \operatorname{Perf}(A) = 0$.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. We denote the algebraic closure of k by $\overline{k}$. By Lemma 3.1 below it is enough to show that $A_{\overline{k}} := A \otimes_k \overline{k}$ has finite representation type. By [5] it is enough to show that $\operatorname{tridim} \underline{\operatorname{mod}} A_{\overline{k}} = 0$. Since we assume that $\operatorname{tridim} \underline{\operatorname{mod}} A = 0$, there exists a finite A-module E such that $\langle E \rangle_1 = \underline{\operatorname{mod}} A$. By the same method of the proof of Theorem 1.1 (2), we see that $\langle E \otimes \overline{k} \rangle_1 = \underline{\operatorname{mod}} A_{\overline{k}}$. This completes the proof of Theorem 1.2. $\square$

Lemma 3.1. *Let A be a finite dimensional k-algebra. If $A_{\overline{k}}$ is of finite representation type, then A is of finite representation type.*

For a finitely generated A-module M, we denote by $\operatorname{add}(M)$ the smallest full subcategory of $\operatorname{mod} A$ which is closed under taking finite direct sums, direct summands and isomorphisms.

Proof. Assume that $A_{\overline{k}}$ is of finite representation type. There exists a finitely generated $A_{\overline{k}}$-module $\overline{N}$ such that $\operatorname{add} \overline{N} = \operatorname{mod} A_{\overline{k}}$. Since $\overline{N}$ has a finite presentation $A_{\overline{k}}^{\oplus n} \to A_{\overline{k}}^{\oplus m} \to \overline{N} \to 0$, by the same method as in the proof of Lemma 2.2, we see that there exists an intermediate field $k \subset K_0 \subset \overline{k}$ which is finite dimensional over k such that there exists a finitely generated A_{K_0}-module N' such that $N' \otimes_{K_0} \overline{k} \cong \overline{N}$ as $A_{\overline{k}}$-modules. We prove that $\operatorname{add}(U(N')) = \operatorname{mod} A$ where $U = U_{K_0/k} : \operatorname{mod} A_{K_0} \to \operatorname{mod} A$ is the forgetful functor. Let M be a finitely generated A-module. Since $M \otimes_k \overline{k}$ belongs to $\operatorname{add}(N' \otimes_{K_0} \overline{k})$, by the same method as in the proof of Lemma 2.2, we see that there exists an intermediate field $K_0 \subset K_1 \subset \overline{k}$ which is finite dimensional over k such that $M \otimes_k K_1$ belongs to $\operatorname{add}(N' \otimes_{K_0} K_1)$. Therefore we have $\operatorname{add}(M \otimes_k K_1) \subset \operatorname{add}(N' \otimes_{K_0} K_1)$. Let $U' := U_{K_1/k} : \operatorname{mod} A_{K_1} \to \operatorname{mod} A$ be the forgetful functor. By the same method as in the proof of Lemma 2.1, we see that $\operatorname{add}(M) = \operatorname{add}(U'(M \otimes_k K_1))$ and that $\operatorname{add}(U(N')) = \operatorname{add}(U'(N' \otimes_{K_0} K_1))$. Hence M belongs to $\operatorname{add}(U(N'))$. $\square$

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