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EIGENVALUES OF WEIGHTED p-LAPLACIAN

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ABSTRACT. In a paper by Z. Lu and J. Rowlett, it is shown that the eigenvalues of the weighted Laplacian can be approximated by eigenvalues of a naturally associated family of narrow graphs. In this paper, we generalize this result to the p-Laplacian. Our approach features overcoming the nonlinearity of the p-Laplacian when $p \neq 2$, which is different from the Laplacian case.

1. INTRODUCTION

It has always been one of the most important and interesting topics to understand how partial differential operators are affected by the geometry of the domain where they are defined. Much work has been done to establish the connection of eigenvalues of the Laplacian to the Ricci curvature on Riemannian manifolds. In this direction, the consideration of other important geometric objects, such as smooth metric spaces and the p-Laplacian, becomes the next area of interest.

Introduced by Bakry and Émery in studying diffusion process on manifolds [2], smooth metric spaces, also known as weighted measure spaces, have recently received considerable interest. One of the reasons is their importance in Perelman's work on the Ricci flow [7]. By using smooth metric spaces, Perelman was able to reformulate the Ricci flow as a gradient flow and establish monotone quantities and comparison theorems. See [3] for a good summary on the topic.

A smooth metric space is a Riemannian manifold equipped with some measure conformal to the usual Riemannian measure. Given a Riemannian manifold (M, g), the triple $(M, g, e^{-\varphi}dV)$ is called a smooth metric space where φ is a smooth function on M and dV is the volume element. In this paper, we always assume $f = e^{-\varphi}$.

The weighted Laplacian Δ_f is defined as

$$\Delta_f = \Delta - \nabla\phi \cdot \nabla,$$

where Δ is the usual Laplacian. Δ_f is a symmetric operator with respect to the weighted measure $e^{-\varphi}dV$. That is, for any $u, v \in C_0^\infty(M)$,

$$-\int_M v\Delta_f u f dV = \int \langle \nabla u, \nabla v\rangle f dV.$$

The Bakry-Émery Ricci tensor is defined to be

$$\mathrm{Ric}_\infty = \mathrm{Ric} + \mathrm{Hess}\,\phi,$$

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where Ric is the Ricci curvature of M. Then the following Bochner formula is true:

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

The relation between Δ_f and Ric_∞ has been extensively studied, and many geometric and topological results have been established in this direction (cf. [10]).

Recently, in [5], Lu and Rowlett showed that there is a natural family of manifolds associated to a compact smooth metric space so that their Neumann eigenvalues converge to the weighted Neumann eigenvalues. In other words, we can get information about the eigenvalues of the weighted Laplacian by applying the known results on eigenvalues of the usual Laplacian.

Theorem 1.1 (Lu-Rowlett)**.** *Let (M, g, ϕ) be a compact smooth metric space. Let*

$$M_\varepsilon \doteq \{(x, y) \mid x \in M, 0 \le y \le \varepsilon f\} \tag{1.1}$$

with $\phi \in \mathcal{C}^2(M)$ and $f \in \mathcal{C}(M \cup \partial M)$. Let $\{\mu_k\}_{k=0}^\infty$ be the eigenvalues of the weighted Laplacian on M. If $\partial M \neq 0$, assume the Neumann boundary condition. Let $\mu_k(\varepsilon)$ be the kth nonzero Neumann eigenvalues of M_ε for $\tilde{\Delta} \doteq \Delta + \partial_y^2$. Then, for $k \ge 0$,

$$\mu_k(\varepsilon) = \mu_k + O(\varepsilon^2).$$

In this paper, we get similar results for a more general operator, the p-Laplacian, which is defined as

$$\Delta_p(u) = \mathrm{div}(|\nabla u|^{p-2}\nabla u),$$

for $1 < p < \infty$. Besides being a generalization of the usual Laplacian, the p-Laplacian occurs naturally in many subjects. Δ_p models the non-Newtonian fluids in physics. It describes dilatant fluids when $p > 2$ and pseudoplastics when $p < 2$, whereas $p = 2$ corresponds to Newtonian fluids. The operator Δ_p with $p \neq 2$ also appears in many other applications, such as reaction-diffusion problems, flow through porous media, nonlinear elasticity, glaceology, and petroleum extraction. See [9] for more details. An interesting connection of the p-Laplacian to inverse mean curvature flow was established by Moser in [6]. Δ_p also naturally arises in mappings between Riemannian manifolds (cf. [8]).

With respect to the weighted measure, we introduce the weighted p-Laplacian denoted by $\Delta_{p,f}$ as follows:

$$\Delta_{p,f}(u) = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^{p-2}\nabla\phi \cdot \nabla.$$

Let μ denote the first nonzero Neumann eigenvalue of Δ_p on a compact smooth metric space. That is, μ is the smallest nonzero constant for which the following equation has a nontrivial solution $v \in \mathcal{C}^{1,\alpha}(M)$ in the weak sense:

$$\Delta_p(v) = -\mu|v|^{p-2}v,$$

where $\frac{\partial \mu}{\partial n} = 0$ on ∂M. Similarly, let $\mu_{p,f}$ denote the first nonzero Neumann eigenvalue of $\Delta_{p,f}$. That is, $\mu_{p,f}$ is the smallest nonzero constant so that there is a nontrivial solution $u \in \mathcal{C}^{1,\alpha}(M)$ for the following equation in the weak sense:

$$\triangle_{p,f}(u) = -\mu_{p,f}|u|^{p-2}v,$$

where $\frac{\partial \mu}{\partial n} = 0$ on ∂M.

The main result of this paper is:

Theorem 1.2 (Main theorem)**.** *Given a compact smooth metric space (M, g, ϕ) with dimension m, let $\mu_{p,f}$ be the first nonzero eigenvalue of the weighted p-Laplacian on M. Assume $p > 1$. If $\partial M \neq \emptyset$, assume the Neumann boundary condition on M. Let $\mu_{p,\varepsilon}$ be the first nonzero Neumann eigenvalue of p-Laplacian $\tilde{\Delta}_p$ on M_ε (defined in Theorem 1.1). Then*

$$\mu_{p,\varepsilon} \leq \mu_{p,f} \leq \mu_{p,\varepsilon} + O(\varepsilon).$$

Remark 1.3. When $p \neq 2$, we do not have similar results for higher eigenvalues of the weighted p-Laplacian as when $p = 2$. We do not know how to define higher eigenvalues for the p-Laplacian.

Remark 1.4. More detailed regularity information of the eigenfunctions is needed to obtain a better convergence rate for $p \neq 2$. Our method yields $O(\varepsilon)$, which is enough to convert a weighted p-Laplacian eigenvalue estimate into a p-Laplacian eigenvalue problem.

The article is organized as follows: first we will briefly discuss the derivation of the weighted p-Laplacian and the variational principles in the Riemannian case and the weighted p-Laplacian case in Section 2. Then we define some auxiliary functions based on eigenfunctions and discuss their properties in Section 3. With these, we construct test functions and prove the main theorem, which is done in Section 4. In Section 5, we discuss the L_∞ estimates for eigenfunctions of the p-Laplacian. Such estimates later play an important part in the convergence of eigenfunctions. We also believe there is an independent interest. We want to point out that the treatment in Section 3 is simpler than what was done in [5]. Another new result in this paper is the absence of the case $p = n$ in the Sobolev inequalities on compact manifolds. We overcome this by using the Sobolev inequalities on compact manifolds in higher dimensions.

2. The weighted p-Laplacian and variational principles

On a compact Riemannian manifold M, by the first Green formula, we have

$$\int_M h\Delta u dV = -\int_M \langle \nabla h, \nabla u\rangle dv + \int_{\partial M} h u_n dV \tag{2.1}$$

for any smooth functions u and v on M, where $u_n = \frac{\partial u}{\partial n}$ with $\frac{\partial}{\partial n}$ as the outward directional derivative. If we replace the measure dV by the weighted measure $f dV$, then

$$\begin{aligned}\int_M h \cdot \Delta u\, f dV &= -\int_M \langle \nabla f h, \nabla u\rangle\, dV + \int_{\partial M} h u_n f\, dV \\ &= -\int_M \langle \nabla h, \nabla u\rangle\, f dV - \int_M \langle \nabla \log f, \nabla u\rangle h\, f dV + \int_{\partial M} u_n h\, f dV.\end{aligned}$$

Therefore we have

$$\int_M h(\Delta u + \langle \nabla \log f, \nabla u\rangle)\, f dV = -\int_M \langle \nabla h, \nabla u\rangle\, f dV + \int_{\partial M} h u_n\, f dV.$$

The weighted Laplacian is defined as

$$\Delta_f u = \Delta u + \langle \nabla u, \nabla \log f\rangle = \Delta u - \langle \nabla u, \nabla \phi\rangle.$$

Similarly, in the case of the p-Laplacian, by the divergence theorem, we have

$$\int_M h\Delta_p u dV = -\int_M \langle \nabla h, |\nabla u|^{(p-2)}\nabla u\rangle dV + \int_{\partial M} |\nabla u|^{(p-2)} h u_n dV \tag{2.2}$$

for any smooth functions u and v on M. With respect to the weighted measure $f dV$, (2.2) becomes

$$\begin{aligned}\int_M h\cdot \Delta_p u\, f dV &= -\int_M \langle \nabla f h, |\nabla u|^{(p-2)}\nabla u\rangle\, dV + \int_{\partial M} |\nabla u|^{(p-2)} h u_n f\, dV \\ &= -\int_M \langle \nabla h, |\nabla u|^{(p-2)}\nabla u\rangle\, f dV - \int_M \langle \nabla \log f, |\nabla u|^{(p-2)}\nabla u\rangle h\, f dV \\ &\quad + \int_{\partial M} |\nabla u|^{(p-2)} u_n h\, f dV.\end{aligned}$$

Therefore, we get the following:

$$\begin{aligned}\int_M h(\Delta_p u + |\nabla u|^{(p-2)}\langle \nabla \log f, \nabla u\rangle)\, f dV &= -\int_M \langle \nabla h, |\nabla u|^{(p-2)}\nabla u\rangle\, f dV \\ &\quad + \int_{\partial M} |\nabla u|^{(p-2)} h u_n\, f dV.\end{aligned} \tag{2.3}$$

Based on the above, the p-Laplacian is defined as follows:

$$\Delta_{p,f} u = \Delta u + |\nabla u|^{(p-2)}\langle \nabla u, \nabla \log f\rangle = \Delta u - |\nabla u|^{(p-2)}\langle \nabla u, \nabla \phi\rangle.$$

Rephrasing (2.3), we get

$$\int_M h\Delta_{p,f} u\, f dV = -\int_M \langle \nabla h, |\nabla u|^{(p-2)}\nabla u\rangle\, f dV + \int_{\partial M} |\nabla u|^{(p-2)} h u_n\, f dV.$$

For any $1 < p < \infty$, we have the following variational characterization for the p-Laplacian:

$$\mu_p = \inf\left\{ \frac{\int_M |\nabla u|^p}{\int_M |u|^p} \,\middle|\, \int_M |u|^{p-2} u = 0 \right\}. \tag{2.4}$$

When $\partial M \neq \emptyset$, the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ is automatically satisfied.

The above variational principles also hold true for the weighted p-Laplacian in a smooth metric space.

Proposition 2.1. *Given a smooth metric space (M, g, ϕ) with $f = e^{-\phi}$, let $\mu_{p,f}$ be the first nonzero eigenvalue of the p-Laplacian on M. When $\partial M \neq \emptyset$, assume the Neumann boundary condition. Then*

$$\mu_{p,f} = \inf\left\{ \frac{\int_M |\nabla u|^p f}{\int_M |u|^p f} \right\}, \tag{2.5}$$

where u is running through functions such that

$$\int_M |u|^{p-2} u\, f = 0.$$

The proof of this proposition is similar to the Laplacian case.

3. Some estimates

In this section, we demonstrate some estimates about eigenfunctions of $\mu_{p,\varepsilon}$ of M_ε. Fixing ε, let $\tilde{\Delta}$ be the p-Laplacian on M_ε and $v = v_\varepsilon$ be the eigenfunction of $\mu_{p,\varepsilon}$:

$$\tilde{\Delta}_p v = -\mu_{p,\varepsilon}\, |v|^{p-2} v.$$

We normalize v so that $\int_{M_\varepsilon} |v|^p = \varepsilon$. Let u be the eigenfunction of $\mu_{p,f}$:

$$\Delta_{p,f} u = -\mu_{p,f} |u|^{p-2} u,$$

with a normalized u so that $\int_M |u|^p f dV = 1$.

Let V and V_f be the volume of M with respect to the measure dV and the weighted measure fdV respectively. Then $V_f \le \|f\|_\infty V$. Let V_ε be the volume of M_ε. Then $V_\varepsilon = \varepsilon V_f \le \varepsilon \|f\|_\infty V$.

For any $0 \le r \le \varepsilon$, define an auxiliary function

$$b(x,r) \doteq v(x, rf(x)).$$

Lemma 3.1. *For $1 \le k \le p$ and $1 < p$, fixed r and any constant a, we have*

$$\left| \varepsilon \int_M |b+a|^k f - \int_{M_\varepsilon} |v+a|^k \right| \le \varepsilon^2 \mu_\varepsilon^{1/p} V^{1-k/p} k (1 + |a| V^{\frac{1}{p}})^k \|f\|_\infty^{2-k/p}.$$

Proof. For any $0 \le r \le \varepsilon$, $0 \le z \le \varepsilon f(x)$,

$$\begin{aligned} \big| |b+a|^k(x,r) - |v+a|^k(x,z) \big| &\le \int_0^{\varepsilon f} \left| \frac{\partial |v+a|^k}{\partial y}(x,y) \right| dy \\ &\le k \int_0^{\varepsilon f} |v+a|^{k-1} \left| \frac{\partial v}{\partial y} \right| dy. \end{aligned} \tag{3.1}$$

Note that for any $0 \le r \le \varepsilon$,

$$\int_{M_\varepsilon} |b+a|^k(x,r) dx dy = \int_M \int_0^{\varepsilon f} |b+a|^k(x,r) dy dx = \varepsilon \int_M |b+a|^k f.$$

Then by (3.1),

$$\begin{aligned} \left| \varepsilon \int_M |b+a|^k f - \int_{M_\varepsilon} |v+a|^k \right| &= \int_{M_\varepsilon} \big| |b(x,r)+a|^k - |v(x,y)+a|^k \big| \\ &\le \int_{M_\varepsilon} k \int_0^{\varepsilon f} |v+a|^{k-1} \left| \frac{\partial v}{\partial y} \right| dy dV_{M_\varepsilon} \\ &\le k\varepsilon \|f\|_\infty \int_{M_\varepsilon} |v+a|^{k-1} \left| \frac{\partial v}{\partial y} \right|. \end{aligned}$$

By the Hölder inequality,

$$\begin{aligned}\int_{M_\varepsilon} |v+a|^{k-1}\left|\frac{\partial v}{\partial y}\right| &\le (\int_{M_\varepsilon} |v+a|^p)^{\frac{k-1}{p}} (\int_{M_\varepsilon} |\nabla v|^p)^{\frac{1}{p}} (\int_{M_\varepsilon} 1)^{1-\frac{k-1}{p}-\frac{1}{p}} \\ &= (\int_{M_\varepsilon} |v+a|^p)^{\frac{k-1}{p}} (\mu_\varepsilon \int_{M_\varepsilon} |v|^p)^{\frac{1}{p}} (V_\varepsilon)^{1-\frac{k-1}{p}-\frac{1}{p}} \\ &= (\int_{M_\varepsilon} |v+a|^p)^{\frac{k-1}{p}} (\mu_\varepsilon \varepsilon)^{\frac{1}{p}} (V_\varepsilon)^{1-\frac{k-1}{p}-\frac{1}{p}} \\ &\le (\int_{M_\varepsilon} |v+a|^p)^{\frac{k-1}{p}} (\mu_\varepsilon \varepsilon)^{\frac{1}{p}} (\varepsilon \|f\|_\infty V)^{1-\frac{k-1}{p}-\frac{1}{p}} \\ &\le \varepsilon \|f\|_\infty^{1-k/p} (1+|a|\,\|f\|_\infty V^{\frac{1}{p}})^{k-1} V^{1-\frac{k}{p}} \mu_\varepsilon^{\frac{1}{p}}.\end{aligned}$$

The last line is because

$$\begin{aligned}\int_{M_\varepsilon} |v+a|^p &\le \left(\left(\int_{M_\varepsilon} |v|^p\right)^{\frac{1}{p}} + |a| V_\varepsilon^{\frac{1}{p}}\right)^p \\ &\le \left(\varepsilon^{\frac{1}{p}} + |a|(\varepsilon\|f\|_\infty V)^{\frac{1}{p}}\right)^p \\ &= \varepsilon(1+|a|\,\|f\|_\infty V^{\frac{1}{p}})^p.\end{aligned}$$

Thus,

$$\left|\varepsilon \int_M |b+a|^k f - \int_{M_\varepsilon} |v+a|^k\right| \le \varepsilon^2 \mu_\varepsilon^{1/p} V^{1-k/p} k (1+|a|V^{\frac{1}{p}})^k \|f\|_\infty^{2-k/p}$$

for $1 \le k < \infty$. □

Corollary 3.2. *Using the same notation as above, we have*

$$\int_M |b+a|^p f \ge 1 - \varepsilon \mu_\varepsilon^{1/p} p \|f\|_\infty (1+|a|V^{\frac{1}{p}})^p \tag{3.2}$$

when $p > 1$.

Proof. Lemma 3.1 implies that when $k = p$,

$$\left|\varepsilon \int_M |b+a|^p f - \int_{M_\varepsilon} |v+a|^p\right| \le \varepsilon^2 \mu_\varepsilon^{1/p} p \|f\|_\infty (1+|a||V|^{\frac{1}{p}})^p. \tag{3.3}$$

Since the function $g(x) = |x|^p$ is convex for $p > 1$, we have

$$f(v+a) - f(v) \ge f'(v)a.$$

Since

$$f'(v) = p|v|^{p-2}v, \text{ if } v \ne 0$$

and

$$\int_{\{v\ne 0\}} |v|^{p-2} v = \int_{M_\varepsilon} |v|^{p-2} v = 0,$$

we obtain

$$\int_{\{v\ne 0\}} |v+a|^p \ge \int_{\{v\ne 0\}} |v|^p.$$

Therefore

$$\begin{aligned}\int_{M_\varepsilon} |v+a|^p - \int_{M_\varepsilon} |v|^p &= \int_{\{v\neq 0\}} |v+a|^p + \int_{\{v=0\}} |a|^p - \int_{\{v\neq 0\}} |v|^p \\ &\geq \int_{\{v=0\}} |a|^p \\ &\geq 0.\end{aligned}$$

Plugging this into (3.3) implies

$$\int_M |b+a|^p f \geq 1 - \varepsilon \mu_\varepsilon^{1/p} V^{1-k/p} k \|f\|_\infty^{2-k/p}.$$

□

Remark 3.3. The case $p = 2$ easily follows from $|b+a|^2 = b^2 + 2ba + a^2$ and convexity, and Lemma 3.1 is used for general $p \neq 2$.

4. Proof of the main theorem

In this section, we will show two inequalities:

$$\mu_{p,\varepsilon} \leq \mu_{p,f}, \tag{4.1}$$

$$\mu_{p,f} \leq \mu_{p,\varepsilon} + O(\varepsilon). \tag{4.2}$$

The main theorem obviously follows from these two inequalities. To obtain these inequalities, we apply the variational principles in Section 2 by carefully choosing test functions for $\mu_{p,\varepsilon}$ and $\mu_{p,f}$.

Lemma 4.1.

$$\mu_{p,\varepsilon} \leq \mu_{p,f}.$$

Proof. Recall that u is the corresponding eigenfunction of $\mu_{p,f}$. For any $(x,y) \in M_\varepsilon$, let

$$h(x,y) = u(x).$$

Then

$$\int_{M_\varepsilon} |h|^{p-2} h = \int_M \int_0^{\varepsilon f} |h|^{p-2} h(x) dy dx = \varepsilon \int_M |u|^{p-2} u f = 0.$$

By the variational principle (2.4),

$$\mu_{p,\varepsilon} \leq \frac{\int_{M_\varepsilon} |\widetilde{\nabla} h|^p}{\int_{M_\varepsilon} |h|^p} = \frac{\varepsilon \int_M |\nabla u|^p f}{\varepsilon \int_M |u|^p f} = \mu_{p,f}.$$

□

Based on this result, we rewrite Corollary 3.2 by replacing μ_ε by $\mu_{p,f}$.

Corollary 4.2. *Using the same notation as in Section 3, for b we have*

$$\int_M |b+a|^p f \geq 1 - \varepsilon \mu_{p,f}^{1/p} p \|f\|_\infty (1 + |a| V^{\frac{1}{p}})^p \tag{4.3}$$

when $p > 1$.

Lemma 4.3.

$$\mu_{p,f} \leq \mu_{p,\varepsilon} + O(\varepsilon).$$

Proof. Recall that

$$b(x,r) \doteq v(x, rf(x)).$$

For any $\delta \in R$, let

$$B(\delta) = \int_M |b+\delta|^{p-2}(b+\delta)f.$$

Since $\lim_{\delta\to\infty} B(\delta) = \infty$ and $\lim_{\delta\to-\infty} B(\delta) = -\infty$, by continuity of $B(\delta)$ there exists a δ so that

$$B(\delta) = 0. \tag{4.4}$$

By the variational principle (2.5),

$$\mu_{p,f} \le \frac{\int_M |\nabla(b+\delta)|^p f}{\int_M |b+\delta|^p f}. \tag{4.5}$$

Because of (4.4), we know that

$$|\delta| \le \|b\|_\infty = \|v\|_\infty.$$

Combining this with the L_∞ estimate in Corollary 5.7 implies that

$$|\delta| \le C(1+\mu_\varepsilon)^{\frac{m+2}{p^2}} \le C(1+\mu_{p,f})^{\frac{m+2}{p^2}},$$

where C depends only on M.

By Corollary 4.2, we have

$$\begin{aligned}\int_M |b+\delta|^p f &\ge 1 - \varepsilon\mu_{p,f}^{1/p} p\|f\|_\infty (1+|\delta|V^{\frac{1}{p}})^p \\ &\ge 1 - \varepsilon\mu_{p,f}^{1/p} p\|f\|_\infty (1+C(1+\mu_{p,f})^{\frac{m+2}{p^2}}|V^{\frac{1}{p}})^p.\end{aligned} \tag{4.6}$$

Therefore

$$\frac{1}{\int_M |b+\delta|^p f} \le 1 + O(\varepsilon), \tag{4.7}$$

with a replaced by δ. Denote this upper bound by A. On the other hand, since

$$|\nabla(b+\delta)| = |\nabla b| = |\nabla_x v(x,rf) + r\frac{\partial v}{\partial y}(x,rf)\nabla f|,$$

we have

$$\int_M |\nabla(b+\delta)|^p f \le (1+\varepsilon\|\nabla f\|_\infty)^p \int_M |\tilde{\nabla} v|^p(x,rf) f. \tag{4.8}$$

Integrating on both sides of (4.8), we obtain

$$\begin{aligned}\int_0^\varepsilon \int_M |\nabla(b+\delta)|^p(x,rf) f \mathrm{dx} &\le (1+\varepsilon\|f\|_\infty)^p \int_0^\varepsilon [\int_M |\widetilde{\nabla} v|^p(x,rf) f \mathrm{dx}]\mathrm{dr} \\ &= (1+\varepsilon\|f\|_\infty)^p \int_{M_\varepsilon} |\widetilde{\nabla} v|^p \\ &= (1+\varepsilon\|f\|_\infty)^p \varepsilon\mu_{p,\varepsilon}.\end{aligned} \tag{4.9}$$

Plugging (4.7) and (4.9) into (4.5) yields

$$\begin{aligned}\varepsilon\mu_{p,f} &\le A\int_0^\varepsilon \int_M |\nabla(b+\delta)|^p f dx dr \\ &\le A(1+\varepsilon\|f\|_\infty)^p \varepsilon\mu_{p,\varepsilon}.\end{aligned}$$

Thus

$$\begin{aligned}\mu_{p,f} &\leq (1+O(\varepsilon))(1+\varepsilon\|f\|_\infty)^p\mu_{p,\varepsilon}\\ &\leq \mu_{p,\varepsilon}+O(\varepsilon).\end{aligned}$$

□

Remark 4.4. The treatment for $p=2$ in [5] is different. Since $\int_M(b+\delta)=0$, Lu and Rowlett obtained the explicit formula $\delta=\frac{\int_M bf}{\int_M f}=O(\varepsilon)$. Then the estimate of $\int_M|b+\delta|^2$ is established by $\int_M|b+\delta|^2=\int_M b^2+2\delta\int_M b+\delta^2 V$.

Their method becomes complicated for $p\neq 2$ because no explicit formula for δ exists for general p. Our treatment is different and much simpler by estimating $\int_M|b+\delta|^p$ directly.

5. L_∞ estimates for the p-Laplacian

In this section, we establish the L_∞ estimates for the first nonzero Neumann eigenfunctions of the p-Laplacian on M_ε. The result is needed in completing the proof of the convergence of eigenfunctions later.

At first, we estimate the L_∞ norm by the L_p norm on general compact manifolds.

5.1. L_∞ estimates for the p-Laplacian on N^n.

First, let us recall the Sobolev inequalities on compact manifolds (cf. [1]).

Theorem 5.1 (The Sobolev Inequality)**.** *On a compact manifold N^n, for any p and q in $\mathbb{R}$ satisfying $\frac{1}{q}=\frac{1}{p}-\frac{1}{n}$ and $1\leq p<q$, there exists a constant $C_{s,p}$ depending on p and the manifold N so that*

$$(\int_N|v|^q)^{1/q}\leq C_{s,p}(\int_N|f|^p+\int_N|\nabla v|^p)^{1/p} \tag{5.1}$$

for any $v\in H_1^p(N)$.

If $\frac{1}{n}>\frac{1}{p}$, then there exists a constant $C_{\infty,p}$ depending on p and the manifold N so that

$$\|v\|_\infty\leq C_{\infty,p}(\int_N|v|^p+\int_N|\nabla v|^p)^{1/p} \tag{5.2}$$

for any $v\in H_1^p(N)$.

For the case of $p=n$, we get the following lemma.

Lemma 5.2. *Given a compact manifold N^n, define $\tilde{N}$ as the direct product $N\times[0,1]$. Then there exists a constant $C_{s,n}$ depending on $\tilde{N}$ and n so that*

$$(\int_N|v|^q)^{1/q}\leq C_{s,n}(\int_N|v|^n+\int_N|\nabla v|^n)^{1/n}$$

for any $v\in H_1^n(N)$.

Proof. For any $v\in H_1^n(N)$, define $v(x,y)=v(x)$ on $\tilde{N}$. By the Sobolev Inequality on $\tilde{N}$, we have

$$(\int_{\tilde{N}}|v|^q)^{\frac{1}{q}}\leq C_{s,n}(\int_{\tilde{N}}|v|^n+\int_{\tilde{N}}|\tilde{\nabla}v|^n)^{\frac{1}{n}}$$

with $q = n(n+1)$ and $C_{s,n}$ as the Sobolev constant on $\tilde{N}$ with respect to the L_n norm. Here $\tilde{\nabla}$ denotes the gradient in $\tilde{N}$. Since for any $k \geq 1$,

$$\|v\|_k(N) = \|v\|_k(\tilde{N}),$$
$$\|\nabla v\|_k(N) = \|\tilde{\nabla} v\|_k(\tilde{N}),$$

we get the following:

$$(\int_N |v|^q)^{\frac{1}{q}} \leq C_{s,n}(\tilde{N})(\int_N |v|^n + \int_N |\nabla v|^n)^{\frac{1}{n}}.$$

□

Remark 5.3. The same Sobolev inequalities hold true for Riemannian manifolds with boundary and with the Neumann boundary condition.

With the help of Theorem 5.1 and Lemma 5.2, we obtain the following L_∞ estimates by standard Moser iteration.

Proposition 5.4. *On a compact manifold N^n with given $\mu \geq 0$, for any $p \in R$, the operator L is defined as*

$$Lv = \Delta_p v + \mu |v|^{p-2} v,$$

with $p > 1$. Then there exists a constant C^∞ so that

$$\|v\|_\infty \leq C^\infty \|v\|_p$$

for any nonnegative subsolution V of L. Also,

$$C^\infty = (C_{s,p})^{\frac{n}{p}} (4\mu)^{\frac{n}{p^2}} (\frac{n}{n-p})^{(n^2-np)} \qquad \text{when } p < n,$$
$$C^\infty = (C_{s,n})^{\frac{n+1}{n}} (4\mu)^{\frac{n+1}{n^2}} (n+1)^{(n+1)} \qquad \text{when } p = n,$$
$$C^\infty = C_{\infty,p}(1+\mu)^{1/p} \qquad \text{when } p > n.$$

Here $C_{s,p}$ and $C_{\infty,p}$ are the Sobolev constants on N corresponding to $p < n$ and $p > n$, respectively, and $C_{s,n}$ is the Sobolev constant on $\tilde{N}$ with respect to the L_n norm.

Proof. As a subsolution,

$$Lv = \Delta_p v + \mu |v|^{p-2} v \geq 0. \tag{5.3}$$

At first, let us consider the case $p > n$. Multiply v on both sides of (5.13) and use integral by parts. Then we get

$$\int |\nabla v|^p \leq \mu \int v^p.$$

Applying the Sobolev Inequality (5.2), we obtain

$$\|v\|_\infty \leq C^\infty \|v\|_p$$

with $C^\infty = C_{\infty,p}(N)(1+\mu)^{1/p}$.

Now for the case $p < n$, multiply v^β on both sides of (5.3) with $\beta \geq 0$ and then use integral by parts. There is

$$\int |\nabla w|^p \leq C_1 \int |w|^p \tag{5.4}$$

with $\gamma = p - 1 + \beta, w = v^{\gamma/p}$ and $C_1 = \mu p^{-p} |\gamma|^p / \beta$.

Let $\chi = \frac{n}{n-p}$ and $\varphi(k) = \|v\|_k$. Applying the Sobolev Inequality (5.1) to (5.4), we have the following:

$$(C_1+1)C^p_{s,p}(N)\int |w|^p \geq (\int |w|^{p\chi})^{1/\chi}.$$

That is, when $\gamma > 0$,

$$C_2\varphi(\gamma) \geq \varphi(\gamma\chi)$$

with $C_2 = \big((C_1+1)C^p_{s,p}(N)\big)^{1/\gamma} = \big((1+\mu p^{-p}\gamma^p/\beta)C^p_{s,p}(N)\big)^{1/\gamma}$, depending on γ. Now do iteration by replacing γ by $\gamma_i = \gamma\chi^i$. Denote C_2 by $C_2(i)$. We get

$$\prod_{i=0}^{i=\infty} C_2(i)\varphi(\gamma) \geq \lim_{m\to\infty}\varphi(\gamma_m) = \|v\|_\infty,$$

with $C_2(i) = \big(C^p_{s,p}(N)(1+\mu p^{-p}\gamma_i^p\beta_i^{-1})\big)^{\frac{1}{r_i}}$.

By direct calculations, we can see that

$$\begin{aligned}\prod_{i=0}^{\infty} C_2(i) &= \prod_{i=0}^{\infty}\big(C^p_{s,p}(N)(1+\mu p^{-p}\gamma_i^p\beta_i^{-1})\big)^{\frac{1}{r_i}}\\ &= (C^p_{s,p}(N))^{\sum_{i=0}^{\infty}\frac{1}{\gamma_i}}\prod_{i=0}^{\infty}A(i)\\ &= (C_{s,p}(N))^{\frac{n}{\gamma}}\prod_{i=0}^{\infty}A(i),\end{aligned}$$

with $A(i) = (1+\mu p^{-p}\gamma_i^p\beta_i^{-1})^{\frac{1}{r_i}}$.

Now let $m_0 = [\log_\chi(p/\gamma)]+1$ and $m_1 = [\log_\chi(\frac{p\mu^{-1/p}}{\gamma})]+1$. Then

$$\begin{aligned}\beta_m &\geq 1 && \text{if and only if} && m \geq m_0,\\ \mu p^{-p}\gamma_m^p &\geq 1 && \text{if and only if} && m \geq m_1.\end{aligned}$$

Now for $A(i)$, there is

$$\begin{aligned}\prod_{i=0}^{\infty}A(i) &= \left(\prod_{i=0}^{m_0}A(i)\right)\left(\prod_{i\geq m_0}^{\infty}A(i)\right)\\ &\leq \prod_{i=0}^{m_0}(\beta_i)^{\frac{-1}{\gamma_i}}\prod_{i=0}^{\infty}\big(1+\mu\gamma_i^p p^{-p}\big)^{\frac{1}{\gamma_i}}\\ &\leq (\gamma+1-p)^{-\gamma m_0(\chi)^{m_0}}\prod_{i=0}^{m_1-1}\big(1+\mu\gamma_i^p p^{-p}\big)^{\frac{1}{\gamma_i}}\prod_{i\geq m_1}^{\infty}\big(1+\mu\gamma_i^p p^{-p}\big)^{\frac{1}{\gamma_i}}\\ &\leq (\gamma+1-p)^{-\gamma m_0(\chi)^{m_0}}2^{\sum_{i=0}^{m_1-1}\frac{1}{\gamma_i}}\prod_{i\geq m_1}^{\infty}\big(1+\mu\gamma_i^p p^{-p}\big)^{\frac{1}{\gamma_i}}\\ &\leq (\gamma+1-p)^{-\gamma m_0(\chi)^{m_0}}\left(4\mu\left(\frac{\gamma}{p}\right)^p\right)^{\frac{n}{\gamma p}}(\chi^p)^{\gamma(\frac{n}{p}^2-\frac{n}{p})}.\end{aligned}$$

The third inequality is because $\chi \geq 1$ and $\beta_i < 1$.

Thus, we get convergence of the coefficient:

$$\prod_{i=0}^{\infty} C_2(i) \leq (C_{s,p})^{\frac{n}{\gamma}} (\gamma+1-p)^{-\gamma m_0 (\chi)^{m_0}} (4\mu(\frac{\gamma}{p})^p)^{\frac{n}{\gamma p}} (\chi^p)^{\gamma((n/p)^2 - n/p)}.$$

So, when $p < n$,

$$C^{\infty}\varphi(\gamma) \geq \lim_{m\to\infty} \varphi(\gamma_m) = \|v\|_{\infty},$$

with

$$C^{\infty} = (C_{s,p}(N))^{\frac{n}{\gamma}} (\gamma+1-p)^{-\gamma m_0 \chi^{m_0}} \left(4\mu(\frac{\gamma}{p})^p\right)^{\frac{n}{\gamma p}} \chi^{p\gamma\left((\frac{n}{p})^2 - \frac{n}{p}\right)}.$$

The proposition will follow by replacing γ by p.

When $p = n$, there is no corresponding Sobolev imbedding theorem like above. But we can still use the same argument as in the case $p < n$. In fact, we multiply v^{β} on both sides of (5.3) and use integral by parts. We still get (5.4). Apply Lemma 5.2 to (5.4), and other parts of the argument will go through exactly the same as in the case $p < n$ with $\chi = n + 1$. □

One consequence of this proposition is the following:

Corollary 5.5. *On a compact manifold N^n with given $\mu \geq 0$, for any $1 \leq p \in \mathbb{R}$, there exists a constant C^{∞} so that*

$$\|v\|_{\infty} \leq C^{\infty} \|v\|_p$$

for any v satisfying $Lv = \Delta_p v + \mu |v|^{p-2} v = 0$ on N. Also,

$$C^{\infty} = C_{s,p}^{\frac{n}{p}} (4\mu)^{\frac{n}{p^2}} (\frac{n}{n-p})^{((n)^2 - np)} \qquad \text{when } p < n,$$

$$C^{\infty} = C_{s,n}^{\frac{n+1}{n}} (4\mu)^{\frac{n+1}{n^2}} (n+1)^{n+1} \qquad \text{when } p = n,$$

$$C^{\infty} = C_{\infty,p} (1+\mu)^{1/p} \qquad \text{when } p > n,$$

with $C_{s,p}$ and $C_{\infty,p}$ as the Sobolev constants on N and $C_{s,n}$ as the Sobolev constant on $\tilde{N}$.

Proof. This is because $v^+ = \max\{v, 0\}$ and $v^- = \max\{-v, 0\}$ are subsolutions when $Lv = 0$. □

5.2. L_{∞} **estimates for the p-Laplacian on M_{ε}.** Let $N = M_{\varepsilon}$ and $n = m + 1$ as in the last subsection. Then $\tilde{N} = \widetilde{M_{\varepsilon}}$.

Define $\tilde{M}$ as the direct product $M \times (0, 1)$. Define

$$\tilde{M}_{\varepsilon} = \{(x, y, z) \mid (x, y) \in \tilde{M}, 0 \leq z \leq \varepsilon f(x)\}.$$

It is easy to see that $\widetilde{M_{\varepsilon}} = \tilde{M}_{\varepsilon}$.

The next lemma shows the relation between the Sobolev constants on M_{ε} with respect to different ε.

Lemma 5.6. *Using the same notation as in Section 3, let $C_{s,p,\varepsilon}$ and $C_{\infty,p,\varepsilon}$ be the Sobolev constants on M_{ε} corresponding to $p < m + 1$ and $p > m + 1$ respectively.*

Let $C_{s,n,\varepsilon}$ be the Sobolev constant on $\widetilde{M_\varepsilon}$, i.e. $\tilde{M}_\varepsilon$:

$$C_{s,p,\varepsilon} = C_{s,p,1}\varepsilon^{\frac{-1}{m+1}},$$
$$C_{\infty,p,\varepsilon} = C_{\infty,p,1}\varepsilon^{\frac{-1}{p}},$$
$$C_{s,m+1,\varepsilon} = C_{s,m+1,1}\varepsilon^{\frac{-1}{m+2}}$$

when $\varepsilon \leq 1$.

Proof. First assume that $p < n = m+1$. As before, let $\chi = \frac{n}{n-p} = \frac{m+1}{m+1-p}$. It is sufficient to show that

$$\|h\|_{p\chi} \leq C_{s,p,1}\varepsilon^{\frac{-m}{m+1}}\left(\int_{M_\varepsilon}|h|^p + \int_{M_\varepsilon}|\nabla h|^p\right)^{\frac{1}{P}},$$

for any function $h(x,z)$ on M_ε.

Define $g(x,y) = h(x,\varepsilon y)$ on M_1. Then by the Sobolev Inequality (5.1), we have

$$\|g\|_{p\chi} \leq C_{s,p,1}\left(\int_{M_1}|g|^p + \int_{M_1}|\nabla g|^p\right)^{1/p}.$$

Since

$$\int_{M_1}|g|^k dxdy = \frac{1}{\varepsilon}\int_{M_\varepsilon}|h|^k dxdz,$$
$$|\nabla g(x,y)|^2 = |\nabla_x h|^2 + \varepsilon^2|\nabla_z h|^2,$$
$$\leq |\nabla h|^2$$

when $\varepsilon \leq 1$, we get

$$\|h\|_{p\chi} \leq C_{s,p,1}\varepsilon^{\frac{-1}{m+1}}\left(\int_{M_\varepsilon}|h|^p + \int_{M_\varepsilon}|\nabla h|^p\right)^{1/p}.$$

The argument also works for the case $p > n = m+1$.

When $p = n = m+1$, we will get the result by applying the same argument to $\tilde{M}_\varepsilon$ with M replaced by $\tilde{M}$. □

Applying this lemma to Corollary 5.5, we get

Corollary 5.7. *Using the same notation as in Section 3, let v be a eigenfunction of Δ_p on M_ε corresponding to eigenvalue $\mu_{p,\varepsilon}$ with $p \geq 1$. Then there exists a constant $C^{\infty,\varepsilon}$ so that*

$$\|v\|_\infty \leq C^{\infty,\varepsilon}\|v\|_p \leq (1+\mu_\varepsilon)^{\frac{m+2}{p^2}}C.$$

Here

$$C^{\infty,\varepsilon} = \varepsilon^{\frac{-1}{p}}(\mu_\varepsilon)^{\frac{m+1}{p^2}}C, \quad \text{with } C = (C_{s,p,1}^{\frac{m+1}{p}})4^{\frac{n+1}{p^2}}\chi^{(m+1)(m+1-p)}, \quad \text{when } p < m+1,$$
$$C^{\infty,\varepsilon} = \varepsilon^{\frac{-1}{m+1}}(\mu_\varepsilon)^{\frac{m+2}{(m+1)^2}}C, \quad \text{with } C = C_{s,m+1,1}^{\frac{m+2}{m+1}}4^{\frac{m+2}{(m+1)^2}}(m+2)^{m+2}, \quad \text{when } p = m+1;$$
$$C^{\infty,\varepsilon} = \varepsilon^{\frac{-1}{p}}(1+\mu_\varepsilon)^{1/p}C, \quad \text{with } C = C_{\infty,p,1}, \quad \text{when } p > m+1.$$

Here $C_{s,p,1}$ and $C_{\infty,p,1}$ are the Sobolev constants on M_1 with respect to the L_p norm for $p < m+1$ and $p > m+1$ respectively. Also, $C_{s,m+1,1}$ is the Sobolev constant on $\tilde{M}_1$ with respect to the L_{m+1} norm.

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