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ON UNIQUENESS IN THE EXTENDED SELBERG CLASS OF DIRICHLET SERIES

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Abstract. We will show that two functions in the extended Selberg class satisfying the same functional equation must be identically equal if they have sufficiently many common zeros.

This paper concerns the question of how L-functions are determined by their zeros. L-functions are Dirichlet series with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ as the prototype and are important objects in number theory. The Selberg class $\mathcal{S}$ of L-functions is the set of all Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of a complex variable $s = \sigma + it$ with $a(1) = 1$, satisfying the following axioms (see [7]):

(i) (Dirichlet series) For $\sigma > 1$, $L(s)$ is an absolutely convergent Dirichlet series.

(ii) (Analytic continuation) There is a nonnegative integer k such that $(s-1)^k L(s)$ is an entire function of finite order.

(iii) (Functional equation) L satisfies a functional equation of type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\bar{s})},$$

where $\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j)$ with positive real numbers Q, λ_j and with complex numbers μ_j, ω with $\mathrm{Re}\mu_j \geq 0$ and $|\omega| = 1$.

(iv) (Ramanujan hypothesis) $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$;

(v) (Euler product) $\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

The degree d_L of an L-function L is defined to be $d_L = 2\sum_{j=1}^{K} \lambda_j$, where K, λ_j are the numbers in axiom (iii).

The Selberg class includes the Riemann zeta-function ζ and essentially those Dirichlet series where one might expect the analogue of the Riemann hypothesis. At the same time, there are a whole host of interesting Dirichlet series not possessing a Euler product (see e.g. [3], [8]). Throughout the paper, all L-functions are assumed to be functions from the extended Selberg class of those only satisfying the axioms (i)-(iii) (see [3]). Thus, the results obtained in the present paper particularly apply to L-functions in the Selberg class.

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It was pointed out in [1] that two L-functions are expected to have few common zeros if they are "independent", and this problem appears to be very difficult (see [1] for the details and for the meaning of independence). On the other hand, L-functions are meromorphic functions, and meromorphic functions possess the well-known uniqueness property by Nevanlina's uniqueness theorem: two nonconstant meromorphic functions f, g in $\mathbf{C}$ must be identically equal if f and g share five distinct values $c_j \in \mathbf{C} \cup \{\infty\}$ in the sense that $f - c_j$ and $g - c_j$ have the same zeros without counting multiplicities (see e.g. [2] or [8]). It is known that two L-functions are equal if they share a value with counting multiplicities ([8, p. 152]). For L-functions satisfying the same functional equation and sharing two complex numbers c_1, c_2, Steuding proved the uniqueness under a condition on the number of the distinct zeros of $L - c_j$ (see [8, p. 152]). Li ([5]) recently removed this condition. Ki ([4]) further considered if this still holds for one shared value and, in particular, showed that two L-functions L_1, L_2 satisfying the same functional equation can be distinct if L_1, L_2 have the same zeros with different multiplicities. This however cannot happen if the multiplicity of a zero of L_1 does not exceed that of L_2 allowing some exceptions ([6]). In the present paper, we show that an L-function L is completely determined by the functional equation and its nontrivial zeros ρ, allowing an exceptional set G of ρ with the largest possible size. As usual, the nontrivial zeros of an L-function are the zeros which are not located at the poles of the Γ factors in the functional equation. Denote by $Z^+(L)$ the set of all nontrivial zeros of L counted with multiplicity. The size of G is measured by the usual counting function $n(r, G)$, the number of points in $G \cap \{|s| < r\}$ counted with multiplicity ($n(r, G) = 0$ if G is empty). Since $n(r, Z(L))$ of the zero set $Z(L)$ of a degree zero L-function L is $O(r)$, it would be tempting to think that the condition $n(r, G) = o(r)$ would be the best to obtain for the exceptional set. However, a quite delicate analysis shows that a sharp condition can be given in terms of the type of G, which is defined as

$$\tau(G) := \limsup_{r\to\infty} \frac{n(r, G)}{r}.$$

More precisely, we have the following:

Theorem. *Two nonconstant functions L_1 and L_2 in the extended Selberg class satisfying the same functional equation are identically equal if $Z^+(L_1)\backslash G \subseteq Z^+(L_2)$ for a set G satisfying that $\tau(G) < \frac{\log 4}{\pi}$. Furthermore, this inequality for $\tau(G)$ is best possible.*

We note that "the same functional equation" condition in the theorem is crucial and cannot be dropped. For example, ζ and ζ^2 clearly satisfy that $Z^+(\zeta) \subset Z^+(\zeta^2)$, but they are not equal.

Proof. Consider the auxiliary function

$$f(s) := \left(s^2 - 1\right)^m s^n \frac{L_2(s) - L_1(s)}{L_1(s)} \frac{L_2(-s) - L_1(-s)}{L_1(-s)} \prod_{\rho\in G} \left(1 - \frac{s^2}{\rho^2}\right). \tag{1}$$

It is easy to check that the infinite product in (1) converges to an entire function (cf. (2) below). Since L_1, L_2 satisfy the same functional equation, $L_1 - L_2$ satisfies the same functional equation. Thus, L_1 and $L_1 - L_2$ have the same trivial zeros that are located at the poles of the Γ factors in the functional equation. These zeros do not produce any poles of f due to cancellation. Hence, there are integers m, n such

that the function f is an entire function (we may then assume that $s = 0$ is not in G). We may increase m so that $s = \pm 1$ are zeros of f. Choose $0 < D_0 < D_1 < 1$ with $\tau(G) < D_0 \frac{\log 4}{\pi}$. Then, there is an $r_0 > 0$ such that $\frac{n(r,G)}{r} < D_0 \frac{\log 4}{\pi}$ for $r \geq r_0$. We deduce that for large $|s|$,

$$\begin{aligned}
&\log \left| \prod_{\rho \in G} \left(1 - \frac{s^2}{\rho^2}\right) \right| \leq \sum_{\rho \in G} \log \left(1 + |\frac{s^2}{\rho^2}|\right) \\
&= \int_0^\infty \log \left(1 + \frac{|s^2|}{r^2}\right) dn(r, Z(G)) \\
&= 2|s|^2 \int_0^\infty \frac{n(r, Z(G))}{r(r^2 + |s|^2)} dr \\
&\leq 2|s|^2 \{ \frac{1}{|s|^2} \int_0^{r_0} \frac{n(r, Z(G))}{r} dr + D_0 \frac{\log 4}{\pi} \int_{r_0}^\infty \frac{1}{r^2 + |s|^2} dr \} \\
&\leq D_1 |s| \log 4.
\end{aligned} \tag{2}$$

Dividing the functional equation of $L_1 - L_2$ by the same functional equation of L_1, we obtain that

$$\frac{L_2(s) - L_1(s)}{L_1(s)} = \frac{\overline{L_2(1-\overline{s})} - \overline{L_1(1-\overline{s})}}{\overline{L_1(1-\overline{s})}}. \tag{3}$$

Noting that $L_1(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}$ with $a(1) = 1$, we have that

$$L_1(s) \to 1 \text{ and } L_1(s) - L_2(s) = O\left(\frac{1}{2^\sigma}\right) \qquad (\sigma \to +\infty).$$

This together with (3) yields that

$$\left| \frac{L_2(s) - L_1(s)}{L_1(s)} \cdot \frac{L_2(-s) - L_1(-s)}{L_1(-s)} \right| = O\left(\frac{1}{4^{|\sigma|}}\right)$$

as $\sigma \to \pm\infty$. Therefore, by (1) and (2) we have that for D with $D_1 < D < 1$,

$$\log |f(s)| \leq D|s| \log 4 - |\sigma| \log 4 + O(1) \qquad (\sigma \to \pm\infty).$$

It is then easy to see that f is bounded on the rays $\arg(s) = \theta, \pi - \theta, \pi + \theta, 2\pi - \theta$, where $0 < \theta < \pi/2$ with $\cos\theta = D$, since on the rays, $|\cos\theta| = \frac{|\sigma|}{|s|} = D$. Note that nonconstant L-functions are of order 1 (see [8, p. 150], [9]) and the infinite product in (1) is also of order 1. Thus, f is of order at most 1. We then have that $f(s) = O\left(e^{|s|^{1+\epsilon}}\right)$ for any $\epsilon > 0$. Hence, f satisfies the conditions of the Phragmén-Lindelöf theorem in each of the sectors bounded by the above rays, and thus f is bounded in each of the sectors and thus in the entire complex plane. Therefore the entire function f must be a constant. But f has a zero. Thus f and then $L_1 - L_2$ must be identically zero.

It remains to prove that the inequality $\tau(G) < \frac{\log 4}{\pi}$ is best possible. To this end, consider the following two L-functions:

$$l_1(s) = 1 + \frac{2}{4^s}, \quad l_2(s) = 1 + \frac{1}{2^s} + \frac{2}{4^s}.$$

Then one can verify that for $j = 1, 2$,

$$2^s l_j(s) = 2^{1-s} \overline{l_j(1-\overline{s})}.$$

Thus, l_1, l_2 are L-functions of degree 0 satisfying the same functional equation. Recall the Riemann-von-Mangoldt formula for L-functions (see [8, p. 145]): for a nonconstant L-function L of degree 0,

$$N_L^0(T) = \frac{T}{\pi}\log(Q^2) + O(\log T),$$

where Q is the number in the functional equation in axiom (iii) and $N_L^0(T)$ denotes the number of zeros (counting multiplicities) of L in the region $\mathrm{Re}s > 0, |\mathrm{Im}s| \le T$. Thus,

$$\frac{n(r, Z(l_1))}{r} \sim \frac{\log 4}{\pi}$$

as $r \to \infty$. Set $L_1 = Ll_1$ and $L_2 = Ll_2$, where L is any L-function (of arbitrarily given degree). Then L_1, L_2 satisfy the same functional equation. Now, take the exceptional set G to be the entire set $Z(l_1)$. Then G satisfies that $\tau(G) = \frac{\log 4}{\pi}$, and it is clear that $Z^+(L_1) \setminus G \subset Z^+(L_2)$. However, $L_1 \neq L_2$. This proves the theorem. □

We conclude the paper by noting that the above theorem is clearly related to the question of when a functional equation in axiom (iii) has a unique L-function solution. Let L be an L-function solution of the functional equation

$$Q^s L(s) = \omega Q^{1-s}\overline{L(1-\overline{s})}.$$

Then

$$\frac{n(r, Z(L))}{r} \sim \frac{\log Q^2}{\pi}$$

by the Riemann-von-Mangoldt formula (cf. the proof of the Theorem). Thus, when $Q^2 < 4$ we can take the entire set $Z(L)$ as an exceptional set, which, by the theorem, implies that two L-functions satisfying the same functional equation with $0 < Q < 2$ must be equal; i.e., L is the unique solution. The inequality $0 < Q < 2$ is best possible, since the uniqueness breaks down when $Q = 2$, as seen in the proof of the Theorem for the functional equation

$$2^s L(s) = 2^{1-s}\overline{L(1-\overline{s})},$$

which has two distinct L-function solutions.

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