POWERS IN FINITE GROUPS
AND A CRITERION FOR SOLUBILITY

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Abstract. We study the set $G^{[k]}$ of $k$th powers in finite groups $G$. We prove that if $G^{[12]}$ is a subgroup, then $G$ must be soluble; moreover, 12 is the minimal number with this property. The proof relies on results of independent interest, classifying almost simple groups $G$ and positive integers $k$ for which $G^{[k]}$ contains the socle of $G$.

1. Introduction

Powers in groups have been extensively studied in connection with the Burnside problems, powerful $p$-groups and $p$-adic analytic groups, and other areas. For a group $G$ and a positive integer $k$, denote by $G^{[k]}$ the set $\{x^k : x \in G\}$ of $k$th powers in $G$. It is known [6] that if $G$ is a powerful $p$-group, then $G^{[p]}$ is a subgroup of $G$. Malcev [8] showed that if $G$ is a finitely generated nilpotent group, then $G^{[k]}$ always contains a subgroup of finite index in $G$. See also [3], where $G^{[k]}$ is studied for finitely generated linear groups.

In this paper we study the power subsets $G^{[k]}$ in finite groups in general and in almost simple groups in particular. One of our main results is the following somewhat surprising solubility criterion.

Theorem 1. Let $G$ be a finite group, and suppose that $G^{[12]}$ is a subgroup of $G$. Then $G$ is soluble.

Some remarks about this result are in order. First, 12 is the minimal number with this property: we shall see below (Proposition 6) that for every $k < 12$ there is an almost simple group $G$ such that $G^{[k]} = \text{soc}(G)$, the socle of $G$. Second, the proof of the theorem shows that the same conclusion holds with 12 replaced by any integer $2^a 3^b$ with $a \geq 2, b \geq 1$, and there are other numbers which also work (see Section 5). Third, the proof relies on the classification of finite simple groups and requires a detailed study of power subsets in almost simple groups, which is of some independent interest (see Theorem 7 below). A further consequence of this is the following.
Theorem 2. Let $G$ be a finite group, and suppose that $G^{[3]}$ and $G^{[4]}$ are both subgroups of $G$. Then $G$ is soluble.

The next result concerns the set of squares in a finite group. Of course if $G^{[2]} = G$, then $G$ has odd order and hence is soluble by the Feit-Thompson theorem. It turns out that finite groups in which the set of squares is a subgroup need not be soluble; however, their non-abelian composition factors are rather restricted:

Theorem 3. Let $G$ be a finite group such that $G^{[2]}$ is a subgroup. Then the non-abelian composition factors of $G$ are among the groups $L_2(q)$ ($q$ odd), $L_2(q^2)$ ($q$ even) and $L_3(4)$.

It is easy to see that if $G^{[k]}$ is a subgroup for all values of $k$, then $G$ must be nilpotent: indeed, if $p$ is a prime divisor of $|G|$ and $k$ is the $p'$-part of $|G|$, then $G^{[k]}$ must be the unique Sylow $p$-subgroup of $G$.

The next result connects general finite groups and non-abelian composition factors as far as power subsets are concerned.

Theorem 4. Let $G$ be a finite group and $k$ a positive integer such that $G^{[k]}$ is a subgroup of $G$. Then for every non-abelian composition factor $T$ of $G$, either $T \subseteq \text{Aut}(T)^{[k]}$ or the exponent of $T$ divides $k$. In particular, if $k$ is odd or has at most two prime divisors, then $T \subseteq \text{Aut}(T)^{[k]}$ for all non-abelian composition factors $T$.

We now discuss our results on almost simple groups, that is, groups whose socle is a non-abelian simple group. Clearly not all elements of a (non-abelian) simple group are squares. Somewhat surprisingly, it turns out that there are simple groups $T$ in which every element is a square in the automorphism group of $T$:

Proposition 5. Let $T$ be one of the simple groups $L_2(q)$ ($q$ odd), $L_2(q^2)$ ($q$ even) or $L_3(4)$. Then every element of $T$ has a square root in $\text{Aut}(T)$. Moreover, there is a group $G$ of the form $T.2$ such that $G^{[2]} = T$.

The group $G$ in the conclusion is, in the respective cases, $\text{PGL}_2(q)$ ($q$ odd), $L_2(q^2) \langle \sigma \rangle$ ($q$ even, $\sigma$ a field automorphism of order 2), or $L_3(4) \langle \sigma \rangle$ ($\sigma$ a graph-field automorphism). Other results on squares in finite simple groups and their proportion can be found in [7].

Our next result gives further examples for simple groups.

Proposition 6. (i) Let $k = p^r > 2$ with $p$ prime, and let $T = L_2(p^{kl})$ for some $l \geq 1$. Then every element of $T$ has a $k^{th}$ root in $\text{Aut}(T)$. Moreover, if $G = T \langle \sigma \rangle$, where $\sigma$ is a field automorphism of order $k$, then $G^{[k]} = T$.

(ii) Let $k = 2p^r$ with $p$ an odd prime, and let $T = L_2(p^{kl/2})$ for some $l \geq 1$. Then every element of $T$ has a $k^{th}$ root in $\text{Aut}(T)$. Moreover, if $G = \text{PGL}_2(p^{kl/2}) \langle \sigma \rangle$, where $\sigma$ is a field automorphism of order $k/2$, then $G^{[k]} = T$.

Our next theorem shows that there are no further examples of this phenomenon.

Theorem 7. Let $T$ be a finite simple group, and let $k > 1$ be a positive integer dividing $|T|$. Suppose $\text{Aut}(T)^{[k]}$ contains $T$. Then $k = p^r$ or $2p^r$ for some prime $p$. Further, if $k = 2$, then $T = L_2(q)$ or $L_3(4)$ is as in Proposition 5 and if $k = p^r > 2$ or $k = 2p^r$ ($p$ odd), then $T = L_2(p^k)$ or $L_2(p^{k/2})$ is as in Proposition 6.
Note that the assumption that $k$ divides $|T|$ can be made without loss of generality, since if $k = ab$ where $a$ divides $|T|$ and $(|T|, b) = 1$, then $\text{Aut}(T)^{|T|}$ contains $T$ if and only if $\text{Aut}(T)^{|a|}$ contains $T$.

The next result is immediate from Theorem 7.

**Corollary 8.** (i) If $T$ is a finite simple group with $T \neq L_2(q)$, $L_3(4)$, and $k$ is a positive integer such that $\text{Aut}(T)^{|k|}$ contains $T$, then $k$ is coprime to $|T|$.

(ii) If $G$ is a finite almost simple group, then $G^{[p]}$ is a subgroup of $G$ for at most one odd prime $p$ dividing $|\text{soc}(G)|$.

The layout of the paper is as follows. Section 2 is devoted to our examples of almost simple groups $G$ with the property that $G^{[k]}$ contains $\text{soc}(G)$ given in Propositions 5 and 6. In Section 3 we show that these are the only such examples, thereby proving Theorem 7 and we also deduce Corollary 8. Section 4 is devoted to general finite groups. We start with the proof of Theorem 4 and use this to deduce Theorems 1, 2 and 3. Finally, in Section 5 we investigate the set of numbers $k$ for which the assumption that $G^{[k]}$ is a subgroup implies that $G$ is soluble.

2. Almost simple groups: Examples

First we prove Proposition 5. Let $T$ be one of the simple groups in the statement of the proposition. Elements of odd order in $T$ are squares, so we need handle only elements of even order.

First consider $T = L_2(q)$ with $q$ odd. Let $G = \text{PGL}_2(q)$. If $x \in T$ is an element of even order, then its order divides $\frac{1}{2}(q + \epsilon)$ for some $\epsilon \in \{\pm 1\}$, and there is an element $y \in G$ of order $q + \epsilon$ such that $x \in \langle y^2 \rangle$. Hence $G^{[2]} = T$.

Now let $T = L_2(q^2)$ with $q$ even, and $G = T\langle \sigma \rangle$ where $\sigma$ is an involutory field automorphism. For $\alpha \in F_{q^2}$, set

$$u(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.$$  

It is well known that every element of even order in $T$ is conjugate to $u(1)$. For $\alpha \in F_{q^2} \setminus F_q$ we have $(u(\alpha)\sigma)^2 = u(\alpha + \alpha^\sigma)$. It follows that $u(1)$, and hence all elements of even order, are squares in $G$, and so $G^{[2]} = T$.

Finally, for $T = L_3(4)$ and $G = T\langle \sigma \rangle$ with $\sigma$ a graph-field automorphism, the conclusion can be checked using [1]. This completes the proof of Proposition 5. \qed

Now we prove Proposition 6. First consider part (i). Let $k = p^r$, $T = L_2(p^{kl})$ and $G = T\langle \sigma \rangle$ as in the statement. Define $u(\alpha)$ as above, for $\alpha \in F := F_{p^{kl}}$. First assume $p$ is odd. Then every element in $T$ of order divisible by $p$ is conjugate to $u(1)$ or $u(\beta)$ with $\beta \in F$ non-square. We have $(u(\alpha)\sigma)^k = u(Tr(\alpha))$, where $Tr$ is the trace map $F \mapsto F_{p^l}$. Since $Tr$ is surjective, this shows that $u(1)$ and $u(\beta)$ are both $k^{th}$ powers in $G$, as required. Finally, for $p = 2$, every element of even order in $T$ is conjugate to $u(1)$, and the same proof applies.

Now consider part (ii). Let $k = 2p^r$ with $p$ an odd prime, and let $G = \text{PGL}_2(p^{kl}/2)\langle \sigma \rangle$ be as in the proposition. For $x \in T = \text{soc}(G)$, Proposition 5 shows that $x = y^2$ for some $y \in \text{PGL}_2(p^{kl}/2)$. If $y$ has order divisible by $p$, then $y$ is in $T$ and has order $p$, and as in (i), there exists $z \in G$ such that $y = z^p$; the same holds trivially if $y$ has order coprime to $p$. Hence $x = z^{2p^r} = z^k$, and the proof is complete. \qed
3. Almost simple groups: Proof of Theorem 7

We begin with a preliminary result for general finite groups which will be used frequently in the proof.

Lemma 3.1. Let $G$ be a finite group with a normal subgroup $T$ such that $G/T$ is cyclic of order $k$. Write $G = T\langle \sigma \rangle$, where $\sigma^k \in T$. Suppose $y \in G^{[k]} \setminus T^{[k]}$. Then there exists $i$ with $1 \leq i \leq k - 1$ such that $y^{\sigma^i}$ is $T$-conjugate to $y$. In particular, if $k$ is prime, then $y^\sigma$ is $T$-conjugate to $y$.

Proof. Write $y = x^k$. There exist $t \in T$ and $1 \leq i \leq k - 1$ such that $x = t\sigma^{-i}$. Observe that

$$y^{\sigma^i} = ((t\sigma^{-i})^k)^{\sigma^i} = ((t\sigma^{-i})^k)^t = y^i.$$

The first assertion follows. For the second assertion, choose $j$ such that $\sigma^j \equiv \sigma \mod T$, and observe that $y^\sigma$ is $T$-conjugate to $y^{(\sigma^j)^i}$, which is $T$-conjugate to $y$. \qed

Now we embark on the proof of Theorem 7. Suppose $T$ is a finite simple group and $k > 1$ is an integer dividing $|T|$ such that $\text{Aut}(T)^{|k|}$ contains $T$.

Lemma 3.2. If $T$ is alternating or sporadic, then $T = A_5$ or $A_6$ and $k = 2$.

Proof. Since $\text{Out}(T)$ is 2 or $2^2$ for these groups, $k$ must be 2. Note that $A_5 \cong L_2(5)$ and $A_6 \cong L_2(9)$ appear in the conclusion by Proposition 5. For $n \geq 7$, $T = A_n$ does not occur, since for example permutations of cycle shape $(4, 2)$ are not squares in $S_n$. Also, for $T$ sporadic, one checks using the character tables in [1] that for those groups $T$ which possess outer automorphisms, there are elements in $T$ which have no square root in $\text{Aut}(T)$. \qed

Lemma 3.3. The conclusion of Theorem 7 holds if $T = L_2(q)$ or $L_3(4)$.

Proof. For $L_3(4)$ the result can be checked using [1]. So suppose that $T = L_2(q)$ and that $T \subseteq \text{Aut}(T)^{|k|}$ for some $k > 1$ dividing $|T|$. Let $p$ be a prime dividing $k$, and let $p^r$ be the $p$-part of $k$.

Assume first that $p$ is odd and does not divide $q$. Then $p$ divides $q - \epsilon$ with $\epsilon = \pm 1$. Let $x \in T$ be an element of order $(q - \epsilon)/(2, q - \epsilon)$. Clearly $x \not\in T^{[p]}$. Hence $x \in (T\langle \sigma \rangle)^{[p]}$, where $\sigma$ is a field automorphism of order $p$. By Lemma 3.1, this implies that $x$ is $T$-conjugate to $x^\sigma$. But this is a contradiction, as the only elements of $\langle x \rangle$ which are $T$-conjugate to $x$ are $x^{\pm 1}$.

If $p$ is odd and divides $q$, then since $T \subseteq \text{Aut}(T)^{|p|}$, there must be an element of order $p^r$ in $\text{Out}(T)$, and hence $q = p^{r+1}$ for some $l \geq 1$.

Now suppose $p = 2$ and $p^r = 2^t \geq 4$. If $q$ is odd, then 4 divides $q - \epsilon$ with $\epsilon = \pm 1$ and we let $x$ be an element of order $(q - \epsilon)/2$. Then $x \not\in T^{[2]}$, and so $x \in (T\langle \sigma \rangle)^{[4]}$ where $\sigma$ induces an outer automorphism of order 4; but $x$ is not $T$-conjugate to $x^\sigma$ for such an automorphism, so this contradicts Lemma 3.1. If $q$ is even, then $T$ has an outer automorphism of order $2^t$, so $q = 2^{2l}$ for some $l$.

Next assume that $p = p^r = 2$. Then either $q$ is odd, or $q$ is even and $T$ has an outer automorphism of order 2 so that $q = 2^{2l}$ for some $l$.

From the above, we conclude that one of the following holds:

- $k = p^r$, $q = p^{r+1}$,
- $k = 2$, $q$ odd,
- $k = 2p^r$, $q = p^{r+1}$, $p$ odd.
These are precisely the possibilities on the conclusion of Theorem 7. □

Lemma 3.4. The group T is not \( L_n(q) \).

Proof. Suppose \( T = L_n(q) \). By assumption \( n \geq 3 \) and \( (n, q) \neq (3, 2), (3, 4) \).

Assume first that \( p \mid q - 1 \) and \( p \geq 3 \). Let \( \lambda \in \mathbb{F}_q^* \) have order \( q - 1 \) and define \( x = \text{diag}(a(\lambda, 1), \lambda^{-2}, 1, \ldots, 1)Z \in T \), where \( Z \) is the group of scalars and

\[
(1) \quad a(\lambda, \beta) = \begin{pmatrix} \lambda & \beta \\ 0 & \lambda \end{pmatrix}.
\]

For \( q > 4 \), the centralizer of \( x \) in \( \text{PGL}_n(q) \) consists of elements of the form \( \text{diag}(a(\alpha, \beta), \gamma, A)Z \), and \( x \) cannot be the \( p^{th} \) power of one of these, as \( \lambda \) is not a \( p^{th} \) power in \( \mathbb{F}_q^* \). Hence \( x \) is not a \( p^{th} \) power in \( \text{PGL}_n(q) \); a similar argument gives the same conclusion when \( q = 4 \). It follows that \( x \) must be a \( p^{th} \) power in a group \( T(\sigma) \), where \( \sigma \) involves a field automorphism of order \( p \) (i.e. \( \sigma \) is a product of a (possibly trivial) diagonal automorphism and such a field automorphism). Then \( x \) is \( T \)-conjugate to \( x^\sigma \), by Lemma 3.1. But this is not the case, as can be seen by consideration of the eigenvalues of \( x \) and \( x^\sigma \).

Now assume that \( p = 2 \) and \( q \) is odd. Let \( A \in \text{GL}_2(q) \) be an element of order \( q^2 - 1 \) with eigenvalues \( \lambda, \lambda^q \) over \( \mathbb{F}_{q^2} \), and define \( x = \text{diag}(A, \lambda^{-q-1}, 1, \ldots, 1)Z \in T \). By considering the centralizer of \( x \) as above, we see that it is not a square in \( \text{PGL}_n(q) \). Therefore \( x \) must be a square in a group \( T(\sigma) \), where \( \sigma \) involves an involutory field, graph or graph-field automorphism of \( T \). A graph automorphism inverts the eigenvalues of \( x \), while an involutory field automorphism sends the eigenvalue \( \lambda^{-q-1} \) to \( \lambda^{-q_0-q_0} \), where \( q = q_0^2 \). Hence we see that \( x \) cannot be \( T \)-conjugate to \( x^\sigma \), contradicting Lemma 3.1.

This deals with the case where \( p \mid q - 1 \), so assume from now on that \( p \) does not divide \( q - 1 \). If \( p > 2 \) the outer automorphisms of \( T \) of order \( p \) are field automorphisms, while if \( p = 2 \) they are field, graph or graph-field automorphisms.

Assume \( p > 2 \). If \( p \nmid q \), take \( x = \text{diag}(a(\lambda, 1), \lambda^{-2}, 1, \ldots, 1)Z \in T \) with \( |\lambda| = q - 1 \) as before. Then \( x \) is not a \( p^{th} \) power in \( T \) and also is not conjugate to \( x^\sigma \) if \( \sigma \) is a field automorphism of order \( p \). Also, if \( p \) does not divide \( q \), choose \( e \) minimal such that \( p \mid q^e - 1 \) and let

\[
x(\lambda) = \text{diag}(\lambda, \lambda^q, \ldots, \lambda^{q^{e-1}}) \in \text{GL}_1(q^e) \leq \text{GL}_e(q)
\]

for \( \lambda \in \mathbb{F}_{q^e}^* \). For \( \lambda \) of order \( \frac{q^e - 1}{q - 1} \), let \( x = \text{diag}(x(\lambda), I_{n-e})Z \in T \). Then we see as usual that \( x \) is not a \( p^{th} \) power in \( T \) and is not conjugate to \( x^\sigma \) if \( \sigma \) is a field automorphism of order \( p \). This handles the case \( p > 2 \).

Finally, let \( p = 2 \). As \( p \) does not divide \( q - 1 \) by assumption, \( q \) is even. If \( q > 4 \) let \( x = \text{diag}(a(\lambda, 1), \lambda^{-2}, 1, \ldots, 1)Z \in T \) with \( |\lambda| = q - 1 \) and argue as above. If \( q = 2 \) or \( 4 \) and \( n \geq 5 \), let \( x(\lambda) \in \text{SL}_3(q) \) be as above with \( e = 3 \) and \( \lambda \in \mathbb{F}_{q^3} \) of order \( q^2 + q + 1 \), and define \( x = \text{diag}(x(\lambda), J_{n-3}) \), where \( J_{n-3} \) is a unipotent Jordan block of size \( n - 3 \). Then \( x \) is not a square in \( T \) (as \( J_{n-3} \) is not a square in \( \text{SL}_{n-3}(q) \)), and \( x \) is not conjugate to \( x^\sigma \) for \( \sigma \) an involutory field, graph or graph-field automorphism of \( T \).
This leaves the cases $T = L_4(2)$ and $L_4(4)$ (since $(n,q) \neq (3,2),(3,4)$ by assumption). The first of these is the alternating group $A_8$ which has already been handled. Also, $L_4(4)$ has an element $x$ of order 30 of the form $\text{diag}(a(\lambda,1),M)$, where $\lambda$ has order 3 and $M \in GL_2(4)$ has order 15 and determinant $\lambda$. We argue in the usual way that $x$ is not a square in $\text{Aut}(T)$.

**Lemma 3.5.** $T$ is not $U_n(q)$.

**Proof.** Suppose $T = U_n(q)$. Then $n \geq 3$ and $(n,q) \neq (3,2)$.

The proof is quite similar to the previous lemma. Assume first that $p|q + 1$ and $n \geq 4$. Let $x = \text{diag}(a(\lambda,\beta),\lambda^{-2},1,\ldots,1)Z \in T$ for $\lambda \in \mathbb{F}_{q^2}$ of order $q + 1$ and suitable $\beta \in \mathbb{F}_{q^2}$ (where $a(\lambda,\beta)$ is as in (1) and matrices are taken relative to a basis with the first three vectors $e,f,d$, where $e,f$ are singular, $(e,f) = 1$ and $d$ is nonsingular and perpendicular to $e,f$). If $q > 2$ we can argue as in the previous lemma that $x$ is not a $p^{th}$ power in $\text{PGU}_n(q)$ and is not conjugate to $x^\sigma$ for any further outer automorphism $\sigma$ of $T$ of order $p$. Also, if $q = 2$, then $p = 3$ and we take $x = \text{diag}(a(\lambda,\beta),\lambda^{-1},\lambda^{-1},1,\ldots,1)Z \in T$ with $|\lambda| = 3$ and argue similarly.

Now assume $p|q + 1$ and $n = 3$ (so $q > 2$). Again take $x = \text{diag}(a(\lambda,\beta),\lambda^{-2})Z \in T$, with $\lambda$ of order $q + 1$. As usual, $x$ is not a $p^{th}$ power in $\text{PGU}_3(q)$ and is not conjugate to $x^\sigma$ for $\sigma$ a field automorphism unless $p = 2$ and $q = 5$. So it remains to handle $T = U_3(5)$ with $p = 2$; this can be done using [1].

Next assume that $p|q$. If $q > 2$, take $x = \text{diag}(a(\lambda,\beta),\lambda^{-2},1,\ldots,1)Z \in T$ with $\lambda$ of order $q + 1$ again and argue as before. In the case where $q = 2$, take $x = \text{diag}(a(\lambda,\beta),\lambda^{-1},\lambda^{-1},1,\ldots,1)Z \in T$ with $|\lambda| = 3$.

It remains to deal with the case where $p$ divides neither $q + 1$ nor $q$. Then $p > 2$, and any outer automorphism of $T$ of order $p$ is a field automorphism. Choose the first factor in the product $(q^2 - 1)(q^3 + 1)(q^4 - 1) \cdots (q^n - (-1)^n)$ that $p$ divides. If it is $q^i + 1$, take $x$ to be a generator of a cyclic torus of $T$ of type $\text{GU}_1(q^i) < \text{GU}_i(q) \leq \text{GU}_n(q)$ (we must intersect this with $SU_n(q)$ and factor out $Z$); and if it is $q^{2i} - 1$, take $x$ to be a generator of a cyclic torus of type $\text{GL}_1(q^{2i}) < \text{GL}_i(q^{2i}) < \text{GU}_n(q)$. Now argue that $x$ is not a $p^{th}$ power in $T$ and is not conjugate to $x^\sigma$ for $\sigma$ a field automorphism of order $p$.

**Lemma 3.6.** $T$ is not $\text{PSp}_{2n}(q)$.

**Proof.** Suppose $T = \text{PSp}_{2n}(q)$. Then $n \geq 2$ and $(n,q) \neq (2,2)$.

Assume $p > 2$. Then any outer automorphism of $T$ of order $p$ is a field automorphism.

If $p|q$, let $A \in \text{Sp}_2(q)$ be an element of order $q + 1$, and define $x = \text{diag}(A,J_{2n-2})Z \in T$, where as before $J_{2n-2}$ is a unipotent Jordan block of size $2n - 2$. Then $C_T(x) \leq (\text{Sp}_2(q) \times \text{Sp}_{2n-2}(q))/Z$, and since $J_{2n-2}$ is not a $p^{th}$ power in $\text{Sp}_{2n-2}(q)$, $x$ is not a $p^{th}$ power in $T$. Also for a field automorphism $\sigma$ of order $p$, $x^\sigma$ is not conjugate to $x$.

If $p$ does not divide $q$, let $e$ be minimal such that $p|q^e - \delta$ for some $\delta = \pm 1$. If $\delta = -1$, let $x$ be a generator of a cyclic torus of $T$ of order $q^e + 1$ (or $(q^e + 1)/2$) in a subgroup of type $\text{Sp}_2(q^e) \leq \text{Sp}_{2e}(q)$. Also, if $\delta = +1$, then $e$ is odd and we let $x$ generate a torus of order $q^e - 1$ (or $(q^e - 1)/2$) in a subgroup of type $\text{GL}_1(q^e) \leq \text{GL}_e(q) \leq \text{Sp}_{2e}(q)$. Then $x$ is not a $p^{th}$ power in $T$ and $x^\sigma$ is not conjugate to $x$ for a field automorphism $\sigma$ of order $p$.

Now assume $p = 2$. Then a non-diagonal involutory outer automorphism of $T$ involves a field automorphism or, if $n = 2$ and $q = 2^{2k+1}$, a graph automorphism.

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Let \( x = \text{diag}(A, J_{2n-2})Z \in T \) again, and argue as before that \( x \) is not a square in \( T \) and \( x^\sigma \) is not conjugate to \( x \) for a field automorphism \( \sigma \) of order 2. Finally, in the case where \( n = 2 \) and \( q = 2^{2k+1} \) we also need to observe that \( x^\sigma \) is not conjugate to \( x \) for \( \sigma \) an involutory graph automorphism. This follows as \( x = su \) with \( s = \text{diag}(A, I_2) \) and \( u = \text{diag}(I_2, J_2) \) a long root element of \( T \), so \( x^\sigma = s^\sigma u^\sigma \) with \( u^\sigma \) a short root element; hence it is not conjugate to \( x \). □

**Lemma 3.7.** \( T \) is not an orthogonal group.

**Proof.** Suppose \( T \) is orthogonal, so \( T = P\Omega(V) = P\Omega_{2n+1}(q) \) (\( q \) odd, \( n \geq 3 \)) or \( P\Omega_{2n}^\pm(q) \) (\( n \geq 4, \epsilon = \pm \)).

First assume that \( p = 2 \) and \( q \) is odd. Let \( A \) be a matrix in \( GL_2(q) \) of order \( q^2 - 1 \) with eigenvalues \( \lambda, \lambda^q \) over \( \mathbb{F}_q^2 \). With respect to a suitable basis, there is an element \( x = \text{diag}(A, A^{-T}, \lambda^{q+1}, \lambda^{-q-1}, I) \) which lies in a subgroup \( GL_3(q) \) of \( T \) (the subgroup of matrices of a square determinant in \( GL_3(q) \)). We argue in the usual way that \( x \) is not a square in \( P\Delta(V) \) (the notation of [5]) and is not conjugate to \( x^\sigma \) if \( \sigma \) involves an involutory field automorphism.

Now suppose \( p = 2 \) and \( q \) is even. In this case we let \( A \) be an element of order \( q + 1 \) in \( \Omega_2^- (q) \) and argue in the usual way with an element \( x = \text{diag}(A, J_{2n-4}, J_2) \) in a subgroup \( \Omega_2^- (q) \times \Omega_{2n-2}^\epsilon(q) \) of \( T \).

Now let \( p > 2 \). If \( p \) divides \( q + 1 \), let \( A \) be an element of order \( q + 1 \) in \( \Omega_2^- (q) \) and let \( x = \text{diag}(A, J_{2n-3}, J_1) \) in a subgroup \( \Omega_2^- (q) \times \Omega_{2n-2}^\epsilon(q) \). Also, if \( p \) does not divide \( q \), choose \( e \) minimal such that \( p^{e+1} - 1 \) for some \( \delta = \pm 1 \). If \( \delta = -1 \), let \( x \) be a generator of a cyclic torus of type \( \Omega_2^-(q^e) \geq \Omega_2^-(q) \), and if \( \delta = +1 \) (so \( e \) is odd), let \( x \) generate a cyclic torus of type \( GL_1(q^e) \geq GL_1(q) \).

With \( x \) as in the previous paragraph, we argue in the usual way that \( x \) is not a \( p \)-th power in \( T \) and that \( x \) is not conjugate to \( x^\sigma \) when \( \sigma \in P\Gamma(V) \) (the notation of [5]) involves a field automorphism of order \( p \). This completes the proof except in the case where \( p = 3 \) and \( T = P\Omega_8^+(q) \), in which case \( \sigma \) could involve a triality automorphism of \( T \).

So assume finally that \( T = P\Omega_8^+(q) \) and \( p = 3 \).

If \( q = 3^3 \), let \( x = \text{diag}(J_5, \lambda, \lambda^{-1}, 1) \) lying in a subgroup of type \( \Omega_3(q) \times \Omega_3(q) \), where \( \lambda \in \mathbb{F}_q^* \) has order \( (q - 1)/2 \). Write \( x = us \) with \( u = J_5 \in \Omega_5(q) \) and \( s = (\lambda, \lambda^{-1}, 1) \in \Omega_3(q) \). Then \( x \notin T^{[3]} \) as \( u \) is not a cube in \( T \). If \( \sigma \) is an outer automorphism of order 3 involving a triality, then \( x \) is not \( T \)-conjugate to \( x^\sigma \) since \( u \) is not conjugate to \( u^\sigma \) (as \( u^\sigma = J_3^2 \) in a subgroup of type \( Sp_4(q) \)); and if \( \sigma \) is a field automorphism, then the same conclusion holds since \( s \) is not conjugate to \( s^\sigma \).

If \( q \) is not a power of 3, let \( 3 \) divide \( q - \epsilon \) (\( \epsilon = \pm 1 \)), let \( A \) be an element of order \( (q - \epsilon)/(2, q - 1) \) in \( \Omega_2(q) \), and let \( x = \text{diag}(A, J_3, J_2) \) (\( q \) even) or \( \text{diag}(A, J_3, J_1) \) (\( q \) odd) lying in a subgroup of type \( \Omega_2^+(q) \times \Omega_6^+(q) \). Now argue as in the previous paragraph. □

**Lemma 3.8.** \( T \) is not an exceptional group of Lie type.

**Proof.** Suppose \( T \) is an exceptional simple group of Lie type over \( \mathbb{F}_q \). Exclude \( G_2(2)' = U_3(3) \) and \( 2G_2(3)' = L_2(8) \).

Assume first that \( p > 2 \). Then the only outer automorphisms of \( T \) of order \( p \) are field automorphisms, together with diagonal (and field-diagonal) automorphisms when \( p = 3 \), \( T = E_6^\epsilon(q) \) and \( 3|q - \epsilon \).
If \( p \mid q \), then except for \( T = 2G_2(q) \), there is a fundamental \( A = SL_2(q) \) in \( T \), with centralizer \( D \) (where \( D = E_7(q), D_6(q), A_5(q), C_3(q), A_1(q) \) or \( A_1(q^2) \)), according as \( T = E_8(q), E_7(q), E_6(q), F_4(q), G_2(q) \) or \( 3D_4(q) \) respectively). Let \( s \in A \) be an element of order \( q + 1 \), and let \( u \in D \) be a regular unipotent element. Define \( x = su \). Then \( C_T(x) \leq AD \), and so \( x \) is not a \( p^{th} \) power in \( T \) (as \( u \) is not a \( p^{th} \) power in \( D \)). Also, \( x \) is not conjugate to \( x^\sigma \) for \( \sigma \) a field automorphism of order \( p \), so this completes the proof in this case, except for \( T = 2G_2(q) \).

For \( T = 2G_2(q) \), \( p = 3 \), \( q = 3^{2k+1} > 3 \), we require a more detailed argument. Adopting the notation of \([2]\) Table 2.4, \( T \) has a Sylow 3-subgroup \( P = \{x(t,u,v) : t,u,v \in \mathbb{F}_q \} \) of order \( 3^3 \) and exponent 9, where

\[
x(t,u,v) \cdot x(t',u',v') = x(t + t', u + u' + t' \theta^3, v + v' - t'u + (t')^2 \theta^3),
\]

with \( \theta \) being the map \( t \to t^{3^k} \). Then \( Z(P) = \{x(0,0,v) : v \in \mathbb{F}_q \} \). If \( y = x(1,0,0) \), then \( y \) has order 9 (so is not a cube in \( T \)), \( y^3 \in Z(P) \) and \( C_T(y) = \langle y \rangle Z(P) \) (see \([2]\)). If \( \sigma \) is an outer automorphism of \( T \) of order 3, then it is a field automorphism and we can take it to act on \( P \) as \( x(t,u,v) \to x(t^{\sigma}, u^{\sigma}, v^{\sigma}) \). Suppose \( y \) is a cube in \( T(\sigma) \), say \( y = (x\sigma)^3 \) with \( x \in T \). Then \( x\sigma \in C_T(\sigma)(y) = \langle y \rangle Z(P) \langle \sigma \rangle \), so \( y = x^k x(0,0,v) \) for some integer \( k \) and \( v \in \mathbb{F}_q \). But then since \( y \) centralizes \( x(0,0,v) \) we have \((x\sigma)^3 = y^{3k} x(0,0,v^{1+\sigma+\sigma^2})\) which has order dividing 3, so it cannot equal \( y \). Hence \( y \) is not a cube in \( T(\sigma) \), completing the proof in this case.

Now assume \( p \) does not divide \( q \) (still with \( p > 2 \)). Postpone the case where \( p = 3, T = E_6^\epsilon(q) \) and \( 3 | q - \epsilon \). From \([3]\) Section 2, we check that with a few exceptions (listed below), there is a cyclic maximal torus of \( T \) of order divisible by \( p \). If we take \( x \) to be a generator of this torus, then \( x \) is not a \( p^{th} \) power in \( T \) and is not conjugate to \( x^\sigma \) if \( \sigma \) is a field automorphism of order \( p \). The exceptions are as follows:

<table>
<thead>
<tr>
<th>( T )</th>
<th>( E_7(q) )</th>
<th>( E_6(q) )</th>
<th>( 2E_6(q) )</th>
<th>( F_4(q) )</th>
<th>( 2G_2(q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( q_4,q_8 )</td>
<td>( q_6 )</td>
<td>( q_3 )</td>
<td>( q_4 )</td>
<td>( q_2 )</td>
</tr>
</tbody>
</table>

Here \( q_i \) denotes a primitive prime divisor of \( q^i - 1 \). For the \( T = E_7(q) \) case, take \( x \) to be an element of order \( q^8 - 1 \) or \( \frac{q^8 + 1}{2(q-1)} \) in a subsystem subgroup \( A_3(q) \) or \( D_4(q) \) in the respective cases \( p = q_4,q_8 \). If \( x = y^p \) for some \( y \in T \), then \( y \) lies in a maximal torus, but we see from \([4]\) that there is no maximal torus in which \( x \) is a \( p^{th} \) power. Hence \( x \) is not a \( p^{th} \) power in \( T \). Also, if \( \sigma \) is a field automorphism of order \( p \), then from the action of \( \sigma \) on \( A_3(q) \) or \( D_4(q) \), we see that \( x \) is not conjugate to \( x^\sigma \).

The cases \( T = E_6^\epsilon(q) \) are handled similarly by taking \( x \) to be an element of order \( \frac{q^6 - 1}{q-1} \) or \( \frac{q^6 + 1}{2(q-1)} \) in a subsystem subgroup \( A_5^\epsilon(q) \). Finally, in the \( F_4(q) \) and \( 2G_2(q) \) cases we take \( x \) of order \( \frac{q^6 - 1}{2(q-1)} \) or \( \frac{q^6 + 1}{2} \) in a maximal torus of the form \( \langle x \rangle \times (2,q-1) \).

Now consider the postponed case where \( p = 3, T = E_6^\epsilon(q) \) and \( 3 | q - \epsilon \). In a subsystem subgroup \( A_1(q)A_1^\epsilon(q) \), take an element \( x = yz \), where \( y \in A_1(q) \) has order \( q - \epsilon \) and \( z \) is a regular unipotent element in \( A_5^\epsilon(q) \). If \( T.3 \) denotes the group generated by inner and diagonal automorphisms of \( T \), then \( C_T.3(x) = \langle y \rangle U \) where \( U \) is a unipotent group, so \( x \) is not a cube in \( T.3 \). Also \( x \) is not conjugate to \( x^\sigma \) when \( \sigma \) involves a field automorphism of order 3.

This completes the case where \( p > 2 \). Now suppose \( p = 2 \). Note that \( T \neq 2B_2(q), 2G_2(q) \) or \( 2F_4(q) \) (\( q > 2 \)), as these have no outer automorphisms of order 2.
Assume \(q\) is odd. For \(T = E_8(q), F_4(q), 3D_4(q)\) or \(G_2(q)\) (\(q \neq 3^k\)), take \(x\) to be a generator of a cyclic maximal torus of even order (which exists by \([4]\)), and argue as usual that \(x\) is not a square in \(T\) and is not conjugate to \(x^\sigma\) for \(\sigma\) an involutory field automorphism. The other groups \(E_7(q), E_6(q), G_2(q)\) (\(q = 3^k\)) possess diagonal or graph automorphisms of order 2, so require a little more care.

For \(T = E_7(q)\) we work in a subsystem subgroup \(A_2(q)A_5(q)\). This has normalizer \(N = A_2(q)A_5(q).2\) in the inner-diagonal group \(T.2\). The outer involution acts diagonally on the \(A_5(q)\) factor and as an inner automorphism on \(A_2(q)\). Take an element \(x\) in the factor \(A_2(q) \cong SL_3(q)\) of order \(q^2 - 1\). Then \(C_{T.2}(x) \leq N\), so we see that \(x\) is not a square in \(T.2\). Also \(x\) is not conjugate to \(x^\sigma\) when \(\sigma\) involves an involutory field automorphism, so this case is done.

For \(T = E_6(q)\), take \(x\) to be an element of order \(q^4 - 1\) in a subsystem subgroup \(A_4(q) \cong SL_5(q)\). No torus in \(T\) has an element of order \(2(q^4 - 1)\) (see \([4]\)), so \(x\) is not a square in \(T\). If \(\sigma\) is a graph automorphism of \(T\), it acts as a graph automorphism on a suitable subgroup \(A_4(q)\), and hence we see that \(x\) is not conjugate to \(x^\sigma\). Also \(x\) is not conjugate to \(x^\sigma\) when \(\sigma\) involves an involutory field automorphism.

Now consider \(T = G_2(q)\) with \(q = 3^k\). Let \(q \equiv \epsilon \mod 4\) with \(\epsilon = \pm 1\). There is a subgroup \(A_1A_1\) in \(T\), a commuting product of two \(SL_2(q)\)’s where \(A_1\) is generated by long root groups and \(A_1\) by short root groups. Let \(x = us\) with \(u \in A_1\) of order 3 and \(s \in A_1\) of order \(q - \epsilon\). Then \(C_T(x) \leq A_1A_1\), and hence we see that \(x \not\in T^{[2]}\). If \(\sigma\) is an involutory outer automorphism of \(T\) involving a graph automorphism, then \(x^\sigma\) is not \(T\)-conjugate to \(x\) (since the long root element \(u\) is not conjugate to the short root element \(u^\sigma\)), and if \(\sigma\) is a field automorphism, then the same conclusion holds as \(s^\sigma\) is not conjugate to \(s\).

Now assume that \(q\) is even (still with \(p = 2\)). Use \([1]\) for the case where \(T = 2F_4(2)'\). Since we have ruled out \(T\) of type \(2B_2\) or \(2F_4\), this leaves \(T\) of type \(E_8, E_7, E_6, F_4, G_2\) or \(3D_4\). For all but the \(E_6\) and \(F_4\) cases we can argue exactly as for the \(p|q\) case done above for \(p > 2\). For \(E_6\) and \(F_4\) there are graph automorphisms to take into account.

In the case where \(T = E_6(q)\), in a subsystem subgroup \(A_1(q)A_5^+(q)\) take \(x = us\), where \(u \in A_1(q)\) is an involution and \(s \in A_5^+(q)\) an element of order \(\frac{q^6 - 1}{q^2 - 1}\). Then \(C_T(x) = C_{A_1}(u)\langle s\rangle\), so \(x\) is not a square in \(T\). Also, a graph automorphism \(\sigma\) normalizing \(A_1(q)A_5^+(q)\) acts as a graph automorphism on \(A_5^+(q)\), hence inverts \(x\), so \(x\) is not \(T\)-conjugate to \(x^\sigma\). Also, \(x\) is not conjugate to \(x^\sigma\) when \(\sigma\) involves an involutory field or graph-field automorphism.

Finally, consider \(T = F_4(q)\). In a subsystem subgroup \(A_2(q)A_2(q)\) take \(x = us\), where \(u\) is a regular unipotent element of the first factor and \(s\) an element of order \(q^2 + q + 1\) in the second. Since \(C_T(s) = A_2(q)\langle s\rangle\), \(x\) is not a square in \(T\). For \(\sigma\) a graph automorphism, \(x^\sigma = u^s s^\sigma\) is not conjugate to \(x\), as \(u\) and \(u^\sigma\) are not conjugate, one being regular in a long root \(A_2\), the other in a short root \(A_2\). As usual, \(x\) is not conjugate to \(x^\sigma\) when \(\sigma\) is an involutory field automorphism. This completes the proof.

4. General finite groups

First we prove Theorem 4. Let \(G\) be a finite group and suppose \(G^{[k]}\) is a subgroup of \(G\). The proof is by induction on \(|G|\). Let \(N\) be a minimal normal subgroup of \(G\). Then \((G/N)^{[k]}\) is a subgroup; hence by induction its non-abelian composition factors satisfy the conclusion of the theorem. If \(N\) is abelian, then the theorem
follows. So we may assume that $N = T^r$ for some non-abelian simple group $T$. It suffices to show that either $T \subseteq \operatorname{Aut}(T)^{[k]}$ or the exponent of $T$ divides $k$. Assume the contrary, and let $t \in T \setminus \operatorname{Aut}(T)^{[k]}$.

Let $\bar{G} = G/C_G(N)$. Then $\bar{G}$ embeds in $\operatorname{Aut}(N) = \operatorname{Aut}(T) \wr S_r$. We identify $N$ with its image in $\bar{G}$.

We claim that the element $n = (t, 1, \ldots, 1) \in T^r = N$ is not a $k^{th}$ power in $\bar{G}$. To see this, suppose $n = x^k$ where $x = (x_1, \ldots, x_r)\sigma$ with each $x_i \in \operatorname{Aut}(T)$ and $\sigma \in S_r$. Then $\sigma^k = 1$. If $\sigma(1) = 1$, then $t = x_1^k$, contradicting the fact that $t$ is not a $k^{th}$ power in $\operatorname{Aut}(T)$. So $\sigma$ has a cycle $(1i_2 \cdots i_s)$ with $s \geq 1$. Calculating the coordinates of $x^k$ in positions 1 and $i_s$, we get $t = x_1x_{i_2} \cdots x_{i_s}$ and $1 = x_1x_1 \cdots x_{i_{s-1}}$, a contradiction.

It follows that $G^{[k]}$ is a normal subgroup of $G$ which does not contain $N$. Hence $G^{[k]} \cap N = 1$. Therefore all $k^{th}$ powers in $N$ are trivial, which means that $k$ is divisible by the exponent of $T$. This contradicts our assumption on $T$ and completes the proof of the first assertion of Theorem 4. The last assertion follows using Burnside’s $p^aq^b$ theorem.

Finally we deduce Theorems [1] [2] and [3]. Suppose $G$ is a finite group such that $G^{[k]}$ is a subgroup, where $k$ divides 12. Then Theorem [4] shows that $T \subseteq \operatorname{Aut}(T)^{[k]}$ for every composition factor $T$ of $G$.

If $k = 2$, then Theorem [7] shows that the non-abelian composition factors of $G$ are among the groups $L_2(q)$ ($q$ odd), $L_2(q^2)$ ($q$ even) and $L_3(4)$, proving Theorem [3].

Now assume that both $G^{[3]}$ and $G^{[4]}$ are subgroups of $G$. Suppose $G$ is not soluble, and let $T$ be a non-abelian composition factor. Since all non-abelian simple groups have order divisible by 4, Theorem [7] shows that $T = L_2(q)$ with $q$ even. Then $T$ has order divisible by 3, so Theorem [7] now gives a contradiction. Hence $G$ is soluble, proving Theorem [2].

Finally, assume that $G^{[12]}$ is a subgroup of $G$. If $T$ is a non-abelian composition factor, then $T \subseteq \operatorname{Aut}(T)^{[12]} \subseteq \operatorname{Aut}(T)^{[4]}$, so again Theorem [7] gives $T = L_2(q)$ with $q$ even. But then 12 divides $|T|$, so Theorem [7] gives a contradiction. Hence $G$ is soluble and Theorem [1] is proved.

5. **GOOD AND BAD NUMBERS**

Define a positive integer $k$ to be good if the assumption that $G^{[k]}$ is a subgroup implies that $G$ is soluble, and bad otherwise. We observed in the Introduction that 12 is the minimal good number.

**Proposition 5.1.** The following numbers are good:

(i) $2^ap^b$ with $a \geq 2$, $b \geq 1$ and $p \in \{3, 5, 17\}$;

(ii) 105.

**Proof.** We copy the proof of Theorem [1]. Let $k$ be one of the numbers in (i) or (ii) and suppose $G^{[k]}$ is a subgroup of $G$. Assume $G$ has a non-abelian composition factor $T$. Then $T \subseteq \operatorname{Aut}(T)^{[k]}$ by Theorem [4]. For $k$ as in (i), Theorem [7] implies that $T = L_2(2^{4r})$ for some $r$, but then $|T|$ is divisible by the primes $p \in \{3, 5, 17\}$, so Theorem [7] gives a contradiction. Finally, assume $k = 105$. If $|T|$ is divisible by 3, then Theorem [7] implies that $T = L_2(3^{3r})$, but then $|T|$ is divisible by 7 and Theorem [7] gives a contradiction. If $|T|$ is coprime to 3, then $T$ is a Suzuki group. Then 5 divides $|T|$ and once again Theorem [7] gives a contradiction.
Proposition 5.2. The following numbers are bad:
(i) $p^a$ and $2p^a$ with $p$ prime;
(ii) numbers coprime to 6;
(iii) $3^a p^b$ with $p > 3$ prime and $a, b \geq 1$.

Proof. (i) This is clear from Proposition 6.
(ii) Let $k$ be coprime to 6. Using Dirichlet’s theorem on primes in arithmetic progression, one can see that there is a prime $p > 3$ such that $T = L_2(p)$ has order coprime to $k$. Then $T[k] = T$, which shows that $k$ is bad.
(iii) Let $k = 3^a p^b$ as in (iii). If $p \neq 5$, then $k$ is coprime to the order of one of the Suzuki groups $Sz(8)$ or $Sz(32)$, so $k$ is bad. If $p = 5$, then $p$ does not divide the order of $T = L_2(3^3)$, so Proposition 6 shows that there is a group $G$ with socle $T$ such that $G[k] = T$. □

It follows quickly that 20 is the smallest even good number greater than 12 and 105 is the smallest odd good number.

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