ON THE SELF-SIMILARITY PROBLEM FOR GAUSSIAN-KRONECKER FLOWS

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Abstract. It is shown that a countable symmetric multiplicative subgroup $G = -H \cup H$ with $H \subset \mathbb{R}^*_+$ is the group of self-similarities of a Gaussian-Kronecker flow if and only if $H$ is additively $\mathbb{Q}$-independent. In particular, a real number $s \neq \pm 1$ is a scale of self-similarity of a Gaussian-Kronecker flow if and only if $s$ is transcendental. We also show that each countable symmetric subgroup of $\mathbb{R}^*$ can be realized as the group of self-similarities of a simple spectrum Gaussian flow having the Foiaş-Stratila property.

1. Introduction

Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is a (measurable) measure-preserving flow acting on a standard probability Borel space $(X, \mathcal{B}, \mu)$. Given $s \in \mathbb{R}^*$, one says that it is a scale of self-similarity of $\mathcal{T}$ if $\mathcal{T}$ is isomorphic to $\mathcal{T}_s := (T_{st})_{t \in \mathbb{R}}$. Denote by $I(\mathcal{T})$ the set of all scales of self-similarities of $\mathcal{T}$. Then $\mathcal{T}$ is called self-similar if $I(\mathcal{T}) \neq \{\pm 1\}$. Classical examples of self-similar flows are given by horocycle flows where $I(\mathcal{T})$ equals either $\mathbb{R}^*$ or $\mathbb{R}^*_{+}$ [19]. A systematic study of the problem of self-similarity has been done recently in [4] and [6]. In particular, $I(\mathcal{T})$ turns out to be a multiplicative subgroup of $\mathbb{R}^*$ ([6]) which is Borel ([4]), and one of the main problems in this domain is to classify all Borel subgroups of $\mathbb{R}^*$ that may appear as groups of self-similarities of ergodic flows; see also [3], [13], [24], [25] for a contribution to other aspects of the problem of self-similarity of ergodic flows. From this point of view the subclass of so-called GAG flows [17] of the class of Gaussian flows is especially attractive since self-similarities appear there as natural invariants; see (1.1) below. By definition, GAG flows are those Gaussian flows whose ergodic self-joinings remain Gaussian. All Gaussian flows with simple spectrum are GAG flows [17]. If $\mathcal{T}^\sigma = (T_t^\sigma)_{t \in \mathbb{R}}$ denotes the Gaussian flow determined by a finite positive (continuous) measure $\sigma$ on $\mathbb{R}^*_+$ and the flow is GAG, then

\begin{equation}
I(\mathcal{T}^\sigma) \text{ is equal to the (multiplicative) group } -I(\sigma) \cup I(\sigma),
\end{equation}

where $I(\sigma) = \{s \in \mathbb{R}^*_+ : \sigma_s \equiv \sigma\}$ and $\sigma_s = (R_s)_*(\sigma)$ denotes the image of $\sigma$ via the map $R_s : x \mapsto sx$ [17]. Recall that $-1$ is always a scale of self-similarity for a Gaussian flow.

In this note we focus on the problem of self-similarities in some subclasses of simple spectrum Gaussian flows. We first recall already known results. Classically,
if $\sigma$ is concentrated on an additively $\mathbb{Q}$-independent Borel set $A \subset \mathbb{R}_+$, then the Gaussian flow $T^\sigma$ has simple spectrum; see [2]. Moreover, the subgroup $H := I(T^\sigma) \cap \mathbb{R}_+$ is an additively $\mathbb{Q}$-independent set. Indeed, suppose that $H$ is not an additively $\mathbb{Q}$-independent set. That is, for some distinct $h_1, \ldots, h_m \in H$ we have

$$(1.2) \quad \sum_{i=1}^{m} k_i h_i = 0 \quad \text{with} \quad k_i \in \mathbb{Z}, \ i = 1, \ldots, m \quad \text{and} \quad \sum_{i=1}^{m} k_i^2 > 0.$$ 

Denote by $H_0 \subset H$ the multiplicative subgroup generated by $h_1, \ldots, h_m$. Since $H_0 \subset I(T^\sigma)$, we have $\sigma_{h} \equiv \sigma$ for $h \in H_0$; thus the Borel set $B = \bigcap_{h \in H_0} hA$ has full $\sigma$-measure, is $\mathbb{Q}$-independent, and is literally $H_0$-invariant. Take any non-zero $x \in B$. Then the elements $h_i x \in B, \ i = 1, \ldots, m$, are distinct. Now, (1.2) yields

$$\sum_{i=1}^{m} k_i (h_i x) = x \sum_{i=1}^{m} k_i h_i = 0,$$

so $B$ is not independent, a contradiction. On the other hand, in [6] it is shown that whenever a countable group $H \subset \mathbb{R}_+$ satisfies:

- For each polynomial $P \in \mathbb{Q}[x_1, \ldots, x_m]$, if there is a collection of distinct elements $h_1, \ldots, h_m$ in $H$ such that $P(h_1, \ldots, h_m) = 0$, then $P \equiv 0$,

then there exists a probability $\sigma$ concentrated on a Borel $\mathbb{Q}$-independent set such that $I(T^\sigma) = -H \cup H$. It is not difficult to see that condition (1.3) is equivalent to saying that $H$ is an additively $\mathbb{Q}$-independent set.

**Theorem 1.1** ([6]). Assume that $G = -H \cup H$, where $H \subset \mathbb{R}_+$ is a countable multiplicative subgroup. Then $G$ can be realized as $I(T^\sigma)$ for a Gaussian flow whose spectral measure $\sigma$ is concentrated on a Borel $\mathbb{Q}$-independent set if and only if $H$ is an additively $\mathbb{Q}$-independent set.

Note that for $H \subset \mathbb{R}_+$ cyclic generated by $s \in \mathbb{R}_+$, the $\mathbb{Q}$-independence condition is equivalent to saying that $s$ is transcendental. Hence, by Theorem 1.1 a real number $s$ can be realized as a scale of self-similarity of a Gaussian flow whose spectral measure is concentrated on a $\mathbb{Q}$-independent Borel set if and only if $s$ is transcendental.

On the other hand, there are no restrictions on $H$ in the class of all Gaussian flows having simple spectrum.

**Theorem 1.2** ([14]). For each countable subgroup $H \subset \mathbb{R}_+$ there exists a simple spectrum Gaussian flow $T^\sigma$ such that $I(T^\sigma) = -H \cup H$.

Note that, in particular, the above result of Danilenko and Ryzhikov brings a positive answer to the open problem [14] of existence of Gaussian flows $T^\sigma$ with simple spectrum such that $\sigma$ is not concentrated on a $\mathbb{Q}$-independent set. Indeed, whenever $H$ is not an additively $\mathbb{Q}$-independent set, by Theorem 1.1 the spectral measure $\sigma$ resulting from Theorem 1.2 cannot be concentrated on a Borel $\mathbb{Q}$-independent set. See also [6] for constructions of Gaussian flows with zero entropy and having uncountable groups of self-similarities.

Our aim is to continue investigations on the realization of countable subgroups as the groups of self-similarities in further restricted classes of Gaussian flows whose spectral measures are classical from the harmonic analysis point of view. Recall some basic notions. For every $s \in \mathbb{R}$ let $\xi_s : \mathbb{R} \to S^1$ be given by $\xi_s(t) = \exp(2\pi ist)$. 

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A finite positive Borel measure $\sigma$ on $\mathbb{R}$ is called Kronecker if for each $f \in L^2(\mathbb{R}, \sigma)$, $|f| = 1$ $\sigma$-a.e., there exists a sequence $(t_n) \subset \mathbb{R}$, $t_n \to \infty$ such that

$$\xi_{t_n} \to f \quad \text{in} \quad L^2(\mathbb{R}, \sigma).$$

Each measure $\sigma$ concentrated on a Kronecker set is a Kronecker measure [12], [18]. Indeed, Kronecker sets are compact subsets of $\mathbb{R}$ on which each continuous function of modulus one is a uniform limit of characters. Kronecker sets are examples of $\mathbb{Q}$-independent sets [18]. In general, as shown in [15], a Kronecker measure is concentrated on a Borel set which is the union of an increasing sequence of Kronecker sets; hence a Kronecker measure is concentrated on a Borel $\mathbb{Q}$-independent set, and the restriction on $H$ in Theorem 1.1 applies. This turns out to be the only restriction as the main result of the note shows.

**Theorem 1.3.** Assume that $G = -H \cup H$, where $H \subset \mathbb{R}_+^*$ is a countable multiplicative subgroup. Then $G$ can be realized as $I(T^{\sigma})_2$ for a Gaussian-Kronecker flow if and only if $H$ is an additively $\mathbb{Q}$-independent set. In particular, $h \in \mathbb{R}_+$ can be a scale of self-similarity for a Gaussian-Kronecker flow if and only if $h$ is transcendental.

An extremal case when two dynamical systems are non-isomorphic is the disjointness in the Furstenberg sense [7]; see also [9], [11], [14], [23] for disjointness results in ergodic theory. We would also like to emphasize that the notion of disjointness turned out to be quite meaningful in the problem of non-correlation with the Möbius function of sequences of dynamical origin [1]: we need that an automorphism $T$ has the property that $T^p$ and $T^q$ are disjoint for any two different primes. In connection with that we will prove the following.

**Theorem 1.4.** Assume that $T^{\sigma} = (T^{\sigma}_t)_{t \in \mathbb{R}}$ is a Gaussian-Kronecker flow. If $s \in \mathbb{Q} \setminus \{\pm 1\}$, then $T^{\sigma}_s$ is disjoint from $T^{\sigma}_1$. For every Gaussian-Kronecker automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ the iterations $T^n$, $T^m$ are disjoint for any two distinct natural numbers $n, m$.

If $s$ is irrational, then there exists a Gaussian-Kronecker flow $T^{\sigma}_s$ such that $T^{\sigma}_s$ and $T^{\sigma}_1$ have a non-trivial common factor.

The importance of Kronecker measures in ergodic theory follows from the remarkable result of Foiaş and Stratila [5] (see also [2], and remarks on that result in [15] and [21]):

$$\text{If} \ (S_t)_{t \in \mathbb{R}} \text{is an ergodic flow of a standard probability Borel space} \ (Y, \mathcal{C}, \nu), \ f \in L^2(Y, \mathcal{C}, \nu) \text{is real and the spectral measure} \ \sigma_f \text{is the symmetrization of a Kronecker measure, then the (stationary) process} \ \{f \circ S_t\}_{t \in \mathbb{R}} \text{is Gaussian.}$$

In [15], any measure $\sigma$ satisfying the assertion (1.5) of the Foiaş-Stratila theorem is called an FS measure. Each Kronecker measure is a Dirichlet measure [18], but as shown in [15], there are FS measures which are not Dirichlet measures (see Figure 1). Moreover, in [21] it is announced that each continuous measure concentrated on an

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2In a sense, we can also control the flows $T^{\sigma}_s$ for $s \notin -H \cup H$. We will prove their disjointness from $T^{\sigma}_1$; see the proof of this theorem.

3A probability Borel measure $\sigma$ on $\mathbb{R}$ is Dirichlet if (1.4) is satisfied for $f = 1$. From the dynamical point of view, Dirichlet measures correspond to rigidity: a flow $T$ is rigid if $T_{t_n} \to \text{Id}$ for some $t_n \to \infty$. 

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independent Helson set is a Kronecker measure (for some examples in [21], the resulting Gaussian flows have no non-trivial rigid factors). We will strengthen Theorem 1.2 to the following result.

**Theorem 1.5.** Any symmetric countable group $G \subset \mathbb{R}^*$ can be realized as the group of self-similarities of a simple spectrum Gaussian flow $T^\sigma$ with $\sigma$ being an FS measure.

In particular, in connection with the aforementioned question from [14], there is an FS measure for which the Gaussian flow has simple spectrum but $\sigma$ is not concentrated on a $\mathbb{Q}$-independent set. These are apparently the first examples of FS measures which are not concentrated on $\mathbb{Q}$-independent Borel sets but yield Gaussian flows with simple spectrum (cf. [15] and [21]).

At the end of this note we will discuss self-similarity properties of Gaussian flows arising from a “typical” measure or from the maximal spectral type of a “typical” flow (cf. the disjointness results from [4]).

**Theorem 1.6.** Assume that $0 \leq a < b$. For a “typical” $\sigma \in \mathcal{P}([a,b])$ the flow $T^\sigma$ is Gaussian-Kronecker such that for each $|r| \neq |s|$ the flows $T^{\sigma r}$ and $T^{\sigma s}$ are disjoint. In particular, $I(T^\sigma) = \{\pm 1\}$.

For a “typical” flow $T$ of a standard probability Borel space $(X, \mathcal{B}, \mu)$, for its maximal spectral type $\sigma_T$ we have: $T^{\sigma_T |_{\mathbb{R}^+}}$ has simple spectrum, and for $|r| \neq |s|$ the flows $T^{(\sigma_T |_{\mathbb{R}^+})_r}$ and $T^{(\sigma_T |_{\mathbb{R}^+})_s}$ are disjoint. In particular, $I(T^{(\sigma_T |_{\mathbb{R}^+})}) = \{\pm 1\}$.

## 2. Notation and basic results

Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is a measurable measure-preserving flow acting on a standard probability Borel space $(X, \mathcal{B}, \mu)$. It then induces a (continuous) one-parameter group of unitary operators acting on $L^2(X, \mathcal{B}, \mu)$ by the formula $T_t f = f \circ T_t$. By Bochner’s theorem, the function $t \mapsto \int_X T_t f \cdot \mathfrak{f} \, d\mu$ determines the so-called spectral measure $\sigma_f$ of $f$ for which $\hat{\sigma}_f(t) = \int_X T_t f \cdot \mathfrak{f} \, d\mu$, $t \in \mathbb{R}$. Usually, one considers only spectral measures of $f \in L^2_0(X, \mathcal{B}, \mu)$, that is, of elements with zero mean (the spectral measure of the constant function $c$ is equal to $|c|^2 \delta_0$). Then $\sigma_f$ is a finite positive Borel measure on $\mathbb{R}$. Among spectral measures there are maximal ones with respect to the absolute continuity relation. Each such maximal measure is called a maximal spectral type measure and, by some abuse of notation, will be denoted by $\sigma_T$. We refer the reader to [11] and [14] for some basics about

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4 A $\subset \mathbb{R}$ is called Helson if for some $\delta > 0$ and each complex Borel measure $\kappa$ concentrated on $A$, the $\sup_{t \in \mathbb{R}} |\int_A e^{2\pi i t x} \, d\kappa(x)|$ is bounded away from the $\delta$-fraction of the total variation of $\kappa$.

5 Measurability means that for each $f \in L^2(X, \mathcal{B}, \mu)$ the map $t \mapsto f \circ T_t$ is continuous.
spectral theory of unitary representations of locally compact Abelian groups in the dynamical context.

Assume that $\mathcal{T}$ is ergodic and let $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ be another ergodic flow (acting on $(Y,C,\nu)$). Any probability measure $\rho$ on $(X \times Y, \mathcal{B} \otimes C)$ which is $(T_t \times S_t)_{t \in \mathbb{R}}$-invariant and has marginals $\mu$ and $\nu$ respectively is called a joining of $\mathcal{T}$ and $\mathcal{S}$. If, additionally, the flow $((T_t \times S_t)_{t \in \mathbb{R}}, \rho)$ is ergodic, then $\rho$ is called an ergodic joining\(^6\)

The ergodic joinings are extremal points in the simplex of all joinings. If the set of joinings is reduced to contain only the product measure, then one speaks about disjointness of $\mathcal{T}$ and $\mathcal{S}$\(^7\), and we will write $\mathcal{T} \perp \mathcal{S}$. Similar notions appear when one considers automorphisms. Note that whenever for some $t \neq 0$, $T_t \perp S_t$, then $\mathcal{T} \perp \mathcal{S}$. Note also that whenever

\begin{equation} \label{eq:2.1} \mathcal{T} \text{ is weakly mixing, then } \mathcal{T} \perp \mathcal{S} \text{ if and only if } T_1 \perp S_1. \end{equation}

Indeed, if $T_1 \not\perp S_1$, then there exists an ergodic joining $\rho$ between them different from the product measure. Then, $\rho \circ (T_t \times S_t)$ for $0 \leq r < 1$ has the same properties. By disjointness of $\mathcal{T}$ and $\mathcal{S}$, $\int_0^1 \rho \circ (T_t \times S_t) \, dr = \mu \otimes \nu$. But $T_1$ is weakly mixing, so $\mu \otimes \nu$ is an ergodic joining of $T_1$ and $S_1$. Therefore $\rho \circ (T_t \times S_t) = \mu \otimes \nu$ for a.e. $r \in [0,1]$, and thus $\rho = \mu \otimes \nu$. We refer the reader to [9] for the theory of joinings in ergodic theory.

A flow $\mathcal{T}$ is called Gaussian if there is a $\mathcal{T}$-invariant subspace $\mathcal{H} \subset L^2_0(X, \mathcal{B}, \mu)$ of the zero mean real-valued functions such that all non-zero variables from $\mathcal{H}$ are Gaussian and the smallest $\sigma$-algebra making all these variables measurable equals $\mathcal{B}$. A Gaussian flow is ergodic if and only if the maximal spectral type on $\mathcal{H}$ is continuous (and then $\mathcal{T}$ is weakly mixing). Since Gaussian variables are real, it is not hard to see that their spectral measures are symmetric; that is, for $f \in \mathcal{H}$, $\sigma_f$ is invariant under the map $R_{-1} : x \mapsto -x$.

A standard way to obtain a (weakly mixing) Gaussian flow is to start with a finite positive continuous Borel measure $\sigma$ on $\mathbb{R}_+$. Consider the symmetrization $\bar{\sigma} = \sigma + (R_{-1})_* \sigma$\(^4\). We let $\mathcal{V} = (V_t)_{t \in \mathbb{R}}$ denote the one-parameter group of unitary operators on $L^2(\mathbb{R}, \bar{\sigma})$ defined by $V_t(f)(x) = e^{2\pi i t x} f(x)$. Then the correspondence

\begin{equation} \label{eq:2.2} f(x) \mapsto f(-x) \end{equation}

yields the unitary conjugation of $\mathcal{V}$ and its inverse. Let $(X, \mathcal{B}, \mu)$ be a Gaussian probability space, that is, a standard probability space together with an infinite dimensional, closed, real and $\mathcal{B}$-generating subspace $\mathcal{H} \subset L^2(X, \mathcal{B}, \mu)$ where all non-zero variables are Gaussian. We then consider $\mathcal{H} + i \mathcal{H}$, the so-called complex Gaussian space, and define an isomorphic copy of $\mathcal{V}$ on it. It is then standard to show (see e.g. [17], Section 2) that $\mathcal{V}$ has a unique extension to a (measurable) flow $\mathcal{T}^\sigma = (T^\sigma_t)$ of $(X, \mathcal{B}, \mu)$ for which $U_{\mathcal{T}^\sigma_t} |_{\mathcal{H} + i \mathcal{H}} = V_t$, $t \in \mathbb{R}$. By the same token, the correspondence \((\ref{eq:2.2})\) extends to an isomorphism of $(X, \mathcal{B}, \mu)$ which conjugates the Gaussian flow and its inverse $(T^\sigma_{-t})_{t \in \mathbb{R}}$.

A Gaussian flow $\mathcal{T}^\sigma$ is called Gaussian-Kronecker (FS resp.) if $\sigma$ is a continuous Kronecker (FS resp.) measure. Following [17], a Gaussian flow $\mathcal{T}^\sigma$ (with the Gaussian space $\mathcal{H}$) is called GAG if for each of its ergodic self-joining $\eta$ the space

\[
\{ f(x) + g(y) : f, g \in \mathcal{H} \}
\]

\(^6\)If $\mathcal{T} = \mathcal{S}$, then we speak about self-joinings.

\(^7\)In general, when $f$ is a measurable map from $(X, \mathcal{B})$ to $(Y, C)$ and $\kappa$ is a probability measure on $X$, then $f_\ast (\kappa)$ is the measure on $Y$ defined by $f_\ast (\kappa)(C) = \kappa (f^{-1}(C))$. 

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consists solely of Gaussian variables (the flow \( (T_t^\sigma \times T_t^\eta)_{t \in \mathbb{R}}, \eta \) is then a Gaussian flow as well). As shown in [17], FS-flows and Gaussian flows with simple spectrum are GAG flows. Relationships between these classes of flows are shown in Figure 2.

![Diagram of different subclasses of GAG flows]

**Figure 2.** Different subclasses of GAG flows

For all these classes of flows we have that if \( T^\sigma \) is in the class, then so is \( T^\sigma_s \) for \( s \neq 0 \).

In general, Gaussian flows given by equivalent measures are isomorphic. It follows from [17] that any isomorphism between a GAG flow \( T^\sigma \) and another Gaussian flow \( T^\nu \) is entirely determined by a unitary isomorphism of restrictions of the unitary actions \( (T_t^\sigma)_{t \in \mathbb{R}} \) and \( (T_t^\nu)_{t \in \mathbb{R}} \) to their Gaussian subspaces. That is, in the GAG situation, \( T^\sigma \) and \( T^\nu \) are isomorphic if and only if \( \sigma \equiv \nu \). If we apply this to \( \sigma \) and \( \sigma_s \) for \( s \in \mathbb{R}_+ \), we will immediately get (1.1) to hold (in the GAG case).

We will now prove the following.

**Proposition 2.1.** Assume that \( T^\sigma \) is GAG. Fix \( s \neq 0 \). Then the sets of self-joinings of \( T^\sigma \) and of self-joinings of \( T_s^\sigma \) are the same. (Hence ergodic self-joinings are also the same.) In particular, the factors and the centralizer of the flow and of the time \( s \)-automorphism are the same.

**Proof.** This follows from the proof of Theorem 6.1 in [10] which asserts that such an equality of the sets of self-joinings takes place whenever each ergodic self-joining of the flow is an ergodic self-joining for the time-\( s \)-automorphism. In the GAG case, by definition, such ergodic joinings for the flow \( T^\sigma \) are Gaussian joinings, so they are automatically ergodic for the \( T_s^\sigma \) [17].

**Corollary 2.2.** Assume that \( T^\sigma \) is GAG. Then \( T_s^\sigma \) is a GAG automorphism for each \( s \neq 0 \).

We will also make use of the following results.

**Theorem 2.3** ([17]). Assume that \( T^\sigma \) is GAG and let \( T^\nu \) be an arbitrary Gaussian flow. Then \( T^\sigma \perp T^\nu \) if and only if \( \tilde{\sigma} \perp \tilde{\eta} \ast \delta_r \) for each \( r \in \mathbb{R} \).

**Proposition 2.4** ([15]). If \( \sigma_1 \) and \( \sigma_2 \) are measures with the FS property and \( T^{\sigma_1} \perp T^{\sigma_2} \), then \( \sigma = \frac{1}{2}(\sigma_1 + \sigma_2) \) is also an FS measure.

Using the elementary fact that the \( L^2 \)-limit of a sequence of Gaussian variables remains Gaussian, we obtain the following.

**Corollary 2.5.** If \( \sigma_i, i \geq 1 \), are FS probability measures such that \( T^{\sigma_i} \perp T^{\sigma_j} \) for \( i \neq j \), then \( \sigma = \sum_{j=1}^{\infty} \frac{1}{2^j} \sigma_j \) is an FS measure.
3. Auxiliary Lemmas

Given a compact subset $X \subset \mathbb{R}$, denote by $\mathcal{P}(X)$ the set of all Borel probability measures concentrated on $X$ endowed with the usual weak topology which is compact and metrizable: if $\{f_n : n \geq 1\}$ is a dense set in $C(X)$, then

\begin{equation}
    d(\sigma, \eta) = \frac{1}{\sum_{n=1}^{\infty} \frac{1}{2^n} \left| \int f_n \, d\sigma - \int f_n \, d\eta \right|}
\end{equation}

defines a metric compatible with the weak topology. Denote $\mathcal{U}(X) = \{f \in C(X) : |f| = 1\}$, which is a closed subset of $C(X)$ in the uniform topology; in particular, $\mathcal{U}(X)$ is a Polish space.

**Lemma 3.1.** Assume that $X = [a, b]$. Let $\{h_0, h_1, \ldots, h_m\} \subset \mathbb{R}^n$ be a $\mathcal{Q}$-independent set. Then for each $f \in \mathcal{U} \left( \bigcup_{j=0}^{m} h_j X \right)$ and $\varepsilon > 0$,

\begin{equation}
    A_{f,\varepsilon}(h_1, \ldots, h_m) := \left\{ \sigma \in \mathcal{P}([a, b]) : (\exists \xi \in \mathbb{R}) \|f - \xi\|_{L^2(\mathbb{R}, \frac{1}{m+1} \sum_{j=0}^{m} \sigma_{h_j})} < \varepsilon \right\}
\end{equation}

is open and dense in $\mathcal{P}([a, b])$.

**Proof.** The set $A_{f,\varepsilon}(h_1, \ldots, h_m)$ is clearly open, so we need to show its denseness in $\mathcal{P}(X)$. Since discrete measures with a finite number of atoms form a dense subset of $\mathcal{P}(X)$, we take $\nu = \sum_{s=1}^{N} a_s \delta_{y_s}$ with $y_s \in [a, b]$, $a_s > 0$, $s = 1, \ldots, N$, and $\sum_{s=1}^{N} a_s = 1$, and fix $\delta > 0$. All we need to do is find a subset $\{x_1, \ldots, x_N\} \subset [a, b]$ such that $|x_s - y_s| < \delta$ for $s = 1, \ldots, N$ and such that the set

$$ L := \bigcup_{j=0}^{m} \{h_j x_1, \ldots, h_j x_N\} \text{ is } \mathcal{Q}-\text{independent.} $$

Indeed, in this case, by Kronecker’s theorem, the set $L$ is a finite Kronecker set, so the measure $\frac{1}{m+1} \sum_{j=0}^{m} \left( \sum_{s=1}^{N} a_s \delta_{x_s} \right)_{h_j}$ is Kronecker, whence $\sum_{s=1}^{N} a_s \delta_{x_s}$ belongs to $A_{f,\varepsilon}(h_1, \ldots, h_m)$ and $\delta$-approximates $\nu$. To show that $x_1, \ldots, x_N$ can be selected so that $L$ is $\mathcal{Q}$-independent, consider the algebraic varieties of the form

$$ \left\{ (z_1, \ldots, z_N) \in X^N : \sum_{j=0}^{m} \sum_{s=1}^{N} k_{js} h_j z_s = 0 \right\} $$

for some non-zero integer matrix $(k_{js})$. Since

$$ \sum_{j=0}^{m} \sum_{s=1}^{N} k_{js} h_j z_s = \sum_{s=1}^{N} \left( \sum_{j=0}^{m} k_{js} h_j \right) z_s $$

and $\sum_{j=0}^{m} k_{js} h_j \neq 0$ whenever $(k_{0s}, \ldots, k_{ms}) \neq (0, \ldots, 0)$ (and there are such vectors since the matrix $(k_{js})$ is not zero), each such variety has $N$-dimensional Lebesgue measure zero. Since there are only countably many such varieties involved, we may discard the union $S$ of them from $[a, b]^N \times \mathbb{R}$. Now, each choice of $(x_1, \ldots, x_N)$ from $(y_1 - \delta, y_1 + \delta) \times \ldots \times (y_N - \delta, y_N + \delta) \setminus S$ satisfies our requirements. \qed
Lemma 3.2. Given $H \subset \mathbb{R}^+_*$, a countable subset which is a $\mathbb{Q}$-independent set, the set of continuous (Kronecker) measures $\sigma \in \mathcal{P}([a,b])$ for which the measure

\begin{equation}
\sum_{h \in H} a_h \sigma_h \quad \text{is a Kronecker measure (on \(\mathbb{R}\))}
\end{equation}

for each choice of $a_h \geq 0$, $\sum_{h \in H} a_h = 1$, is a $G_\delta$ and dense subset of $\mathcal{P}([a,b])$.

Proof. Denote by $\mathcal{P}_c([a,b])$ the set of continuous measures which is a $G_\delta$ and dense subset of $\mathcal{P}([a,b])$. Let $H = \{h_0, h_1, h_2, \ldots\}$. For every $m \geq 0$ fix a countable dense family $\{g_{i}^{(m)} : i \geq 1\} \subset \mathcal{U}(\bigcup_{i=0}^{m} h_i[a,b])$. Then, by Lemma 3.1 the set

$$
\mathcal{P}_c([a,b]) \cap \bigcap_{i=1}^{\infty} \bigcap_{p=1}^{\infty} A_{g_{i}^{(m)}}^{\frac{1}{p}}(h_1, \ldots, h_m)
$$

is $G_\delta$ and dense in $\mathcal{P}([a,b])$, and it remains to show that this is precisely the set of measures satisfying (3.3). Indeed, given $m \geq 1$, the set

$$
\mathcal{K}_m(H) := \mathcal{P}_c([a,b]) \cap \bigcap_{i=1}^{\infty} \bigcap_{p=1}^{\infty} A_{g_{i}^{(m)}}^{\frac{1}{p}}(h_1, \ldots, h_m)
$$

is precisely the set of continuous Kronecker measures $\sigma \in \mathcal{P}([a,b])$ such that the measure $\frac{1}{m+1} \sum_{i=0}^{m} \sigma_{h_i}$ is a Kronecker measure (on the real line). Moreover, each measure absolutely continuous with respect to a Kronecker measure is also a Kronecker measure [15]. Therefore the set $\mathcal{K}_m(H)$ is equal to the set of all Kronecker measures $\sigma \in \mathcal{P}([a,b])$ such that $\sum_{i=0}^{m} b_i \sigma_{h_i}$ is Kronecker for an arbitrary choice of $b_i \geq 0$, $\sum_{i=0}^{m} b_i = 1$. Finally, for each $m \geq 1$,

$$
\frac{1}{m+1} \sum_{i=0}^{m} a_{h_i} \sigma_{h_i} \ll \frac{1}{m+1} \sum_{i=0}^{m} \sigma_{h_i},
$$

so if for each $m \geq 1$ the measure $\frac{1}{m+1} \sum_{i=0}^{m} \sigma_{h_i}$ is Kronecker, then so is $\sum_{h \in H} a_h \sigma_h$. \(\mathbb{Q}\)-independent set.

Remark 3.3. The idea of the above proofs is taken from a letter that was sent to us by T.W. Körner. In this letter, Körner shows that given a transcendental number $h \in \mathbb{R}$, for a “typical” (in the Hausdorff metric) closed subset $K \subset [a,b]$ the set $K \cup hK$ is Kronecker and uncountable. The proofs are the same since finite sets are dense in the Hausdorff metric, and if $h$ is transcendental, then given distinct $y_1, \ldots, y_N \in [a,b]$ and $\delta > 0$ we can find $q_i \in \mathbb{Q}$ so that for $x_i := h^{2i} q_i$ we have $|x_i - y_i| < \delta$ for $i = 1, \ldots, N$ and clearly the set $\{x_1, \ldots, x_N, hx_1, \ldots, hx_N\}$ is $\mathbb{Q}$-independent. It only remains to notice that uncountable closed subsets are typical in the Hausdorff metric.

Note also that using the proofs of Lemmas 3.1 and 3.2, given $H \subset \mathbb{R}^+_*$, a countable multiplicative subgroup which is additively $\mathbb{Q}$-independent, we obtain that a typical (with respect to the Hausdorff distance) closed subset $K \subset [a,b]$ has the property that for each finite subset $C \subset H$ the set $\bigcup_{h \in C} hK$ is Kronecker, so the set $\bigcup_{h \in H} hK$ is a $\mathbb{Q}$-independent $F_\sigma$-set.

We will also need the following “compact $\mathbb{Q}$-independent set” version of Lemma 3.2.
**Lemma 3.4.** Assume that $K \subset \mathbb{R}$ is a compact uncountable set. Assume that $H \subset \mathbb{R}_+^*$ is a countable set which is additively $\mathbb{Q}$-independent. Assume moreover that the set $\bigcup_{h \in H} hK$ is $\mathbb{Q}$-independent. Then the set of continuous (Kronecker) measures $\sigma$ concentrated on $K$ for which the measure

\[(3.4) \quad \sum_{h \in H} a_h \sigma_h \text{ is a Kronecker measure}
\]

for each choice of $a_h \geq 0$, $\sum_{h \in H} a_h = 1$, is a $G_\delta$ and dense subset of $\mathcal{P}(K)$.

**Proof.** This follows from the proofs of Lemmas 3.1 and 3.2 where in addition the proof of Lemma 3.1 is simplified. Indeed, for any choice of $\{y_1, \ldots, y_N\} \subset K$ the set $\bigcup_{j=0}^m (h_j y_1, \ldots, h_j y_N)$ is $\mathbb{Q}$-independent by assumption (so we may take $x_i = y_i$ for $i = 1, \ldots, N$). \(\square\)

**Remark 3.5.** For any non-trivial compact interval $[a, b] \subset \mathbb{R}$ denote by $\mathcal{P}_c([a, b])(\mathbb{R})$ the set of measures $\nu \in \mathcal{P}_c(\mathbb{R})$ such that $\nu([a, b]) > 0$. Since the map $\mathcal{P}_c(\mathbb{R}) \ni \nu \mapsto \nu([a, b]) \in \mathbb{R}$ is continuous, the set $\mathcal{P}_c([a, b])(\mathbb{R})$ is open and dense in $\mathcal{P}_c(\mathbb{R})$.

Let us consider the map $\Delta = \Delta([a, b]) : \mathcal{P}_c([a, b])(\mathbb{R}) \to \mathcal{P}_c([a, b])$ such that $\Delta(\nu)$ is the conditional probability measure $\nu(\cdot | [a, b])$. This map is continuous, and the preimage of any dense subset of $\mathcal{P}_c([a, b])$ is dense in $\mathcal{P}_c([a, b])(\mathbb{R})$. Indeed, let $A \subset \mathcal{P}_c([a, b])$ be dense and take any $\nu \in \mathcal{P}_c([a, b])(\mathbb{R})$. Then there exists a sequence $(\tilde{\nu}_n)_{n \leq 1}$ in $A$ such that $\tilde{\nu}_n \to \Delta(\nu)$ weakly. For every $n \geq 1$ define $\nu_n \in \mathcal{P}_c([a, b])(\mathbb{R})$ so that the restriction of $\nu_n$ to $[a, b]$ is $\nu([a, b])\tilde{\nu}_n$ and the measures $\nu_n$ and $\nu$ coincide on $\mathbb{R} \setminus [a, b]$. Then $\Delta(\nu_n) = \tilde{\nu}_n \in A$ and $\nu_n \to \nu$ weakly. Consequently, the preimage $\Delta^{-1}A$ of any $G_\delta$ dense subset $A \subset \mathcal{P}_c([a, b])$ is $G_\delta$ dense in $\mathcal{P}_c([a, b])(\mathbb{R})$.

Before we prove a certain disjointness property of Kronecker measures, we will need the following general observation.

**Lemma 3.6.** Let $(X, \mathcal{B})$ be a measurable space and let $\varphi : X \to X$ be a measurable map. Let $\sigma$ be a finite positive continuous Borel measure on $X$ such that the map $\varphi : (X, \sigma) \to (X, \varphi_* \sigma)$ is almost everywhere invertible. Assume that $\sigma(\{x \in X : \varphi(x) = x\}) = 0$ and that the measures $\sigma$ and $\varphi_* \sigma$ are not mutually singular. Then there exists a measurable set $A \in \mathcal{B}$ such that $\sigma(A) > 0$, $\sigma(A \cap \varphi^{-1}A) = 0$ and the measures $\sigma$ and $\varphi_* \sigma$ restricted to $A$ are equivalent.

**Proof.** By assumption, there exists $Y \in \mathcal{B}$ such that $\sigma(Y) > 0$ and the measures $\sigma$ and $\varphi_* \sigma$ restricted to $Y$ are equivalent. It follows that if $A \in \mathcal{B}$, $A \subset Y$, $\sigma(A) > 0$, then the measures $\sigma$ and $\varphi_* \sigma$ restricted to $A$ are also equivalent.

**Case 1.** Suppose that there exists $B \in \mathcal{B}$ such that $B \subset Y$ and $\sigma(B \setminus \varphi(B)) > 0$. Set $A := B \setminus \varphi(B)$. Then $\sigma(A) > 0$ and $A \cap \varphi^{-1}A = (B \setminus \varphi(B)) \cap (\varphi^{-1}B \setminus B) = 0$. Since $A \subset B \subset Y$, our claim follows.

**Case 2.** Suppose that for every $B \in \mathcal{B}$ with $B \subset Y$ we have $\sigma(B \setminus \varphi(B)) = 0$. As $\sigma$ and $\varphi_* \sigma$ restricted to $Y$ are equivalent, it follows that

\[(3.5) \quad 0 = \varphi_* \sigma(B \setminus \varphi(B)) = \sigma(\varphi^{-1}B \setminus B) \quad \text{for every } B \subset Y.
\]

We now show that there exists $A \in \mathcal{B}$ such that $A \subset Y$, $\sigma(A) > 0$ and $\sigma(A \cap \varphi^{-1}A) = 0$, which gives our assertion. Suppose that, contrary to our claim, for every $A \in \mathcal{B}$ with $A \subset Y$ the condition $\sigma(A) > 0$ implies $\sigma(A \cap \varphi^{-1}A) > 0$. It follows that

\[(3.6) \quad \sigma(B \setminus \varphi^{-1}B) = 0 \quad \text{for every measurable } B \in \mathcal{B} \text{ with } B \subset Y.
\]
Otherwise for some \( B \) as above, \( A := B \setminus \varphi^{-1}(B) \subset Y \) would be of positive \( \sigma \)-measure, and since

\[
(B \setminus \varphi^{-1}B) \cap \varphi^{-1}(B \setminus \varphi^{-1}B) = (B \setminus \varphi^{-1}B) \cap (\varphi^{-1}B \setminus \varphi^{-2}B) = \emptyset,
\]

we would get \( \sigma(A \cap \varphi^{-1}A) = 0 \), a contradiction.

Now, (B.5) combined with (B.6) gives \( \sigma(B \triangle \varphi^{-1}B) = 0 \) for every \( B \in \mathcal{B} \) with \( B \subset Y \). It follows that \( \varphi(x) = x \) for \( \sigma \)-a.e. \( x \in Y \), contrary to our assumption. \( \square \)

For any real \( s \) let \( \theta_s : \mathbb{R} \to \mathbb{R}, \theta_s(t) = t + s \). Recall that for every \( n \in \mathbb{Z} \) and \( z_1, z_2 \in S^1 \) we have

\[
|z_1^n - z_2^n| \leq |n||z_1 - z_2|.
\]

**Lemma 3.7.** Let \( \sigma \) be a continuous Kronecker measure on \( \mathbb{R} \). Then for every \( s \in \mathbb{Q}^* \setminus \{1\} \) and \( r \in \mathbb{R} \) we have \( \sigma \perp \sigma_x \ast \delta_r \).

**Proof.** Suppose that, contrary to our claim, there exist \( s \in \mathbb{Q}^* \setminus \{1\} \) and \( r \in \mathbb{R} \) such that \( \sigma \not\perp \sigma_x \ast \delta_r \). Let \( \varphi := \theta_x \circ R_r \). Then \( \varphi : \mathbb{R} \to \mathbb{R} \) is an invertible map with one fixed point and \( \sigma_x \ast \delta_r \equiv \varphi_x \sigma \). By Lemma 3.6 there exists a Borel set \( A_0 \subset \mathbb{R} \) such that \( \sigma(A_0) > 0 \), \( \sigma(A_0 \setminus \varphi^{-1}A_0) = 0 \) and the measures \( \sigma_x \ast \delta_r \) restricted to \( A_0 \) are equivalent. Thus \( \sigma_x \varphi^{-1}A_0 > 0 \). Let \( A_1, A_2 \subset A_0 \) be disjoint Borel subsets such that \( \sigma_x \varphi^{-1}A_1 > 0 \) and \( \sigma_x \varphi^{-1}A_2 > 0 \).

Let \( s = q/p \) with \( p \) and \( q \) relatively prime integer numbers. Choose \( z_0 \in S^1 \) such that \( z_0^q \neq 1 \). Let us consider the measurable map \( f : \mathbb{R} \to S^1 \) such that \( f(x) = z_0 \) if \( x \in \varphi^{-1}A_2 \) and \( f(x) = 1 \) otherwise. Since \( \sigma \) is a Kronecker measure, there exists a sequence \( (t_n)_{n \in \mathbb{N}} \) of real numbers such that \( \xi_{t_n} \to f \) in \( L^2(\mathbb{R}, \sigma) \). Thus \( \xi_{t_n} \circ \varphi^{-1} \to f \circ \varphi^{-1} \) in \( L^2(\mathbb{R}, \varphi_x \sigma) \). Since

\[
\begin{align*}
g_0^1(x) &:= \chi_{A_0}(x)|\exp(2\pi it_n x) - 1| \leq |\xi_{t_n}(x) - f(x)|, \\
g_0^2(x) &:= \chi_{A_1}(x)|\exp(2\pi it_n s^{-1}(x - r)) - 1| \leq |\xi_{t_n}(\varphi^{-1}x) - f(\varphi^{-1}x)|, \\
g_n^1(x) &:= \chi_{A_2}(x)|\exp(2\pi it_n s^{-1}(x - r)) - z_0| \leq |\xi_{t_n}(\varphi^{-1}x) - f(\varphi^{-1}x)|,
\end{align*}
\]

it follows that \((g_0^0)\) tends to zero in measure \( \sigma \) and the sequences \((g_0^1), (g_0^2)\) tend to zero in measure \( \varphi_x \sigma \). As \( \sigma_x \equiv \varphi_x \sigma \) on \( A_0 \) and \( A_1, A_2 \subset A_0 \), the sequences \((g_n^1), (g_n^2)\) tend to zero in measure \( \sigma \) as well. Fix

\[
0 < \varepsilon < \frac{|z_0^q - 1|}{2(|p| + |q|)}.
\]

Then there exist measurable sets \( A_k' \subset A_k \), \( k = 0, 1, 2 \), and \( n \in \mathbb{N} \) such that for \( k = 0, 1, 2 \),

\[
\sigma(A_k \setminus A_k') < \frac{1}{4} \min(\sigma(A_1), \sigma(A_2)) \text{ and } g_n^k(x) < \varepsilon \text{ for all } x \in A_k'.
\]

Therefore for \( k = 1, 2 \) we have

\[
\sigma(A_k \setminus A_0') \leq \sigma(A_0 \setminus A_0') < \frac{1}{4} \sigma(A_k) \text{ and } \sigma(A_k \setminus A_k') < \frac{1}{4} \sigma(A_k),
\]
so \( \sigma(A_0 \cap A'_k) > \sigma(A_k)/2 > 0 \). Choose two real numbers \( x_1 \in A_0 \cap A'_1 \) and \( x_2 \in A_0 \cap A_2 \). Then

\[
\begin{align*}
|\exp(2\pi it_n x_1) - 1| &= g_n^0(x_1) < \varepsilon, \\
|\exp(2\pi it_n x_2) - 1| &= g_n^0(x_2) < \varepsilon,
\end{align*}
\]

In view of (3.7),

\[
\begin{align*}
|\exp(2\pi it_n p x_1) - 1| &< |p|\varepsilon, \\
|\exp(2\pi it_n p x_2) - 1| &< |p|\varepsilon,
\end{align*}
\]

Hence

\[
|\exp(2\pi it_n pr) - 1| < (|p| + |q|)\varepsilon, \\
|\exp(2\pi it_n pr) z_0^q - 1| < (|p| + |q|)\varepsilon,
\]

so

\[
1 - z_0^q < 2(|p| + |q|)\varepsilon,
\]

contrary to (3.8). □

Let us now consider the space \( \mathcal{P}(\mathbb{R}) \) of all Borel probability measures on \( \mathbb{R} \) endowed with the weak topology.

By \( \text{supp}(\sigma) \) we always mean the topological support of the measure \( \sigma \). Let us recall that

\[
\text{if } \sigma \in \mathcal{P}(\mathbb{R}) \text{ has } \text{supp}(\sigma) = \mathbb{R}, \text{ then the set } \{\nu \in \mathcal{P}(\mathbb{R}) : \nu \ll \sigma\} \text{ is dense in } \mathcal{P}(\mathbb{R}).
\]

Denote by \( \mathcal{P}_c(\mathbb{R}) \) the set of all continuous members of \( \mathcal{P}(\mathbb{R}) \) (this is a \( G_\delta \) and dense subset of \( \mathcal{P}(\mathbb{R}) \)).

The proof of the lemma below is a slight modification of the proof of Lemma 3.1 from [4].

**Lemma 3.8.** The set

\[
S = \{\sigma \in \mathcal{P}_c(\mathbb{R}) : \sigma \perp \sigma * \delta_t \text{ for each } 1 \neq s \in \mathbb{R}^*, t \in \mathbb{R}\}
\]

is \( G_\delta \) and dense in \( \mathcal{P}(\mathbb{R}) \).

**Proof.** Denote by \( \mathcal{I} \) the family of open subset of \( \mathbb{R} \) which are finite unions of open intervals. Recall that for two measures \( \sigma, \nu \in \mathcal{P}(\mathbb{R}), \)

\[
\sigma \perp \nu \iff \forall n \in \mathbb{N} \exists O \in \mathcal{I} \sigma(O) < 1/n \text{ and } \nu(O) > 1 - 1/n.
\]

For any compact rectangle \( I \times J \subset (\mathbb{R}^* \setminus \{1\}) \times \mathbb{R} \), denote by \( \mathcal{V}(I \times J) \) the set of all finite covers of \( I \times J \) by compact rectangles contained in \( (\mathbb{R}^* \setminus \{1\}) \times \mathbb{R} \). Notice that for each open subset \( O \in \mathcal{I}, \) the map

\[
\mathcal{P}_c(\mathbb{R}) \times \mathbb{R}^* \times \mathbb{R} \ni (\sigma, s, r) \mapsto \sigma * \delta_r(O) \in \mathbb{R}
\]

is continuous. Therefore, given a compact rectangle \( F \subset (\mathbb{R}^* \setminus \{1\}) \times \mathbb{R} \) and an open subset \( O \in \mathcal{I}, \) the map

\[
f_{F,O} : \mathcal{P}_c(\mathbb{R}) \ni \sigma \mapsto \left( \sigma(O), \max_{(s,r) \in F} \sigma(s, r) * \delta_r(O) \right) \in \mathbb{R}^2
\]
is continuous. Let
\[
\tilde{S} = \bigcap_{I \neq J} \bigcap_{n \in \mathbb{N}} \bigcup_{\kappa \in \mathcal{V}(I \times J)} \bigcap_{F \in \kappa} \bigcap_{O \in \mathcal{I}} f_{F,O}^{-1} \left( (1 - 1/n, \infty) \times (-\infty, 1/n) \right),
\]
where \( I \) and \( J \) run over closed intervals with rational endpoints. Then \( \tilde{S} \) is a \( G_\delta \) set.

We claim that \( \tilde{S} = S \). Indeed, let \( \sigma \in S \). Let \( I \neq J \) and \( J \subset \mathbb{R} \) be compact intervals and \( n \in \mathbb{N} \). By assumption and \( (3.10) \), for every \( (s_0, r_0) \in I \times J \) there exists an open set \( O_{s_0, r_0} \in \mathcal{I} \) such that
\[
\sigma(O_{s_0, r_0}) > 1 - 1/n \quad \text{and} \quad \sigma_{s_0} * \delta_{r_0}(O_{s_0, r_0}) < 1/n.
\]
Since the map \( (3.11) \) is continuous, there exist open rectangles \( U'_{s_0, r_0} \subset U_{s_0, r_0} \subset \mathbb{R}^2 \) such that \( (s_0, r_0) \in U'_{s_0, r_0} \) and a compact rectangle \( F_{s_0, r_0} \subset (\mathbb{R}^* \setminus \{1\}) \times \mathbb{R} \) satisfying \( U'_{s_0, r_0} \subset F_{s_0, r_0} \subset U_{s_0, r_0} \) such that
\[
\sigma_{s_0} * \delta_{r_0}(O_{s_0, r_0}) < 1/n \quad \text{for all} \quad (s, r) \in U_{s_0, r_0}.
\]
Thus \( \sigma \in \tilde{S} \).

Suppose that \( \sigma \in \tilde{S} \) and fix \( s_0 \in \mathbb{R}^* \setminus \{1\} \), \( r_0 \in \mathbb{R} \) and \( n \in \mathbb{N} \). Next choose \( I \neq J \) and \( J \subset \mathbb{R} \) compact intervals such that \( (s_0, r_0) \in I \times J \). By assumption, there exists a finite cover \( \kappa := \{ F_{s_1, r_1}, \ldots, F_{s_k, r_k} \} \) of \( I \times J \). It follows that
\[
f_{F_{s_j, r_j}, O_{s_j, r_j}}(\sigma) \in (1 - 1/n, \infty) \times (-\infty, 1/n) \quad \text{for all} \quad j = 1, \ldots, k,
\]
thus \( \sigma \in \tilde{S} \).

Choosing \( F \in \kappa \) for which \( (s_0, r_0) \in F \) and applying \( (3.10) \) we have that \( \sigma \) and \( \sigma_{s_0} * \delta_{r_0} \) are orthogonal, so \( \sigma \in S \).

It remains to show that \( S \) is dense. To this end we use the proof of Proposition 3.4 in [4]. Namely, in this proposition there is a construction of a weakly mixing flow \( T \) such that for a certain sequence of real numbers \( u_k \to \infty \) we have: for each \( l \in \mathbb{N} \),
\[
\xi_{u_k} \to 10^{-l} \quad \text{for} \quad d = 1 - 10^{-l} \quad \text{and}
\]
\[
\xi_{u_k} \to 0 \quad \text{uniformly in} \quad c \in [1, 10^l]
\]
(the convergence takes place in the weak operator topology). It follows that
\[
\sigma_{T_d} \perp \sigma_{T_c} * \delta_l
\]
for all \( t \in \mathbb{R} \); indeed, \( (3.12) \) and \( (3.13) \) mean respectively
\[
\xi_{u_k} \to 10^{-l} \quad \text{weakly in} \quad L^2(\mathbb{R}, \sigma_{T_d})
\]
and
\[
\xi_{u_k} \to 0 \quad \text{weakly in} \quad L^2(\mathbb{R}, \sigma_{T_c} * \delta_l)
\]
It is easy to see that the latter condition implies
\[
\xi_{u_k} \to 0 \quad \text{weakly in} \quad L^2(\mathbb{R}, \sigma_{T_c} * \delta_l)
\]
for each \( t \in \mathbb{R} \), and the mutual singularity \( (3.14) \) follows.
Now, in view of (3.14), \( \sigma_T \perp \sigma_{T_{i/d}} \ast \delta_{t/d} \), and since in (3.13) \( c \) can be replaced by \(-c\), it follows that \( \sigma_T \in \mathcal{S} \). It is also clear that \( \mathcal{S} \) is closed under taking absolutely continuous measures. Since \( \text{supp} \sigma_T = \mathbb{R}^8 \) the result follows from (3.9).

Also recall the following basic observation.

**Lemma 3.9.** Let \( \mathcal{I} = (s_j)_{j \geq 1} \) be a sequence of positive numbers and let \( \mathcal{J} = (g_j)_{j \geq 1} \) be a uniformly bounded sequence of continuous functions. Then the set

\[
\mathcal{W}_{\mathcal{I}, \mathcal{J}} = \{ \nu \in \mathcal{P}(\mathbb{R}) : (\exists t_n \to \infty) (\forall j \geq 1) \quad \xi_{s_j t_n} \to g_j \text{ weakly in } L^2(\mathbb{R}, \nu) \}
\]

is \( \mathcal{G}_\delta \) in \( \mathcal{P}(\mathbb{R}) \).

**Proof.** Let \((f_m)_{m \geq 1}\) be a sequence of continuous functions on \( \mathbb{R} \) uniformly bounded by 1, which is linearly dense in \( L^2(\mathbb{R}, \nu) \) for every \( \nu \in \mathcal{P}(\mathbb{R}) \). Set

\[
\mathcal{R}(n, \varepsilon) = \left\{ \mu \in \mathcal{P}(\mathbb{R}) : \sum_{m,j \geq 1} \frac{1}{2m+j} \left| \int_{\mathbb{R}} \left( e^{2\pi is_j nx} - g_j(x) \right) f_m(x) \, d\mu(x) \right| < \varepsilon \right\}.
\]

The set \( \mathcal{R}(n, \varepsilon) \) is open. To complete the proof it suffices to notice that

\[
\mathcal{W}_{\mathcal{I}, \mathcal{J}} = \bigcap_{Q \ni \varepsilon > 0} \bigcap_{m \geq 1} \bigcap_{n \geq m} \mathcal{R}(n, \varepsilon). \tag{3.15}
\]

**Lemma 3.10.** Let \( H \subset \mathbb{R}^+ \) be a countable multiplicative subgroup. Then for a typical \( \nu \in \mathcal{P}(\mathbb{R}) \) the measure \( \eta := \sum_{h \in H} a_h \nu_h \) (with \( a_h > 0 \) and \( \sum_{h \in H} a_h = 1 \)) yields a Gaussian flow \( \mathcal{T}^\eta \) with simple spectrum.

**Proof.** Set \( G = -H \cup H \) and let \( H = \{ s_i : i \geq 0 \} \) (\( s_0 = 1 \)). In [4], Danilenko and Ryzhikov constructed a rank-1 flow \( \mathcal{T} \) preserving a \( \sigma \)-finite measure \( \mu \) (the flow acts on \( (X, \mathcal{B}, \mu) \)) such that if \( \sigma = \sigma_T \) denotes its maximal spectral type on \( L^2(X, \mathcal{B}, \mu) \), then the Gaussian flow

\[
\mathcal{T} \left( \sum_{i \geq 1} \frac{1}{a_i} \sigma_i \right) \text{ has simple spectrum.}
\]

To prove this, they used the following properties of \( \mathcal{T} \):

a) \( T_{\sqrt{2}s} \in \text{WCP}(T_s) \) for each \( s \in H \),

b) \( \frac{1}{q} I + \frac{q-1}{q} T_s \in \text{WCP}(T_q) \) for each \( s \in H \) and \( q \in \mathbb{N} \),

c) for each finite sequence \( s_1 < s_2 < \cdots < s_k \) of elements of \( H \) and each \( 1 \leq l_0 \leq k \) there exists \( t_j \to \infty \) such that

(i) \( T_{t_j s_j} \to \frac{1}{2\pi} I \) if \( 1 \leq l \leq k, l \neq l_0 \),

(ii) \( T_{t_j s_{l_0}} \to \frac{1}{2\pi} T_{s_{l_0}} \).

Notice that conditions a), b) and c) can be expressed as follows in terms of weak convergence of continuous and bounded functions in \( L^2(\mathbb{R}, \sigma) \):

a’) for each \( s \in H \) there exists a sequence \( n_k \to \infty \) such that

\[
\xi_{sn_k} \to \xi_{\sqrt{2}s},
\]

b’) for each \( s \in H \) and \( q \in \mathbb{N} \) there exists a sequence \( n_k \to \infty \) such that

\[
\xi_{sn_k} \to \frac{1}{q} + \frac{q-1}{q} \xi_s,
\]

\footnote{This fact is well known for \( \mathbb{Z} \)-actions, e.g. [20], Chapter 3, and can be easily rewritten using special representation of flows. See also the proof of Theorem A in [24].}

\footnote{An operator \( Q \) belongs to the weak closure of powers \( \text{WCP}(R) \) if for an increasing sequence \( (m_j) \) of integers, \( R^{m_j} \to P \) in the weak operator topology.}
c') for each finite sequence \( s_1 < s_2 < \cdots < s_k \) of elements of \( H \) and each \( 1 \leq l_0 \leq k \) there exists \( t_j \to \infty \) such that

(i) \( \xi_{t_j s_j} \to \frac{1}{2k} \) if \( 1 \leq l \leq k, l \neq l_0 \),

(ii) \( \xi_{t_j s_{l_0}} \to \frac{1}{2k} \xi_{s_{l_0}} \).

The arguments used in the proof of Theorem 4.4 in [4] show that for each continuous probability measure \( \sigma \) on \( \mathbb{R} \) conditions a'), b') and c') imply the simplicity of spectrum of the flow \( T^{(\sum_{k \geq 1} \frac{1}{2k} \sigma_k)_L} \). Moreover, by Lemma 3.9 the set of measures \( \nu \in \mathcal{P}(\mathbb{R}) \) satisfying these conditions is \( G_\delta \). We will now show that it is also dense in \( \mathcal{P}(\mathbb{R}) \). Notice that conditions a'), b') and c') also hold in \( L^2(\mathbb{R}, \nu) \) for any \( \nu \ll \sigma \).

Since \( \sigma_T \) is the maximal spectral type of a rank-1 infinite measure-preserving flow \( T \), the Gelfand spectrum of the corresponding Koopman representation is equal to \( \mathbb{R} \). It follows that the topological support of \( \sigma_T \) is full, and therefore the result follows from (3.9).

\[ \Box \]

4. Proofs of theorems

Proof of Theorem 1.3 (based on Lemmas 3.2 and 3.8). Using these two lemmas, for a “typical” (continuous, Kronecker) measure \( \sigma \in \mathcal{P}([a, b]) \) we have (with \( a_h > 0 \), and \( \sum_{h \in H} a_h = 1 \))

\[ -H \cup H \subset I(T^n), \]

where \( \eta := \sum_{h \in H} a_h \sigma_h \) is a Kronecker measure and, moreover,

\[ \sigma_s \perp \sigma \ast \delta_t \]

for each non-zero real \( s \neq 1 \) and arbitrary \( t \in \mathbb{R} \). All we need to show is that when \( s \not\in -H \cup H \), then \( \eta_s \neq \eta \). However if \( s \not\in H \), then even more is true: \( \eta \perp \eta_s \ast \delta_t \) for an arbitrary \( t \in \mathbb{R} \) and \( s \not\in \{0, 1\} \). It follows that

\[ \overline{\eta}_s \perp \overline{\eta} \ast \delta_t \]

for each \( s \not\in -H \cup H \) and \( t \in \mathbb{R} \). In view of Theorem 2.3 it follows that \( T^n \) is disjoint from \( T^n_s \) (isomorphic to \( T^n_1 \)) for \( s \not\in -H \cup H \) (cf. footnote 2). In particular, \( -H \cup H = I(T^n) \) and the result follows.

\[ \Box \]

Proof of Theorem 1.4 (based on Lemma 3.4). Given \( H \subset \mathbb{R}_+^* \) a multiplicative subgroup which is an additively \( \mathbb{Q} \)-independent set, in [9] there is a construction of a perfect compact set \( K \) such that \( \hat{K} := \bigcup_{h \in H} hK \) is independent and for \( \hat{K} := -\hat{K} \cup \hat{K} \) the following holds: \( (r\hat{K} + t) \cap \hat{K} \) is countable whenever \( |r| \not\in H \) and \( t \in \mathbb{R} \) is arbitrary. Using Lemma 3.4 find a (continuous, Kronecker) measure \( \sigma \in \mathcal{P}(K) \) such that \( \eta := \sum_{h \in H} a_h \sigma_h \) is a Kronecker measure. Then \( \eta \) is concentrated on \( \hat{K} \). All we need to show is that if \( |r| \not\in H \), then the symmetrization of \( \eta_r \) is not equivalent to the symmetrization of \( \eta \). This is, however, clear, since the symmetrization of \( \eta_r \) is a continuous measure concentrated on \( r\hat{K} \). As in the previous proof we deduce that for \( s \not\in -H \cup H \) we obtain disjointness of the corresponding flows.

\[ \Box \]

Proof of Theorem 1.4 First notice that directly from Lemma 3.7 it follows that whenever \( \sigma \) is a Kronecker measure, then for each \( r_1, r_2 \in \mathbb{Q}^* \), \( r_1 \neq r_2 \), we have

\[ \sigma_{r_1} \perp \sigma_{r_2} \ast \delta_t \]

for each \( t \in \mathbb{R} \).
It follows that \( \bar{\sigma}_r \perp \bar{\sigma}_s * \delta_t \) for all \( t \in \mathbb{R} \), so by Theorem 2.3, the Gaussian-Kronecker flows \( T^\sigma_\tau \) and \( T^\sigma_\tau \) are disjoint. In view of (2.1), it follows that \( T^\sigma_\tau \perp T^\sigma_\tau \), thus \( T^\sigma_\tau \perp T^\sigma_\tau \).

Now suppose that \( T = T_\sigma : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) is a Gaussian-Kronecker automorphism, i.e. \( \sigma = \sigma_0 + \sigma_0 \) for a continuous Kronecker measure \( \sigma_0 \in \mathcal{P}(\mathbb{T}) \).

Denote by \( \sigma' \) the image of \( \sigma_0 \) via the map \( \mathbb{T} \ni z \mapsto \text{Arg}(z)/2\pi \in [0,1) \). Then \( \sigma' \) is a continuous Kronecker measure on \( \mathbb{R} \) such that \( (\xi_1)_* \sigma' = \sigma \) and \( \sigma' \perp \delta_m \perp \sigma' \) for all \( m \in \mathbb{N} \). Denote by \( \mathcal{H} \) the Gaussian space of the flow \( T^\sigma \). Then the Koopman operator of \( T^\sigma_1 \) has simple spectrum on \( \mathcal{H} \) and its spectral type is \( (\xi_1)_* \sigma' = \sigma \); see Appendix in [16]. Since the spectral type of \( \xi_1 \) (with respect to \( T^\sigma_1 \)) is isomorphic to \( \sigma \), it follows that \( \xi_1 \circ (T_1^\sigma)^n, n \in \mathbb{Z}, \) span the space \( \mathcal{H} \). Thus \( T_\sigma^\sigma \) is isomorphic to \( T_\sigma \).

By the first assertion of the theorem, it follows that \( T^\sigma_\tau \) is disjoint from \( T^\sigma_\tau \) for any pair of distinct natural numbers.

In order to prove the second part of the theorem note that if \( s \) is irrational, then the set \( \{1, s\} \) is \( \mathbb{Q} \)-independent, so by Lemma 3.2 we can find a (continuous, Kronecker) measure \( \sigma \in \mathcal{P}([a, b]) \) such that \( \eta := \frac{1}{2} (\sigma + \sigma_s) \) is a Kronecker measure. Since \( \sigma_s \ll \eta \) and \( \sigma_s \ll \eta_s \) the Gaussian-Kronecker flows \( T^\eta \) and \( T^\eta \) have a common non-trivial (Gaussian) factor. Its time one map is a common non-trivial factor of \( T^\eta \) and \( T^\eta \), and it remains to notice that the Gaussian automorphism \( T^\eta \) is isomorphic to \( T^\eta \).

**Proof of Theorem 1.5** Let \( H = G \cap \mathbb{R}^*_+ \) and let \( (a_h)_{h \in H} \) be positive numbers such that \( \sum_{h \in H} a_h = 1 \). By Lemmas 3.8 3.10 and Lemma 3.2 (applied to \( H = \{1\} \)) combined with Remark 3.5 there exists \( \nu' \in \mathcal{P}_c(\mathbb{R}) \) such that

1. \( \nu'_s \perp \nu' * \delta_h \) for all \( s \in \mathbb{R}^* \setminus \{1\} \) and \( t \in \mathbb{R} \),
2. the Gaussian flow \( T^{(\sum_{h \in H} a_h \nu'_s)} \) has simple spectrum,
3. \( \nu := \Delta(\nu') \in \mathcal{P}_c([a, b]) \) is a Kronecker measure

(it is, in fact, for a “typical” \( \nu' \in \mathcal{P}_c(\mathbb{R}) \) the properties (i)-(iii) hold). Since conditions (i) and (ii) also hold for any measure absolutely continuous with respect to \( \nu \), the measure \( \nu' \) satisfies (i) and (ii) as well. Therefore, setting \( \sigma := \sum_{h \in H} a_h \nu_s \), by (ii), the Gaussian flow \( T^\sigma \) has simple spectrum. The same argument as in the proof of Theorem 1.3 shows that (i) together with (ii) implies that \( I(T^\sigma) = -H \cup H \) and \( T^{\nu_s} \perp T^{\nu_r} \) whenever \( |r| \neq |s| \). Each Kronecker measure \( \nu_h, h \in H \), is an FS measure, so, by Corollary 2.5 it follows that \( \sigma = \sum_{h \in H} a_h \nu_s \) is an FS measure, which completes the proof.

**Proof of Theorem 1.6** The first part follows from Lemma 3.8 along the same lines as the first proof of Theorem 1.3 (for \( H = \{1\} \)).

In view of Corollary 2 in [16], a typical flow \( T \) has the SC property, which is equivalent to the fact that \( T^{\sigma_T} \) has simple spectrum. In particular, it implies that \( T^{\sigma_T} \) is GAG.

In order to prove that \( \sigma_T \perp (\sigma_T)_s * \delta_r \), \( s \in \mathbb{R}^* \setminus \{1\}, r \in \mathbb{R} \), for a typical flow \( T \), we follow the proof of Theorem 3.2 from [4] (using Lemma 3.8 and the existence of a flow satisfying (3.14)). Since \( T^{\sigma_T} \) is GAG for a typical flow \( T \), by Theorem 2.3 it follows that \( T^{(\sigma_T)_s} \) and \( T^{(\sigma_T)_r} \) are disjoint wherever \( |r| \neq |s| \).

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10\( \mathbb{T} \) stands for \{ \( z \in \mathbb{C} \mid |z| = 1 \) \}.

11The SC property means that if we set \( \sigma = \sigma_T \), then for each \( n \geq 2 \) the conditional measures of the disintegration of \( \sigma^{\otimes n} \) over \( \sigma^{\ast n} \) via the map \( \mathbb{R}^\ast \ni (x_1, \ldots, x_n) \mapsto x_1 + \cdots + x_n \in \mathbb{R} \) are purely atomic with \( n! \) atoms.
**Question.** Is there a Kronecker measure $\sigma \in \mathcal{P}(\mathbb{R}^+)$ such that $I(\mathcal{T}^\sigma)$ is uncountable?

This question is to be compared with Ryzhikov’s question as to whether there is a weakly mixing, non-mixing flow with an uncountable group of self-similarities; see [3], Problem (1).


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