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A GENERAL STABILITY THEOREM WITH APPLICATIONS

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Abstract. The author presents a generalization of recent stability theorems. Polynomials whose coefficients are successive derivatives of a class of orthogonal functions evaluated at $x = c$, where c is a constant, are shown to fit in this general framework. Special reference is made to the ones related to the classical orthogonal polynomials. Related families of polynomials with real negative roots are also introduced.

1. Introduction

In two recent papers [2, 3], new properties of a class of Jacobi polynomials [7] and generalized Laguerre polynomials [7] were presented. As a result, new classes of stable polynomials and polynomials with real negative roots were introduced. The purpose of this paper is to provide a general stability theorem and to show how the previous results fit in this general framework. A first attempt for this generalization was also made in [1].

Stability analysis plays an important role in many areas of pure and applied mathematics such as the Routh-Hurwitz problem [13], geometry of polynomials [8], matrix analysis [12], operators preserving stability [6] and numerical analysis [4, 5, 10, 11].

A real polynomial, $p(z)$, is said to be *stable* (or a *Hurwitz polynomial*) if all the zeros of $p(z)$ lie in the open left half-plane, $\mathrm{Re}(z) < 0$. The characterization of stable polynomials that is most useful for the purpose of this paper is given by the *Hermite-Biehler Theorem* [11] in terms of positive pairs, defined below.

Definition 1. Two real polynomials $\Omega_1(z)$ and $\Omega_2(z)$ of degree n and $n-1$ (or n), respectively, form a *positive pair* if: (a) the zeros $z_1, \ldots, z_n$ of Ω_1 and $z'_1, \ldots, z'_{n-1}$ (or $z'_1, \ldots, z'_n$) of Ω_2 are real, negative and distinct; (b) the zeros strictly interlace (or alternate) as follows: $z_1 < z'_1 < \cdots < z'_{n-1} < z_n < 0$ (or $z'_1 < z_1 < \cdots < z'_n < z_n < 0$); and (c) the highest coefficients of $\Omega_1(z)$ and $\Omega_2(z)$ are of like sign.

Theorem 2 (Hermite-Biehler Theorem). *A real polynomial $p(z) = \Omega_1(z^2) + z\Omega_2(z^2)$ is stable if and only if $\Omega_1(z)$ and $\Omega_2(z)$ form a positive pair.*

An outline of this paper is as follows: In Section 2, a general stability theorem is introduced (Theorem 3). Next, it is shown that the theorem is valid for a family

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of generalized Laguerre and Jacobi polynomials. In Section 3, related families of polynomials that have only real negative zeros are proved with the aid of the Hermite-Biehler Theorem (Theorem 6).

2. New classes of stable polynomials

Below, a general stability theorem is presented which is the main result of this paper.

Theorem 3. *Let $(a,b) \subseteq \mathbb{R}$, z a complex number, $z \neq 0$, and $W : (a,b) \to \mathbb{R}_{\geq 0}$ a differentiable weight function. Also, let $f(x;z) : (a,b) \to \mathbb{C}$, where $f, \frac{df}{dx} \in L^2_W := \left\{ g : \int_a^b |g|^2 W(x)dx < \infty \right\}$ and $f(x;z)$ is a solution to*

$$f(x;z) - z\frac{df(x;z)}{dx} = q(x;z), \tag{1}$$

where $q(x;z) : (a,b) \to \mathbb{C}$. Further, suppose that

$$\int_a^b \operatorname{Re}\left(zf(x;z)\overline{q(x;z)} \right) W(x)dx = -\nu \operatorname{Re}(z) - \delta \tag{2}$$

for some $\nu \geq 0$, where $\overline{q}$ denotes the complex conjugate of q.

(a) If $W(x)$ is non-decreasing, $\delta \geq 0$ and

$$\lim_{x\to a^+} \{|f(x;z)|^2 W(x)\} \geq \lim_{x\to b^-} \{|f(x;z)|^2 W(x)\},$$

then $\operatorname{Re}(z) < 0$, while

(b) if $W(x)$ is non-increasing, $\delta \leq 0$ and

$$\lim_{x\to a^+} \{|f(x;z)|^2 W(x)\} \leq \lim_{x\to b^-} \{|f(x;z)|^2 W(x)\},$$

then $\operatorname{Re}(z) > 0$.

Proof. Rearrange equation (1) in the form

$$\frac{1}{z}(f(x;z) - q(x;z)) = \frac{df(x;z)}{dx}. \tag{3}$$

For simplicity let f and q denote $f(x;z)$ and $q(x;z)$ and $\overline{f}$ and $\overline{q}$ denote their complex conjugates. Multiply equation (3) by $W(x)\overline{f}$ and add to this its complex conjugate to obtain

$$\left(\frac{1}{z} + \frac{1}{\overline{z}}\right)|f|^2 W(x) - \left(\frac{\overline{f}q}{z} + \frac{f\overline{q}}{\overline{z}}\right) W(x) = \frac{d|f|^2}{dx} W(x). \tag{4}$$

Next, integrate equation (4) over (a,b) to get

$$\left(\frac{1}{z} + \frac{1}{\overline{z}}\right)\int_a^b |f|^2 W(x)dx - \int_a^b \left(\frac{\overline{f}q}{z} + \frac{f\overline{q}}{\overline{z}}\right) W(x)dx \;=\; \int_a^b \frac{d|f|^2}{dx} W(x)dx. \tag{5}$$

The conditions of Theorem 3 along with an integration by parts argument on the right hand side of equation (5) yields

$$\frac{2\operatorname{Re}(z)}{|z|^2}\left(\int_a^b |f|^2 W(x)dx + \nu\right) = \lim_{x\to b^-}\{|f|^2 W(x)\} - \lim_{x\to a^+}\{|f|^2 W(x)\} - \int_a^b |f|^2 \frac{dW}{dx}dx - \frac{2\delta}{|z|^2}. \tag{6}$$

Since $\nu \geq 0$, the bracket on the left hand side of equation (6) is positive. Hence, hypothesis *(a)* of the theorem asserts that the right hand side of equation (6) is negative and thus $\mathrm{Re}(z) < 0$, while hypothesis *(b)* asserts that the right hand side is positive and thus $\mathrm{Re}(z) > 0$. □

Remark 1. Notice that $f(x;z) = \sum_{k=0}^{\infty} D^k q(x;z)\, z^k$, where $D^k q(x;z) := \frac{d^k q(x;z)}{dx^k}$, provided the series is convergent. In case $q(x;z)$ is a polynomial in x, the series is finite. Also observe that if $c \in (a,b)$ and hypothesis *(a)* of Theorem 3 holds, then $f(c;z) = \sum_{k=0}^{\infty} D^k q(c;z)\, z^k$ is stable. If instead hypothesis *(b)* of Theorem 3 holds, $f(c;-z) = \sum_{k=0}^{\infty} D^k q(c;z)\, (-z)^k$ is stable. Further, if $f(x;z)$ is defined for an interval larger than (a,b) where equation (1) is satisfied, the above hold for all c in the new interval.

Recall that the generalized Laguerre polynomials [7] are defined by $L_n^{(\alpha)}(x) := \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}$, where $\alpha > -1$. They are orthogonal over the interval $(0,\infty)$ with respect to the weight function $w(x) = x^\alpha e^{-x}$. The next theorem, which gives stable polynomials generated by a family of generalized Laguerre polynomials, was originally proved in [2]. These polynomials fit the general framework of Theorem 3 with an additional condition.

Theorem 4 ([2])**.** *Let $\lambda, \mu \geq 0$. For fixed α, $-1 < \alpha \leq 1$, define the polynomial*

$$p_n^{(\alpha)}(z) = \sum_{k=0}^{n} (-1)^k\, z^k D^k L_n^{(\alpha)}(0) \qquad (D := \frac{d}{dx}). \tag{7}$$

Then the polynomial

$$P_n(z;\alpha,\lambda,\mu) = p_n^{(\alpha)}(z) + (\lambda + \mu z^2) p_{n-1}^{(\alpha)}(z) \tag{8}$$

is stable, where $L_n^{(\alpha)}(x)$ denotes the generalized Laguerre polynomial of degree $n \geq 2$.

Proof. Let the interval of Theorem 3 be $(a,b) = (0,\infty)$ and $q(x;z)$ of equation (1) be $q(x,z) = L_n^{(\alpha)}(x) + (\lambda+\mu z^2) L_{n-1}^{(\alpha)}(x)$. By Remark 1, $f(x;z) = \sum_{k=0}^{n} D^k L_n^{(\alpha)}(x) z^k + (\lambda + \mu z^2) \sum_{k=0}^{n-1} z^k D^k L_{n-1}^{(\alpha)}(x)$, which satisfies equation (1) for $x \in \mathbb{R}$. Also, define $W(x) = x^{-1} w(x) = x^{\alpha-1} e^{-x}$, where $w(x)$ is the weight function of the generalised Laguerre polynomial. For simplicity use f, q and $L_n^{(\alpha)}$ in place of $f(x;z)$, $q(x;z)$ and $L_n^{(\alpha)}(x)$. Then we can write

$$\begin{aligned}
&\frac{2}{|z|^2} \int_0^\infty \mathrm{Re}(z f \overline{q(x;z)}) W(x)\, dx \\
&= \int_0^\infty \left(\frac{f}{\overline{z}x} \left(L_n^{(\alpha)} + (\lambda + \mu \overline{z}^2) L_{n-1}^{(\alpha)} \right) + \frac{\overline{f}}{zx} \left(L_n^{(\alpha)} + (\lambda + \mu z^2) L_{n-1}^{(\alpha)} \right) \right) x^\alpha e^{-x}\, dx \\
&\quad = \int_0^\infty \left(\frac{f}{\overline{z}x} + \frac{\overline{f}}{zx} \right) L_n^{(\alpha)} x^\alpha e^{-x}\, dx + \lambda \int_0^\infty \left(\frac{f}{\overline{z}x} + \frac{\overline{f}}{zx} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x}\, dx \\
&\qquad + \mu \int_0^\infty \left(\frac{f\overline{z}}{x} + \frac{\overline{f} z}{x} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x}\, dx := I_0 + \lambda I_1 + \mu I_2.
\end{aligned} \tag{9}$$

If we further demand that z also satisfies $f(0;z) = 0$, then

$$I_0 = \int_0^\infty \left(\frac{f}{\overline{z}x} + \frac{\overline{f}}{zx} \right) L_n^{(\alpha)} x^\alpha e^{-x}\, dx = 0, \tag{10}$$

since $L_n^{(\alpha)}(x)$ is orthogonal to all polynomials with degree less than n. Also notice that Lemma 7 in [2, eq. 21 and eq. 22] gives

$$I_1 = \int_0^\infty \left(\frac{f}{\overline{z}x} + \frac{\overline{f}}{zx} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x}\, dx = -\frac{1}{n} \left(\frac{1}{z} + \frac{1}{\overline{z}} \right) \frac{\Gamma(n+\alpha)}{(n-1)!} \tag{11}$$

and

$$I_2 = \int_0^\infty \left(\frac{f\overline{z}}{x} + \frac{\overline{f}z}{x} \right) L_{n-1}^{(\alpha)} x^\alpha e^{-x}\, dx = -\frac{1}{n} \left(z + \overline{z} \right) \frac{\Gamma(n+\alpha)}{(n-1)!}. \tag{12}$$

Combining equations (9), (10), (11) and (12), a simple calculation shows

$$\int_0^\infty \mathrm{Re}(z f \overline{q(x;z)}) W(x)\, dx = -\frac{\Gamma(n+\alpha)}{n(n-1)!} \left(\lambda + \mu |z|^2 \right) \mathrm{Re}(z). \tag{13}$$

This shows that $\nu = \frac{\Gamma(n+\alpha)}{n(n-1)!} \left(\lambda + \mu |z|^2 \right) \geq 0$ and $\delta = 0$. Moreover, since $f(0;z) = 0$, observe that

$$\lim_{x \to 0^+} \{ |f(x;z)|^2 W(x) \} = \lim_{x \to 0^+} \{ |f(x;z)|^2 x^{\alpha-1} e^{-x} \} = 0 \tag{14}$$

for $\alpha > -1$, while

$$\lim_{x \to \infty} \{ |f(x;z)|^2 W(x) \} = \lim_{x \to \infty} \{ |f(x;z)|^2 x^{\alpha-1} e^{-x} \} = 0. \tag{15}$$

Finally, since

$$\frac{d}{dx} \{ W(x) \} = (\alpha - 1 - x)\, x^{\alpha-1} e^{-x} \leq 0 \tag{16}$$

for $-1 < \alpha \leq 1$, $W(x)$ is non-increasing, and by hypothesis *(b)* of Theorem 3 $\mathrm{Re}(z) > 0$. Thus $f(0;-z) = P_n(z;\alpha,\lambda,\mu)$ is stable. □

Recall that the Jacobi polynomials [7, 14] are defined by

$$P_n^{(\alpha,\beta)}(x) := \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^{n-k} (x+1)^k,$$

where $\alpha,\ \beta > -1$ and are orthogonal over the interval $(-1,1)$ with respect to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$. They satisfy the orthogonality relation

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta\, P_m^{(\alpha,\beta)} P_n^{(\alpha,\beta)}\, dx = \begin{cases} 0, & m \neq n, \\ h_n^{\alpha,\beta}, & m = n, \end{cases} \tag{17}$$

where $h_n^{\alpha,\beta} > 0$ is given in [14, 1.5.5], as well as the three-term recurrence relation

$$a_{1,n} P_{n+1}^{(\alpha,\beta)}(x) = (a_{2,n} + a_{3,n} x) P_n^{(\alpha,\beta)}(x) - a_{4,n} P_{n-1}^{(\alpha,\beta)}(x) \tag{18}$$

with positive constants $a_{1,n}$, $a_{2,n}$, $a_{3,n}$ and $a_{4,n}$. They also satisfy the derivative relation [9]:

$$\frac{d}{dx} P_{n+1}^{(\alpha,\beta)}(x) = B_n P_n^{(\alpha,\beta)}(x) + p_{n-1}(x) \tag{19}$$

with $B_n = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+\alpha+\beta+1)}$ and $p_{n-1}(x)$ a polynomial of degree $n-1$. The Jacobi polynomials are generalizations of several families of orthogonal polynomials such as the Chebyshev polynomials of the first and second kind ($\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, respectively), the Legendre polynomials ($\alpha = \beta = 0$), and the Gegenbauer (ultraspherical) polynomials ($\alpha = \beta = \nu - \frac{1}{2}$).

The next theorem, which gives stable polynomials generated by a family of Jacobi polynomials [7], was originally proved in [4]. These polynomials fit the general framework of Theorem 3 with an additional condition.

Theorem 5 ([4])**.** *Let $\mu \geq 0$. For fixed α, β, $-1 < \alpha, \beta \leq 1$, define the polynomial*

$$\phi_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} z^k D^k P_n^{(\alpha,\beta)}(1) \qquad (D := \frac{d}{dx}). \tag{20}$$

Then the polynomial

$$\Phi_n(z;\alpha,\beta,\mu) = \phi_n^{(\alpha,\beta)}(z) + \mu z^2 \phi_{n-1}^{(\alpha,\beta)}(z) \tag{21}$$

is stable, where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of degree $n \geq 3$.

Proof. Let the interval of Theorem 3 be $(a,b) = (-1,1)$ and $q(x;z)$ of equation (1) be $q(x,z) = P_n^{(\alpha,\beta)}(x) + \mu z^2 P_{n-1}^{(\alpha,\beta)}(x)$. By Remark 1, $f(x;z) = \sum_{k=0}^{n} D^k P_n^{(\alpha,\beta)}(x) z^k + \mu z^2 \sum_{k=0}^{n-1} z^k D^k P_{n-1}^{(\alpha,\beta)}(x)$, which satisfies equation (1) for $x \in \mathbb{R}$. Also, define $W(x) = \frac{1+x}{1-x} w(x) = (1-x)^{\alpha-1}(1+x)^{\beta+1}$, where $w(x)$ is the weight function of the Jacobi polynomial. For simplicity use f, q and $P_n^{(\alpha,\beta)}$ in place of $f(x;z)$, $q(x;z)$ and $P_n^{(\alpha,\beta)}(x)$. Then we can write

$$\begin{aligned} &\frac{2}{|z|^2} \int_{-1}^{1} \operatorname{Re}(z f \overline{q(x;z)}) W(x)\, dx \\ &= \int_{-1}^{1} \left(\frac{f}{\overline{z}} \left(P_n^{(\alpha,\beta)} + \mu \overline{z}^2 P_{n-1}^{(\alpha,\beta)} \right) + \frac{\overline{f}}{z} \left(P_n^{(\alpha,\beta)} + \mu z^2 P_{n-1}^{(\alpha,\beta)} \right) \right) W(x)\, dx \\ &= \int_{-1}^{1} \left(\frac{f}{\overline{z}} + \frac{\overline{f}}{z} \right) P_n^{(\alpha,\beta)}(x) W(x)\, dx + \mu \int_{-1}^{1} \left(f\overline{z} + \overline{f} z \right) P_{n-1}^{(\alpha,\beta)} W(x)\, dx := I_1 + \mu I_2. \end{aligned} \tag{22}$$

If we further demand that z satisfies $f(1;z) = 0$, then equations A.19 and A.23 in [4] give

$$I_1 = \int_{-1}^{1} \left(\frac{f}{\overline{z}} + \frac{\overline{f}}{z} \right) P_n^{(\alpha,\beta)}(x) W(x)\, dx = -\frac{2\operatorname{Re}(z)}{|z|^2} h_n^{(\alpha,\beta)}, \tag{23}$$

where $h_n^{(\alpha,\beta)} = \int_{-1}^{1} \left(P_n^{(\alpha,\beta)}(x) \right)^2 (1-x)^{\alpha}(1+x)^{\beta} dx$.

Also notice that equations A.20 and A.23 in [4] give

$$\begin{aligned} I_2 &= \int_{-1}^{1} \left(f\overline{z} + \overline{f} z \right) P_{n-1}^{(\alpha,\beta)}(x) W(x)\, dx \\ &= -2\operatorname{Re}(z) \left(\frac{2a_{3,n-1}}{a_{1,n-1}} + \mu |z|^2 \right) h_{n-1}^{(\alpha,\beta)} - 2B_{n-1}|z|^2 h_{n-1}^{(\alpha,\beta)}. \end{aligned} \tag{24}$$

Combining equations (22), (23), and (24), a simple calculation shows

$$\begin{aligned} \int_{-1}^{1} \operatorname{Re}(z f \overline{q(x;z)}) W(x)\, dx = &-\left(h_n^{(\alpha,\beta)} + \mu \left(\frac{2a_{3,n-1}}{a_{1,n-1}} + \mu |z|^2 h_{n-1}^{(\alpha,\beta)} \right) \right) \operatorname{Re}(z) \\ &- B_{n-1}|z|^4 h_{n-1}^{(\alpha,\beta)}. \end{aligned} \tag{25}$$

This shows that $\nu = h_n^{(\alpha,\beta)} + \mu\left(\frac{2a_{3,n-1}}{a_{1,n-1}} + \mu|z|^2 h_{n-1}^{(\alpha,\beta)}\right) > 0$ and $\delta = B_{n-1}|z|^4 h_{n-1}^{(\alpha,\beta)} > 0$. Moreover, observe that

$$\lim_{x\to -1^+}\{|f(x;z)|^2 W(x)\} = \lim_{x\to -1^+}\{|f(x;z)|^2(1-x)^{\alpha-1}(1+x)^{\beta+1}\} = 0 \tag{26}$$

for $\beta > -1$, while since $f(1;z) = 0$,

$$\lim_{x\to 1^-}\{|f(x;z)|^2 W(x)\} = \lim_{x\to 1^-}\{|f(x;z)|^2(1-x)^{\alpha-1}(1+x)^{\beta+1}\} = 0 \tag{27}$$

for $\alpha > -1$. Finally, since

$$\frac{d}{dx}\{W(x)\} = (\beta - \alpha + 2 - (\beta+\alpha)x)\,(1-x)^{\alpha-2}(1+x)^{\beta} \geq 0 \tag{28}$$

for $-1 < \alpha, \beta \leq 1$, $W(x)$ is non-decreasing, and by hypothesis *(a)* of Theorem 3 $\mathrm{Re}(z) < 0$. Thus $f(1;z) = \Phi_n(z;\alpha,\beta,\mu)$ is stable. □

Remark 2. Theorem 5 ensures that the polynomial (21) is stable for the special cases of the Chebyshev polynomials of the first and second kind, the Legendre polynomials and the Gegenbauer polynomials for $-\frac{1}{2} < \nu \leq \frac{3}{2}$.

Theorem 3 is quite general since it does not demand $q(x;z)$ to be a polynomial. Even in the case that $q(x;z)$ is a polynomial, the only known examples of functions f satisfying Theorem 3 are the ones described by Theorems 4 and 5. These observations have led to the formulation of the following open problem:

Open Problem 1. Determine whether there exist more functions $f(x;z)$ that fit in the general framework of Theorem 3.

3. Polynomials with real negative roots

With the aid of Remark 1 and Theorem 3 and the Hermite-Biehler Theorem, we obtain new families of polynomials that have only real and negative zeros.

Theorem 6. *Let $f(x;z)$ and $q(x;z)$ be as in Theorem 3, where $q(x;z)$ is now a polynomial of degree n. If $f(c;z)$ is stable, then the polynomials*

$$\Phi_1(z) = \sum_{k=0}^{\left[\frac{n}{2}\right]} D^{2k} q(c;z)\, z^k \quad \textit{and} \quad \Phi_2(z) = \sum_{k=0}^{\left[\frac{n}{2}\right]} D^{2k+1} q(c;z)\, z^k \qquad \left(D^k := \frac{d^k}{dx^k}\right) \tag{29}$$

form a positive pair. If instead $f(c;-z)$ is stable, then the polynomials $\Phi_1(z)$ and $-\Phi_2(z)$ form a positive pair.

The theorem follows from a direct application of the Hermite Biehler Theorem on the stable polynomials $f(c;z)$ and $f(c;-z)$, respectively.

Remark 3. For the special case of the generalized Laguerre polynomials, Theorems 6 and 4 imply that the polynomials

$$\sum_{k=0}^{n} z^k D^{2k} L_n^{(\alpha)}(0) + (\lambda + \mu z^2)\sum_{k=0}^{n} z^k D^{2k} L_{n-1}^{(\alpha)}(0)$$

and

$$-\sum_{k=0}^{n} z^k D^{2k+1} L_n^{(\alpha)}(0) - (\lambda + \mu z^2)\sum_{k=0}^{n} z^k D^{2k+1} L_{n-1}^{(\alpha)}(0)$$

form positive pairs; that is, they have real negative roots that interlace. This result was originally proved in [2].

Remark 4. For the special case of the Jacobi polynomials, Theorems 6 and 5 imply that the polynomials $\sum_{k=0}^{n} z^k D^{2k} P_n^{(\alpha,\beta)}(1) + \mu z^2 \sum_{k=0}^{n} z^k D^{2k} P_{n-1}^{(\alpha,\beta)}(1)$ and $\sum_{k=0}^{n} z^k D^{2k+1} P_n^{(\alpha,\beta)}(1) + \mu z^2 \sum_{k=0}^{n} z^k D^{2k+1} P_{n-1}^{(\alpha,\beta)}(1)$ form positive pairs, as well. This result was originally proved in [4].

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