PSEUDO-DIAGONALS AND UNIQUENESS THEOREMS

GABRIEL NAGY AND SARAH REZNIKOFF

(Communicated by Marius Junge)

ABSTRACT. We examine a certain type of abelian C*-subalgebra that allows one to give a unified treatment of two uniqueness theorems: for graph C*-algebras and for certain reduced crossed products.

This note is meant to complement the paper [NR], which provides one of the two main examples of our results, by offering a conceptual treatment of the Uniqueness Theorem for the C*-algebras associated with graphs satisfying condition (L) and its generalization found in [Sz]. As pointed out in many places in graph C*-algebra literature (see for instance [Re1]), for graphs satisfying condition (L), a natural abelian C*-subalgebra (which is referred to in [NR] as the "diagonal") turns out to give substantial information about the ambient (graph) C*-algebra. A similar treatment was proposed by Kumjian in [Ku], where he introduced the notion of C*-diagonals.

In this paper we explain how, by considerably weakening several hypotheses in Kumjian's definitions (in particular by getting rid of normalizers entirely), one can still obtain several key results, the most significant one being an "abstract" uniqueness property (see Theorem 3.1 below). Another illustration of this general approach is given in the context of reduced crossed products of abelian C*-algebras by essentially free actions of discrete groups, where we recover another uniqueness result due to Archbold and Spielberg [AS].

Our treatment focuses on the unique state extension property, as discussed in [KS] and [An], by weakening the global requirement made in the so-called Extension Property discussed in [An].

1. NOTATION AND PRELIMINARIES

Notation. For any C*-algebra $\mathcal{A}$, we denote by $S(\mathcal{A})$ its set of states and by $P(\mathcal{A})$ the set of pure states. Given a C*-subalgebra $\mathcal{B} \subset \mathcal{A}$, using the Hahn-Banach Theorem, it follows that any $\phi \in S(\mathcal{B})$ has at least one extension to some $\psi \in S(\mathcal{A})$, so the set

$$S_\phi(\mathcal{A}) = \{\psi \in S(\mathcal{A}) : \psi\big|_{\mathcal{B}} = \phi\}$$

is non-empty, convex and weak*-compact. Remark that if $\phi \in P(\mathcal{B})$, then every extreme point in $S_\phi(\mathcal{A})$ is in fact an extreme point in $S(\mathcal{A})$; thus the intersection $S_\phi(\mathcal{A}) \cap P(\mathcal{A})$ is non-empty.

With this observation in mind, we define the space

$$P_1(\mathcal{B}\uparrow\mathcal{A}) = \{\phi \in P(\mathcal{B}): \operatorname{card} S_\phi(\mathcal{A}) = 1\}.$$

1.1. Remark. It is fairly easy to see that the sets P_1 enjoy the following functorial property:

(F) *If $\pi : \mathcal{A} \to \mathcal{M}$ is a $*$-homomorphism, and $\phi \in S(\pi(\mathcal{B}))$ is such that $\phi \circ \pi \in P_1(\mathcal{B}\uparrow\mathcal{A})$, for some C*-subalgebra $\mathcal{B} \subset \mathcal{A}$, then $\phi \in P_1(\pi(\mathcal{B}) \uparrow \pi(\mathcal{A}))$.*

Concerning the space $P_1(\mathcal{B}\uparrow\mathcal{A})$, one has the following important result of Anderson.

1.2. Theorem ([An, Thm. 3.2]). *Assume $\mathcal{A}$ is unital and the C*-subalgebra $\mathcal{B} \subset \mathcal{A}$ contains the unit of $\mathcal{A}$. For a pure state $\phi \in P(\mathcal{B})$, the following conditions are equivalent:*

(i) *$\phi \in P_1(\mathcal{B}\uparrow\mathcal{A})$;*

(ii) *there exists a map $\psi : \mathcal{A} \to \mathbb{C}$ (no extra condition is assumed) such that*
  (∗) *for every $a \in \mathcal{A}$, for every $\varepsilon > 0$, there exists a positive element $b \in \mathcal{B}$ such that $\|b\| = \phi(b) = 1$ and $\|bab - \psi(a)b^2\| < \varepsilon$.*

Moreover, if ψ satisfies condition (∗), then ψ is the unique (pure) state on $\mathcal{A}$ that extends ϕ.

Comment. Condition (ii) implies both (i) and the uniqueness of ψ, even in the non-unital case. Indeed, if ψ satisfies condition (∗), then we can define, for each $a \in \mathcal{A}$, some sequence $\{b_n(a)\}_{n=1}^\infty \subset \mathcal{B}$ of positive elements with $\|b_n(a)\| = \phi(b_n(a)) = 1$ such that

$$\lim_{n\to\infty} \|b_n(a)ab_n(a) - \psi(a)b_n(a)^2\| = 0,$$

so for every $\eta \in S_\phi(\mathcal{A})$ we will get

$$\lim_{n\to\infty} \left[\eta(b_n(a)ab_n(a)) - \psi(a)\phi(b_n(a)^2)\right] = 0.$$

Since for any positive element $b \in \mathcal{B}$, by the Cauchy-Schwartz Inequality we have $\phi(b)^2 \leq \|\phi\| \cdot \phi(b^2)$, it follows that $\phi(b_n(a)^2) = 1$, so the above equality yields

(1) $$\psi(a) = \lim_{n\to\infty} \eta(b_n(a)ab_n(a)).$$

Since for every $\eta \in S(\mathcal{A})$ and every positive element $x \in \mathcal{A}$ with $\|x\| = \eta(x) = 1$, we have (again using Cauchy-Schwartz) $\eta((1-x)a) = \eta(a(1-x)) = 0$, thus $\eta(a) = \eta(xax)$, $\forall a \in \mathcal{A}$, we see that $\eta(b_n(a)ab_n(a)) = \eta(a)$, $\forall \eta \in S_\phi(\mathcal{A})$, so equality (1) forces $S_\phi(\mathcal{A}) = \{\psi\}$.

An application of the above theorem, which perhaps clarifies the status of the set $P_1(\mathcal{B}\uparrow\mathcal{A})$ a little better in the case when $\mathcal{B}$ is *abelian*, is as follows. (Although this result may be known, we were not able to find a proof in the literature, so one is supplied here for the reader's convenience.)

1.3. Proposition. *Assume $\mathcal{B}$ is an abelian (possibly non-unital) C*-subalgebra of a (possibly non-unital) C*-algebra $\mathcal{A}$. For a pure state $\phi \in P(\mathcal{B})$, conditions (i) and (ii) from Theorem 1.2 are also equivalent to the condition*

(ii') *there exists a map $\psi : \mathcal{A} \to \mathbb{C}$ and a net $(b_\lambda)_{\lambda\in\Lambda}$ of positive elements in $\mathcal{B}$ such that $\|b_\lambda\| = \phi(b_\lambda) = 1$, $\forall \lambda \in \Lambda$, and $\lim_{\lambda\in\Lambda} \|b_\lambda ab_\lambda - \psi(a)b_\lambda^2\| = 0$, $\forall a \in \mathcal{A}$.*

As in Theorem 1.2, the map ψ is the unique (pure) state extension of ϕ.

Proof. Since the implication $(ii') \Rightarrow (ii)$ is trivial, while the implication $(ii) \Rightarrow (i)$ has been discussed above, we only need to consider the implication $(i) \Rightarrow (ii')$. Assume $\phi \in P(\mathcal{A})$ has a unique state extension $\psi \in S(\mathcal{A})$. Let us adjoin a (new) unit $\mathbf{1}$ to both $\mathcal{A}$ and $\mathcal{B}$, and let us consider the canonical extensions $\tilde{\phi} \in S(\tilde{\mathcal{B}})$ and $\tilde{\psi} \in S(\tilde{\mathcal{A}})$ of ϕ and ψ, respectively, to the unitized algebras. Of course, $\tilde{\phi}$ is a pure state on $\tilde{\mathcal{B}}$, and $\tilde{\psi}$ is its unique extension to a (pure) state on $\tilde{\mathcal{A}}$, so we can now apply Theorem 1.2 to $\tilde{\phi}$ and $\tilde{\psi}$. Choose then, for every $a \in \mathcal{A}$ and every $\varepsilon > 0$, a positive element $x_{a,\varepsilon} \in \tilde{\mathcal{B}}$ so that $\|x_{a,\varepsilon}\| = \tilde{\phi}(x_{a,\varepsilon}) = 1$ and such that

$$(2) \qquad \|x_{a,\varepsilon} a x_{a,\varepsilon} - \psi(a) x_{a,\varepsilon}^2\| < \varepsilon.$$

(Note that, since we only use $a \in \mathcal{A}$, $\tilde{\psi}$ is replaced by ψ.)

We define, for every finite set $\mathcal{F} = \{a_1, \ldots, a_n\} \subset \mathcal{A}$ and any $\varepsilon > 0$, the element $x_{\mathcal{F},\varepsilon} = \prod_{j=1}^n x_{a_j,\varepsilon} \in \tilde{\mathcal{B}}$. Since $\tilde{\mathcal{B}}$ is abelian, this product is unambiguously defined and positive. Furthermore, since all the $x_{a_j,\varepsilon}$'s have norm 1, by suitably multiplying both sides of (2), we also get

$$(3) \qquad \|x_{\mathcal{F},\varepsilon} a x_{\mathcal{F},\varepsilon} - \psi(a) x_{\mathcal{F},\varepsilon}^2\| < \varepsilon, \ \ \forall\, a \in \mathcal{F},$$

with all $x_{\mathcal{F},\varepsilon}$ having norm ≤ 1.

Of course, since $\tilde{\phi}$ is a pure state on $\tilde{\mathcal{B}}$, which is abelian, it follows that $\tilde{\phi}$ is multiplicative on $\tilde{\mathcal{B}}$, so in particular we also have $\tilde{\phi}(x_{\mathcal{F},\varepsilon}) = 1$, which in turn forces $\|x_{\mathcal{F},\varepsilon}\| = 1$.

Up to this point, we have almost all that we want: a net $(x_\lambda)_{\lambda \in \Lambda}$ of positive elements in $\tilde{\mathcal{B}}$ such that

(a) $\|x_\lambda\| = \tilde{\phi}(x_\lambda) = 1, \forall\, \lambda \in \Lambda$;
(b) $\lim_{\lambda \in \Lambda} \|x_\lambda a x_\lambda - \psi(a) x_\lambda^2\| = 0, \forall\, a \in \mathcal{A}$.

To produce the desired b_λ's (in $\mathcal{B}$), we regard $\mathcal{B}$ as $C_0(\Omega)$, for some locally compact space, so ϕ will be an evaluation map $\phi : C_0(\Omega) \ni f \longmapsto f(\omega) \in \mathbb{C}$. Then, if we take $f \in \mathcal{B}$ to be any positive element with $\|f\| = 1 = f(\omega)$, the assignment $b_\lambda = f x_\lambda$ will do the job. □

2. Pseudo-diagonals

One key condition we wish to isolate is as follows.

Definition. Assume $\mathcal{A}$ is a C*-algebra. A C*-subalgebra $\mathcal{B} \subset \mathcal{A}$ is said to have the *Almost Extension Property* if the set $P_1(\mathcal{B}{\uparrow}\mathcal{A})$ is weak*-dense in $P(\mathcal{B})$.

Let us say that $\mathcal{B} \subseteq \mathcal{A}$ has the (honest) Extension Property if $P_1(\mathcal{B}{\uparrow}\mathcal{A}) = P(\mathcal{B})$; of course, if $\mathcal{B}$ has the Extension Property, then it also has the Almost Extension Property.

2.1. **Example.** Given any set Σ, the C*-algebra $\ell^\infty(\Sigma)$, viewed as a C*-subalgebra of $\mathcal{B}(\ell^2(\Sigma))$ (as multiplication operators), has the Almost Extension Property, since all pure states of the form $\mathrm{ev}_s : \ell^\infty(\Sigma) \ni f \longmapsto f(s) \in \mathbb{C}$, $s \in \Sigma$, have unique state extensions to $\mathcal{B}(\ell^2(\Sigma))$, and the set $\{\mathrm{ev}_s : s \in \Sigma\}$ is dense in $P(\ell^\infty(\Sigma)) = \beta\Sigma$ (the Stone-Cech compactification of Σ). The famous Kadison-Singer Problem asks whether the inclusion $\ell^\infty(\mathbb{N}) \subset \mathcal{B}(\ell^2(\mathbb{N}))$ has in fact the (honest) Extension Property.

2.2. *Remark.* Using the functorial property (F) from Remark 1.1, it is immediate that whenever $\mathcal{B} \subset \mathcal{A}$ has the honest Extension Property and $\pi : \mathcal{A} \to \mathcal{M}$ is a *-homomorphism, it follows that $\pi(\mathcal{B}) \subset \pi(\mathcal{A})$ also has the honest Extension Property.

This implication may fail if we use the Almost Extension Property, as seen for instance in Example 2.3 below. However, again using Remark 1.1, the implication does hold if π is injective on $\mathcal{B}$.

As Anderson has shown, in the case when $\mathcal{B}$ is an *abelian* C*-subalgebra of $\mathcal{A}$ with the (honest) Extension Property, there exists a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$, which is in fact unique. This may no longer be the case if we consider only the Almost Extension Property, as seen in the following.

2.3. **Example.** Consider $\mathcal{A} = C([0,1])$ (the algebra of continuous functions on $[0,1]$) and the algebra $C(\mathbb{T})$ (the algebra of continuous functions on the unit circle), identified as a C*-subalgebra of $\mathcal{A}$ as

$$\mathcal{B} = \{f \in \mathcal{A} : f(0) = f(1)\}.$$

Obviously, all evaluation maps $\mathrm{ev}_s : C(\mathbb{T}) \ni f \longmapsto f(s) \in \mathbb{C}$, $s \in (0,1)$, have unique extensions to states on $C([0,1])$, but there is no conditional expectation of $C([0,1])$ onto $\mathcal{B}$.

If we consider the *-homomorphism $\pi : \mathcal{A} \ni f \longmapsto (f(0), f(1)) \in \mathbb{C} \oplus \mathbb{C}$, then obviously the inclusion $\pi(\mathcal{B}) \subset \pi(\mathcal{A})$ fails to have the Almost Extension Property.

2.4. *Remark.* If $\mathcal{B} \subset \mathcal{A}$ has the Almost Extension Property and there exists a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$, then it is unique. Indeed, if $E, F : \mathcal{A} \to \mathcal{B}$ are two conditional expectations, then for $\phi \in P_1(\mathcal{B} \uparrow \mathcal{A})$ we have $\phi \circ E = \phi \circ F$ (by uniqueness of state extensions), so for every $a \in \mathcal{A}$ we have $\phi\big(E(a) - F(a)\big) = 0$, $\forall \phi \in P_1(\mathcal{B} \uparrow \mathcal{A})$. By density this forces $\phi\big(E(a) - F(a)\big) = 0$, $\forall \phi \in P(\mathcal{B})$, and then we get

$$\phi\big(E(a) - F(a)\big) = 0, \ \ \forall a \in \mathcal{A}, \ \phi \in \mathcal{A}^*,$$

which forces $E = F$.

Definition. In the case where $\mathcal{B} \subset \mathcal{A}$ has the Almost Extension Property and there exists a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$, we declare the inclusion $\mathcal{B} \subset \mathcal{A}$ to have the *Canonical Almost Extension Property*, and the (unique) conditional expectation E will be called the *associated expectation*.

Comment. If $\mathcal{B} \subset \mathcal{A}$ has the Almost Extension Property, the "candidate" for the associated conditional expectation E is constructed as follows. Start off by considering the unique state extension as a map $\theta_0 : P_1(\mathcal{B} \uparrow \mathcal{A}) \to \mathcal{A}^*$, and define its natural linear continuous extension $\theta : \ell^1(P_1(\mathcal{B} \uparrow \mathcal{A})) \to \mathcal{A}^*$. Then take the dual map

$$(4) \qquad\qquad \theta^* : \mathcal{A}^{**} \to \ell^\infty(P_1(\mathcal{B} \uparrow \mathcal{A})),$$

together with the embedding

$$(5) \qquad\qquad J : \mathcal{B} \ni b \longmapsto (\phi(b))_{\phi \in P_1(\mathcal{B} \uparrow \mathcal{A})} \in \ell^\infty(P_1(\mathcal{B} \uparrow \mathcal{A})).$$

If $\theta^*(\mathcal{A}) \subset \mathrm{Range}\, J$, then E is simply $J^{-1} \circ \theta^*\big|_{\mathcal{A}}$. Note that both maps (4) and (5) are completely positive (because they are positive and take values in a commutative C*-algebra), and thus they are both contractive.

2.5. *Remark.* In the case when $\mathcal{B}$ is *abelian*, the map J, as in (5), is an isometric $*$-homomorphism, so by Tomiyama's Theorem [To], the condition $\theta^*(\mathcal{A}) \subset \text{Range } J$ is sufficient to conclude that E is a conditional expectation.

For future use, let us record the following technical result.

2.6. **Lemma.** *Assume that $\mathcal{B}$ is abelian and we have an inclusion $\mathcal{B} \subset \mathcal{A}$ with the Canonical Almost Extension Property, with associated expectation E. If we consider the seminorm $p : \mathcal{A} \to [0, \infty)$, defined by*

$$p(a) = \sup \{\|ab - ba\| \, : \, b \in \mathcal{B}, \, \|b\| \leq 1\}, \;\; a \in \mathcal{A},$$

then

$$(6) \qquad \|E(a)^2 - E(a^2)\| \leq 2\|a\| \cdot p(a), \;\; \forall\, a \in \mathcal{A}.$$

Proof. Fix $a \in \mathcal{A}$, as well as some $\phi \in P_1(\mathcal{B} \uparrow \mathcal{A})$, so that the unique state on $\mathcal{A}$ that extends ϕ is $\phi \circ E$. According to Proposition 1.3, there exists a net $(b_\lambda)_{\lambda \in \Lambda}$ of positive elements in $\mathcal{B}$, with $\|b_\lambda\| = \phi(b_\lambda) = 1$, so that

$$(7) \qquad \lim_{\lambda \in \Lambda} \|b_\lambda a b_\lambda - \phi(E(a)) b_\lambda^2\| = 0,$$

$$(8) \qquad \lim_{\lambda \in \Lambda} \|b_\lambda a^2 b_\lambda - \phi(E(a^2)) b_\lambda^2\| = 0.$$

On the one hand, since we have

$$\|[b_\lambda a b_\lambda]^2 - [\phi(E(a)) b_\lambda^2]^2\|$$
$$= \|b_\lambda a b_\lambda [b_\lambda a b_\lambda - \phi(E(a)) b_\lambda^2] + [b_\lambda a b_\lambda - \phi(E(a)) b_\lambda^2] \phi(E(a)) b_\lambda^2\|$$
$$\leq \big[\|b_\lambda a b_\lambda\| + \|\phi(E(a)) b_\lambda^2\| \big] \cdot \|b_\lambda a b_\lambda - \phi(E(a)) b_\lambda^2\|$$
$$\leq 2\|a\| \cdot \|b_\lambda a b_\lambda - \phi(E(a)) b_\lambda^2\|,$$

we also get

$$(9) \qquad \lim_{\lambda \in \Lambda} \|(b_\lambda a b_\lambda)^2 - [\phi(E(a)) b_\lambda^2]^2\| = 0.$$

On the other hand, if we multiply (inside the norm) the left-hand side of (8) on each side by b_λ, we also get

$$(10) \qquad \lim_{\lambda \in \Lambda} \|b_\lambda^2 a^2 b_\lambda^2 - \phi(E(a^2)) b_\lambda^4\| = 0.$$

Finally, since we have

$$\|b_\lambda^2 a^2 b_\lambda^2 - [b_\lambda a b_\lambda]^2\| = \|b_\lambda^2 a [a b_\lambda - b_\lambda a] b_\lambda - b_\lambda [a b_\lambda - b_\lambda a] b_\lambda a b_\lambda\|$$
$$\leq \big[\|b_\lambda^2 a\| \cdot \|b_\lambda\| + \|b_\lambda\| \cdot \|b_\lambda a b_\lambda\| \big] \cdot \|a b_\lambda - b_\lambda a\| \leq 2\|a\| \cdot p(a),$$

we now get

$$\|[\phi(E(a)) b_\lambda^2]^2 - \phi(E(a^2)) b_\lambda^4\| \leq \|b_\lambda^2 a^2 b_\lambda^2 - \phi(E(a^2)) b_\lambda^4\|$$
$$+ \|[b_\lambda a b_\lambda]^2 - [\phi(E(a)) b^2]^2\| + \|b_\lambda^2 a^2 b_\lambda^2 - [b_\lambda a b_\lambda]^2\|$$
$$\leq \|b_\lambda^2 a^2 b_\lambda^2 - \phi(E(a^2)) b_\lambda^4\| + \|(b_\lambda a b_\lambda)^2 - [\phi(E(a)) b^2]^2\| + 2\|a\| \cdot p(a),$$

so when we use (9) and (10), we get

$$(11) \qquad \limsup_{\lambda \in \Lambda} \|[\phi(E(a)) b_\lambda^2]^2 - \phi(E(a^2)) b_\lambda^4\| \leq 2\|a\| \cdot p(a).$$

Of course, when we apply ϕ to the quantity inside the norm and use the obvious equality $\phi(b_\lambda^4) = \phi(b_\lambda)^4 = 1$, we get

$$\phi([\phi(E(a))b_\lambda^2]^2 - \phi(E(a^2))^2 b_\lambda^4) = \phi(E(a^2) - E(a)^2),$$

so inequality (11) yields

$$|\phi(E(a^2) - E(a)^2)| \leq 2\|a\| \cdot p(a).$$

If we take supremum over all $\phi \in P_1(\mathcal{B} \uparrow \mathcal{A})$, by density, we get (6).   □

Definition. Suppose $\mathcal{A}$ is a C*-algebra. An abelian C*-subalgebra $\mathcal{B} \subset \mathcal{A}$ is called a *pseudo-diagonal in $\mathcal{A}$* if

(i) the inclusion $\mathcal{B} \subset \mathcal{A}$ has the Canonical Almost Extension Property;
(ii) the associated expectation $E : \mathcal{A} \to \mathcal{B}$ is *faithful*: i.e. it satisfies the implication $E(a^*a) = 0 \Rightarrow a = 0$.

Condition (i) alone is not sufficient, as seen in the following.

2.7. Example. Consider $\mathcal{B} = C([0,1])$, the characteristic function $f_0 = \chi_{\{0\}} \in \ell^\infty([0,1])$, and the C*-subalgebra $\mathcal{A} = \mathcal{B} + \mathbb{C}f_0$, which can be characterized as the algebra of all bounded functions on $[0,1]$ that are continuous on $(0,1]$ and have limit at 0. As it turns out, the inclusion $\mathcal{B} \subset \mathcal{A}$ has the Canonical Almost Extension Property. Indeed, all maps $\mathrm{ev}_s \in P(\mathcal{B})$, $s \in (0,1]$, have unique state extensions, and we have a conditional expectation $E : \mathcal{A} \to \mathcal{B}$, defined as

$$(E(f))(t) = \begin{cases} f(t), & \text{if } t > 0, \\ \lim_{s \to 0} f(s), & \text{if } t = 0. \end{cases}$$

However, E fails to be faithful, since $E(f_0) = 0$.

Comment. The use of the term "pseudo-diagonal" in our definition is meant to hint at Kumjian's definition of so-called *C*-diagonals*, which includes the existence of a faithful conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$, plus a certain feature of the normalizer set

$$(12) \qquad \mathcal{N}(\mathcal{B}) = \{a \in \mathcal{A} : a\mathcal{B}a^* \cup a^*\mathcal{B}a \subset \mathcal{B}\}.$$

(Actually, Kumjian uses a certain distinguished subset in $\mathcal{N}(\mathcal{B})$, which he requires to generate $\mathcal{A}$ as a C*-algebra.) As shown in [Ku], C*-diagonals have in fact the (honest) Extension Property; thus C*-diagonals are in fact pseudo-diagonals.

2.8. Example. Assume $\mathcal{A} = C^*(E)$, the graph C*-algebra of some countable graph $E = (E^0, E^1)$, which is the universal C*-algebra generated by $\{P_v\}_{v \in E^0} \cup \{S_e\}_{e \in E^1}$, subject to the Cuntz-Krieger relations:

$(13) \qquad P_v^* = P_v = P_v^2, \;\; \forall\, v \in \mathcal{E}^0;$

$(14) \qquad S_e^* S_e = P_{s(e)}, \;\; \forall\, e \in \mathcal{E}^1;$

$(15) \qquad S_e^* S_f = 0$, for any two *distinct* edges $e, f \in \mathcal{E}^1$;

$(16) \qquad$ if $v \in \mathcal{E}^0$ has $r^{-1}(v)$ *non-empty and finite*, then $S_v = \sum\limits_{e \in r_1^{-1}(v)} S_e S_e^*.$

(Here $s, r : E^1 \to E^0$ denote the source and range maps.) Define for every path $\alpha = e_1 e_2 \cdots e_n$ (the convention here is as in [Ra], i.e. $s(e_i) = r(e_{i+1})$, $\forall\, i$) the projection $R_\alpha = S_{e_1} \cdots S_{e_n} S_{e_n}^* \cdots S_{e_1}^*$ (with the convention that if $\alpha = v \in E^0$,

i.e. α is a path of length zero, then $R_\alpha = P_v$), and define the "diagonal" algebra $\Delta(E)$ to be the (abelian) C*-subalgebra generated by all R_α's.

As it turns out, $\Delta(E)$ is a pseudo-diagonal if and only if the following condition is satisfied:

(L) every cycle in E has an entry.

As pointed out in [NR], in the absence of condition (L), the correct substitute for $\Delta(E)$ is its commutant

$$M(E) = \{x \in C^*(E) : xa = ax, \ \forall \, a \in \Delta(E)\},$$

which turns out to be a pseudo-diagonal.

Comment. It should be pointed out here that even in very simple cases, such as the one described below, the pseudo-diagonal $M(E)$ may fail to have the (honest) Extension Property, so there are instances when $P_1(\mathcal{B}\!\uparrow\!\mathcal{A})$, although dense in $P(\mathcal{B})$, fails to coincide with $P(\mathcal{B})$. In other words, there are pseudo-diagonals which are not C*-diagonals.

For instance, if one considers the Toeplitz C*-algebra $\mathcal{T}$ (which can be identified as $C^*(E)$ for a graph with two vertices v and w and two edges e and f with $s(e) = v$ and $r(e) = s(f) = r(f) = w$), the pseudo-diagonal $\mathcal{B} = \Delta(E) = M(E)$ is identified with $C(\mathbb{N} \cup \{\infty\})$, where $\mathbb{N} \cup \{\infty\}$ is the Alexandrov compactification of $\mathbb{N}$, and $P_1(\mathcal{B}\!\uparrow\!\mathcal{T}) = \mathbb{N} \subsetneq \mathbb{N} \cup \{\infty\} = P(\mathcal{B})$.

In preparation for our next example (see Theorem 3.1 below), let us introduce the following.

Notation. Assume X is some locally compact (Hausdorff) space and G is a (discrete) group acting on X (by homeomorphisms):

$$(17) \qquad\qquad G \times X \ni (g, x) \longmapsto g \cdot x \in X.$$

Associated with this action, one defines the full crossed product $C_0(X) \times G$ as follows. Start off with the convolution $*$-algebra $C_0(X)[G]$ of finite formal sums of the form $A = \sum_{g \in G} \mathbf{u}_g a_g$, with $a_g \in C_0(X)$, and $\{\mathbf{u}_g\}_{g \in G}$, some formal unitaries, satisfying the identity

$$(18) \qquad\qquad \mathbf{u}_g a \mathbf{u}_{g^{-1}} = \alpha_g(a), \ \ a \in C_0(X), \, g \in G,$$

where $\alpha_g : C_0(X) \to C_0(X)$, $g \in G$, are the automorphisms given by

$$(\alpha_g(a))(x) = a(g^{-1} \cdot x), \ \ a \in C_0(X), \, x \in X, \, g \in G.$$

One equips this $*$-algebra with the maximal C*-norm $\| \cdot \|_{\max}$ (which can be shown to exist) and defines the C*-algebra $C_0(X) \times G$ to be the completion of $C_0(X)[G]$ in this norm. Of course, $C_0(X) \times G$ contains $C_0(X)$ as a C*-subalgebra, and it is not hard to see that there exists a conditional expectation E of $C_0(X) \times G$ onto $C_0(X)$, hereafter referred to as the *standard* expectation, which acts on the dense subalgebra $C_0(X)[G]$ as

$$(19) \qquad\qquad E(\sum_{g \in G} \mathbf{u}_g a_g) = a_e.$$

In general, E may fail to be faithful; one way to correct this is to consider the *reduced* norm C*-norm $\| \cdot \|_{\mathrm{red}}$ on $C_0(X)[G]$ so that when we take $C_0(X) \times_{\mathrm{red}} G$ to be the completion, we have a faithful conditional expectation E_{red} acting as in (19).

By construction, we have a surjective $*$-homomorphism $\pi_{\mathrm{red}} : C_0(X) \times G \to C_0(X) \times_{\mathrm{red}} G$ satisfying the identity $E_{\mathrm{red}} \circ \pi_{\mathrm{red}} = E$.

Since π_{red} acts as the identity on $C_0(X)$, by Remark 1.1, we have the inclusion

$$(20) \qquad P_1(C_0(X) \uparrow C_0(X) \times G) \subset P_1(C_0(X) \uparrow C_0(X) \times_{\mathrm{red}} G).$$

2.9. Lemma. *Use the notation as above, and fix some point $x \in X$. Let ψ be a state on $C_0(X) \times G$ that extends the evaluation state $\mathrm{ev}_x : C_0(X) \ni a \longmapsto a(x) \in \mathbb{C}$. If $g \in G$ is such that $g \cdot x \neq x$, then*

$$\psi(\mathbf{u}_g a) = 0, \quad \forall\, a \in C_0(X).$$

Proof. Fix $a \in C_0(X)$ and $g \in G$ as above, so we also have $g^{-1} \cdot x \neq x$. Let $f \in C_0(X)$ be a non-negative function, with $\|f\| = 1$, $f(x) = 1$ and $f(g^{-1} \cdot x) = 0$.

On the one hand, using the Cauchy-Schwarz Inequality – in the form $|\psi(T)|^2 \leq \psi(T^*T)$ – we have $|\psi(\mathbf{u}_g(a - af))|^2 \leq \psi\left(|a - af|^2\right) = |a - af|^2(x) = 0$, so we get

$$(21) \qquad\qquad \psi(\mathbf{u}_g a) = \psi(\mathbf{u}_g af).$$

On the other hand, using the Cauchy-Schwarz Inequality – in the form $|\psi(T)|^2 \leq \psi(TT^*)$ – we also have

$$|\psi(\mathbf{u}_g a)|^2 = |\psi(\mathbf{u}_g af)|^2 \leq \psi\left(\mathbf{u}_g |af|^2 \mathbf{u}_g^*\right),$$

so using (18) we obtain

$$|\psi(\mathbf{u}_g a)|^2 \leq \psi\left(\alpha_g(|af|^2)\right) = \mathrm{ev}_x\left(\alpha_g(|af|^2)\right) = |af|^2(g^{-1} \cdot x) = 0,$$

with the last equality due to the fact that $f(g^{-1} \cdot x) = 0$. $\qquad\square$

Notation. For every $g \in G \smallsetminus \{e\}$, let us consider the set moved by g:

$$D_g = \{x \in X : g \cdot x \neq x\}.$$

Of course, all sets D_g, $g \in G \smallsetminus \{e\}$, are open in X.

2.10. Corollary. *With the notation as above, for every $x \in \bigcap_{g \in G \smallsetminus \{e\}} D_g$, the evaluation state $\mathrm{ev}_x \in P(C_0(X))$ belongs to $P_1\left(C_0(X) \uparrow C_0(X) \times G\right)$.*

Proof. Fix $x \in \bigcap_{g \in G \smallsetminus \{e\}} X_g$, as well as some state ψ on $C_0(X) \times G$ that extends ev_x, and let us prove the equality

$$(22) \qquad\qquad \psi = \mathrm{ev}_x \circ E.$$

This follows immediately from Lemma 2.9, which implies that for any finite sum $A = \sum_g \mathbf{u}_g a_g \in C_0(X)[G]$, we have $\psi(A) = \psi(a_e) = \mathrm{ev}_x(a_e) = \mathrm{ev}_x(E(A))$, so the states ψ and $\mathrm{ev}_x \circ E$ agree on a dense C*-subalgebra. $\qquad\square$

Definition. An action (17) is said to be *essentially free* if for every $g \in G \smallsetminus \{e\}$, the set $X_g = \{x \in X : g \cdot x = x\}$ has empty interior. Equivalently, for every $g \in G \smallsetminus \{e\}$, the set D_g is dense in X.

2.11. Theorem. *If the action of a countable group G on X is essentially free, then*

(i) *the inclusion $C_0(X) \subset C_0(X) \times G$ has the Canonical Almost Extension Property, with E as the associated expectation;*

(ii) *$C_0(X)$ is a pseudo-diagonal in $C_0(X) \times_{\mathrm{red}} G$, with associated expectation E_{red}.*

Proof. Using (20), it suffices to prove only condition (i). By Corollary 2.10, it suffices to prove that the set $\bigcap_{g \in G \smallsetminus \{e\}} D_g$ is dense in X or, equivalently, the union of the complements $\bigcup_{g \in G \smallsetminus \{e\}} X_g$ has empty interior. Since each X_g is closed with empty interior and the union is countable, the desired conclusion follows from Baire's Theorem for locally compact spaces. $\qquad\square$

2.12. *Remark.* If a countable group G acts essentially freely on X, then Remark 1.1 (the last statement) yields the following fact:

($*$) *Whenever (π, u) is a covariant representation of $(C_0(X), G)$ with π injective, it follows that the inclusion $\pi(C_0(X)) \subset C^*(\pi(C_0(X)) \cup u(G))$ has the Almost Extension Property.*

We conclude this section with an off-topic discussion of a variant of the results from Example 2.8 through Corollary 2.10, which answers a question of Paulsen [Pa]. The starting point is the following property, which is much stronger than essential freeness.

Definition. One says that an action (17) is *free* if $D_g = X$, $\forall\, g \in G \smallsetminus \{e\}$.

2.13. *Remark.* If G acts freely on X, even when G is uncountable, by the same proof as in Corollary 2.10 it follows that the inclusion $C_0(X) \subset C_0(X) \times G$ has the honest Extension Property. Furthermore, as argued in Remark 1.1, this gives the following fact:

($**$) *Whenever (π, u) is a covariant representation of $(C_0(X), G)$ (no extra assumption), it follows that the inclusion $\pi(C_0(X)) \subset C^*(\pi(C_0(X)) \cup u(G))$ has the honest Extension Property.*

Of course, when we consider the reduced crossed product, i.e. the quotient $*$-homomorphism π_{red}, it follows that $C_0(X)$ is a pseudo-diagonal in $C_0(X) \times_{\mathrm{red}} G$, with the honest Extension Property.

2.14. **Example.** Fix a discrete group G and identify $\ell^\infty(G) = C(\beta G)$. We claim that when we let G act on $\ell^\infty(G)$ by translation, the corresponding action of G on βG is free. To see this, we fix some $g \in G \smallsetminus \{e\}$, as well as some point $x \in \beta G$ represented by an ultrafilter $\mathcal{U}$ in G, and we show that the ultrafilter that correponds to $g \cdot x$, which is obviously $g\mathcal{U} = \{gU : U \in \mathcal{U}\}$, is distinct from $\mathcal{U}$. Let Γ be the subgroup generated by g, and let $T \subset G$ be a complete set of representatives for the quotient space Γ/G, so that every element $h \in G$ can be written uniquely as $h = st$, with $s \in \Gamma$ and $t \in T$. Now consider the following two cases: (i) Γ is finite, say $\Gamma = \{e, g, g^2, \ldots, g^k\}$; (ii) Γ is cyclic, so for every element $h \in G$, there exists a unique integer $n(h) \in \mathbb{Z}$ and a unique $t \in T$, such that $h = g^{n(h)} t$.

Case (i) is trivial, because we have a finite partition $G = T \cup gT \cup \cdots \cup g^k T$, so there exists some $j \in \{0, \ldots, k\}$ such that the set $U = g^j T \in \mathcal{U}$, in which case the set $gU = g^{j+1} T$ (which belongs to $g\mathcal{U}$), cannot belong to $\mathcal{U}$, simply because $U \cap gU = \varnothing$.

In case (ii) we use the map $n : G \to \mathbb{Z}$ to partition $G = G_e \cup G_o$, where

$$G_e = \{h \in G : n(h) \text{ even}\},$$
$$G_o = \{h \in G : n(h) \text{ odd}\},$$

which satisfy the equalities $gG_e = G_o$ and $gG_o = G_e$, so either G_e belongs to $\mathcal{U}$, in which case $G_o = gG_o$ belongs to $g\mathcal{U}$ but not to $\mathcal{U}$, or vice-versa.

To summarize, we have shown that the inclusion $\ell^\infty(G) \subset \ell^\infty(G) \times G$ has the honest Extension Property, and furthermore, for any covariant representation (π, u) of $(\ell^\infty(G), G)$, the inclusion $\pi(\ell^\infty(G)) \subset C^*(\pi(\ell^\infty(G)) \cup u(G))$ also has the honest Extension Property.

3. Abelian cores

In this section we derive the main results from this article, which deal with the automatic maximality of a pseudo-diagonal as an abelian subalgebra as well as a uniqueness property for the ambient C*-algebra. As pointed out, for instance in [KS], if $\mathcal{B} \subset \mathcal{A}$ has the (honest) Extension Property (with $\mathcal{B}$ abelian, of course), then $\mathcal{B}$ is a maximal abelian subalgebra. As we saw in Example 2.7, this may not necessarily be the case if only the Canonical Almost Extension Property is assumed. As we shall prove shortly, pseudo-diagonals are in fact maximal abelian. In addition to this feature, pseudo-diagonals also have another key property, geared toward uniqueness for the ambient C*-algebra $\mathcal{A}$, so it is convenient to introduce the following terminology.

Definition. Assume $\mathcal{A}$ is a C*-algebra. An abelian C*-subalgebra $\mathcal{B} \subset \mathcal{A}$ is called an *abelian core for* $\mathcal{A}$ if the following conditions are satisfied:

  (E$_1$) there exists a unique conditional expectation E of $\mathcal{A}$ onto $\mathcal{B}$;
  (E$_2$) the conditional expectation E of (E$_1$) is faithful;
  (M) $\mathcal{B}$ is a maximal abelian subalgebra in $\mathcal{A}$;
  (U) a $*$-representation $\pi : \mathcal{A} \to B(H)$ is injective whenever its restriction to $\mathcal{B}$ is injective.

Comment. Although we are mainly interested in the implication "pseudo-diagonal" $\Rightarrow$ "abelian core", it will be beneficial to prove a slight generalization (Theorem 3.1 below) in order to accommodate the following situation. Assume that a C*-algebra $\mathcal{A}$ can be written as

$$(23) \qquad \mathcal{A} = \overline{\bigcup_{\sigma \in \Sigma} \mathcal{A}_\sigma},$$

where $(\mathcal{A}_\sigma)_{\sigma \in \Sigma}$ is an increasing net of C*-subalgebras (i.e. $\nu \succ \sigma \Rightarrow \mathcal{A}_\sigma \subset \mathcal{A}_\nu$). Assume also that we have an abelian C*-subalgebra $\mathcal{B} \subset \mathcal{A}$, which is contained in all $\mathcal{A}_\sigma$'s. It is pretty clear that the net $(P_1(\mathcal{B} \uparrow \mathcal{A}_\sigma))_{\sigma \in \Sigma}$ is decreasing and, furthermore, we have the equality

$$\mathcal{P}_1(\mathcal{B} \uparrow \mathcal{A}) = \bigcap_{\sigma \in \Sigma} P_1(\mathcal{B} \uparrow \mathcal{A}_\sigma),$$

so, even in the case when all inclusions $\mathcal{B} \subset \mathcal{A}_\sigma$, $\sigma \in \Sigma$, have the Almost Extension Property, the inclusion $\mathcal{B} \subset \mathcal{A}$ might fail to have it.

3.1. **Theorem.** *Assume $\mathcal{A}$ is a C*-algebra, written for an increasing net $(\mathcal{A}_\sigma)_{\sigma \in \Sigma}$ of C*-algebras. Assume also that $\mathcal{B} \neq \{0\}$ is an abelian C*-subalgebra in $\bigcap_{\sigma \in \Sigma} \mathcal{A}_\sigma$ such that there exists a faithful conditional expectation E of $\mathcal{A}$ onto $\mathcal{B}$. If all inclusions $\mathcal{B} \subset \mathcal{A}_\sigma$, $\sigma \in \Sigma$, have the Almost Extension Property, then $\mathcal{B}$ is an abelian core in $\mathcal{A}$.*

Proof. Denote, for each $\sigma \in \Sigma$, the restriction of E to $\mathcal{A}_\sigma$ by E_σ. It is clear that $\mathcal{B}$ is a pseudo-diagonal in $\mathcal{A}_\sigma$ (with the associated expectation E_σ), for each $\sigma \in \Sigma$.

The proof will be carried on in steps that correspond to the conditions listed in the preceding Definition.

(E_1)-(E_2) Assume F is another conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. It is clear that, for each $\sigma \in \Sigma$, the restriction $F_\sigma = F\big|_{\mathcal{A}_\sigma} : \mathcal{A}_\sigma \to \mathcal{B}$ is a conditional expectation of $\mathcal{A}_\sigma$ onto $\mathcal{B}$, so using Remark 2.2 (for the inclusion $\mathcal{B} \subset \mathcal{A}_\sigma$), it follows that $F_\sigma = E_\sigma$. This argument shows that F coincides with E on the dense subalgebra $\bigcup_{\sigma \in \Sigma} \mathcal{A}_\sigma$, so by continuity we have $F = E$; thus both conditions (E_1), (E_2) are satisfied.

(M) All we have to prove here is that the commutant

$$\mathcal{B}' = \{a \in \mathcal{A} : ab = ba, \ \forall \, b \in \mathcal{B}\}$$

is contained in $\mathcal{B}$. Since $\mathcal{B}'$ is a C*-algebra, it suffices to show that all self-adjoint elements $x \in \mathcal{B}'$ belong to $\mathcal{B}$. Fix such an element x, and choose for every integer $n \geq 1$ an element $a_n \in \bigcup_{\sigma \in \Sigma} \mathcal{A}_\sigma$, with $\|x - a_n\| \leq \frac{1}{n}$. Using the seminorm p from Lemma 2.6, we have $\big|p(x) - p(a_n)\big| \leq 2\|x - a_n\| \leq \frac{2}{n}$, which implies $p(a_n) \leq \frac{2}{n}$, so if we choose $\sigma_n \in \Sigma$ so that $a_n \in \mathcal{A}_{\sigma_n}$, then by applying Lemma 2.6 to the inclusion $\mathcal{B} \subset \mathcal{A}_{\sigma_n}$, we get the inequalities $\|E(a_n^2) - E(a_n)^2\| \leq 2\|a_n\| \cdot p(a_n) \leq \frac{4}{n}\left(\|x\| + \frac{1}{n}\right)$, which imply that

$$(24) \qquad \lim_{n \to \infty} \|E(a_n^2) - E(a_n)^2\| = 0.$$

Since by continuity we have $\lim_{n \to \infty} E(a_n) = E(x)$ and $\lim_{n \to \infty} E(a_n^2) = E(x^2)$, using (24) it follows that $\|E(x^2) - E(x)^2\| = 0$, so we get the identity

$$(25) \qquad E(x^2) = E(x)^2.$$

By the properties of conditional expectations this yields:

$$E([x - E(x)]^2) = E(x^2 - xE(x) - E(x)x + E(x)^2) = E(x^2) - E(x)^2 = 0.$$

In other words, if we consider the (self-adjoint) element $y = x - E(x)$, we have the equality $E(y^*y) = 0$, which by the faithfulness of E implies $y = 0$, which means that $x = E(x) \in \mathcal{B}$.

(U) Fix a $*$-representation $\pi : \mathcal{A} \to B(H)$, whose restriction $\pi\big|_{\mathcal{B}} : \mathcal{B} \to B(H)$ is injective, and let us show that π itself is injective (thus isometric). Denote $\operatorname{Ker} \pi$ simply by $\mathcal{J}$, so what we must show is the equality $\mathcal{J} = \{0\}$, given that $\mathcal{J} \cap \mathcal{B} = \{0\}$. By construction, it suffices to prove that for each $\sigma \in \Sigma$, the restriction $\pi_\sigma = \pi\big|_{\mathcal{A}_\sigma} : \mathcal{A}_\sigma \to B(H)$ is injective (thus isometric), which is equivalent to the equality $\mathcal{J} \cap \mathcal{A}_\sigma = \{0\}$. Fix σ and some $x \in \mathcal{J} \cap \mathcal{A}_\sigma$, and let us prove that $x = 0$. We can, of course, assume that x is positive, so by the faithfulness of E, it suffices to prove that $E_\sigma(x) = 0$ (in $\mathcal{B}$). By the density of $P_1(\mathcal{B} \!\uparrow\! \mathcal{A}_\sigma)$ in $P(\mathcal{B})$, it suffices to prove that $\phi(E_\sigma(x)) = 0$, $\forall \, \phi \in P_1(\mathcal{B} \!\uparrow\! \mathcal{A}_\sigma)$. Fix such a ϕ, and use Proposition 1.3 to find a net $(b_\lambda)_{\lambda \in \Lambda}$ of positive elements in $\mathcal{B}$ such that $\|b_\lambda\| = \phi(b_\lambda) = 1$, $\forall \, \lambda$, and such that

$$(26) \qquad \lim_{\lambda \in \Lambda} \|b_\lambda x b_\lambda - \phi(E_\sigma(x)) b_\lambda^2\| = 0.$$

Remark now that since $b_\lambda x b_\lambda$ belongs to $\mathcal{J} (= \operatorname{Ker} \pi)$, for every λ, by applying π in (26) we get

$$(27) \qquad 0 = \lim_{\lambda \in \Lambda} \|\pi(\phi(E_\sigma(x)) b_\lambda^2)\| = \lim_{\lambda \in \Lambda} |\phi(E_\sigma(x))| \cdot \|\pi(b_\lambda)^2\|.$$

But now, since we know that π is injective (hence isometric) on $\mathcal{B}$, we have

$$\|\pi(b_\lambda)^2\| = \|\pi(b_\lambda)\|^2 = \|b_\lambda\|^2 = 1, \ \ \forall\, \lambda \in \Lambda,$$

and then (27) forces $\phi(E(x)) = 0$, and we are done. $\qquad\square$

3.2. Corollary. *If $\mathcal{B}$ is a pseudo-diagonal in $\mathcal{A}$, then $\mathcal{B}$ is an abelian core in $\mathcal{A}$.*

3.3. Example. Using the set-up from Example 2.8, it follows that $M(E)$ is an abelian core in $C^*(E)$. As pointed out in [NR], the fact that $M(E)$ has property (U) gives an alternative proof of Szymanski's Uniqueness Theorem from [Sz], which in turn is a generalization of the Uniqueness Theorem for Cuntz algebras.

The last application (compare with [AS, Corollary of Theorem 1]) is the following combined consequence of Theorems 2.11 and 3.1.

3.4. Corollary. *If a (not necessarily countable) discrete group G acts essentially freely on X, then $C_0(X)$ is an abelian core in $C_0(X) \times_{\mathrm{red}} G$.*

In particular, if the action of G on X is also minimal, i.e. the only G-invariant open subsets of X are $\varnothing$ and X, then $C_0(X) \times_{\mathrm{red}} G$ is simple.

Proof. As pointed out right before Lemma 2.9, we do have a faithful conditional expectation E_{red} of $C_0(X) \times_{\mathrm{red}} G$ onto $C_0(X)$. Let Σ be the collection of all countable subgroups of G, equipped with the order $H \succ K \Leftrightarrow H \supset K$, $H, K \in \Sigma$. Since we work with discrete groups, for any subgroup $H \subset G$, we have an inclusion $C_0(X) \times_{\mathrm{red}} H \subset C_0(X) \times_{\mathrm{red}} G$, so we can consider the net $(C_0(X) \times_{\mathrm{red}} H)_{H \in \Sigma}$ of C*-subalgebras in $C_0(X) \times_{\mathrm{red}} G$, which clearly satisfies the equality (even without closure!)

$$C_0(X) \times_{\mathrm{red}} G = \bigcup_{H \in \Sigma} C_0(X) \times_{\mathrm{red}} H.$$

The first statement now follows from Theorem 3.1, since by Theorem 2.11, each inclusion $C_0(X) \subset C_0(X) \times_{\mathrm{red}} H$ has the Almost Extension Property.

To prove the second statement, assume the action is minimal, and let us show that if $\mathcal{J}$ is a closed two-sided ideal in $C_0(X) \times_{\mathrm{red}} G$, then either

(a) $\mathcal{J} = C_0(X) \times_{\mathrm{red}} G$ or
(b) $\mathcal{J} = \{0\}$.

Since $\mathcal{J} \cap C_0(X)$ is a G-invariant closed two-sided ideal in $C_0(X)$, by minimality it follows that either (a') $\mathcal{J} \cap C_0(X) = C_0(X)$ or (b') $\mathcal{J} \cap C_0(X) = \{0\}$. In case (a') it follows immediately that $\mathcal{J} = C_0(X) \times_{\mathrm{red}} G$. In case (b'), since $C_0(X)$ is an abelian core, by property (U) it follows that $\mathcal{J} = \{0\}$. $\qquad\square$

References

[An] Joel Anderson, *Extensions, restrictions, and representations of states on C^*-algebras*, Trans. Amer. Math. Soc. **249** (1979), no. 2, 303–329, DOI 10.2307/1998793. MR525675 (80k:46069)

[AS] R. Archbold, J. Spielberg, *Topologically free actions and ideals in discrete C*-dynamical systems*, Proc. Edinburgh Math. Soc. (2) **37** (1994), 119–124. MR1258035 (94m:46101)

[KS] Richard V. Kadison and I. M. Singer, *Extensions of pure states*, Amer. J. Math. **81** (1959), 383–400. MR0123922 (23 #A1243)

[Ku] Alexander Kumjian, *On C^*-diagonals*, Canad. J. Math. **38** (1986), no. 4, 969–1008, DOI 10.4153/CJM-1986-048-0. MR854149 (88a:46060)

[Na] G. Nagy, *Reduced crossed products via Hilbert modules*, preprint, 2010.

[NR] Gabriel Nagy and Sarah Reznikoff, *Abelian core of graph algebras*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 889–908, DOI 10.1112/jlms/jdr073. MR2927813

[Pa]   V. Paulsen, *Three approaches to Kadison-Singer*, preprint, 2005.

[Ra]   Iain Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, vol. 103, Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the Amer. Math. Soc., Providence, RI, 2005. MR2135030 (2005k:46141)

[Re1]   Jean Renault, *Cartan subalgebras in C*-algebras*, Irish Math. Soc. Bull. **61** (2008), 29–63. MR2460017 (2009k:46135)

[Re2]   Jean Renault, *Examples of masas in C*-algebras*, Operator structures and dynamical systems, Contemp. Math., vol. 503, Amer. Math. Soc., Providence, RI, 2009, pp. 259–265, DOI 10.1090/conm/503/09903. MR2590626

[Sz]   Wojciech Szymański, *General Cuntz-Krieger uniqueness theorem*, Internat. J. Math. **13** (2002), no. 5, 549–555, DOI 10.1142/S0129167X0200137X. MR1914564 (2003h:46083)

[To]   Jun Tomiyama, *On the projection of norm one in W*-algebras*, Proc. Japan Acad. **33** (1957), 608–612. MR0096140 (20 #2635)

DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS 66506
E-mail address: `nagy@math.ksu.edu`

DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS 66506
E-mail address: `sararez@math.ksu.edu`