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FUGLEDE-KADISON DETERMINANTS FOR OPERATORS IN THE VON NEUMANN ALGEBRA OF AN EQUIVALENCE RELATION

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ABSTRACT. We calculate the Fuglede-Kadison determinant for operators of the form $\sum_{i=1}^{n} M_{f_i} L_{g_i}$, where L_{g_i} are unitaries or partial isometries coming from Borel (partial) isomorphisms g_i on a probability space which generate an ergodic equivalence relation and where M_{f_i} are multiplication operators. We obtain formulas for the cases when the relation is treeable or the f_i's and g_i's satisfy some restrictions.

1. INTRODUCTION

The determinant of an (invertible) operator was first introduced in 1952 in the paper [FK]. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace. To summarize the construction let $T \in M$ be normal and $|T| = \sqrt{T^*T}$. By the spectral theorem one can represent T as an integral where $E(\lambda)$ is a projection-valued measure:

$$T = \int_{\sigma(T)} \lambda dE(\lambda).$$

In this setting $\mu_T = \tau \circ E$ becomes a probability measure on the complex plane whose support is the spectrum $\sigma(T)$.

For any $T \in M$ the Fuglede-Kadison determinant is defined by

$$\Delta(T) = \exp\left(\int_0^\infty \log t \, d\mu_{|T|}\right).$$

Notice that if T is invertible, then

$$\Delta(T) = \exp(\tau(\log |T|)),$$

where $\log |T|$ comes from functional calculus.

The determinant has been used recently in the calculation of Brown measures and applied to the invariant subspace problem by Haagerup and Schultz (see [HSc] where Δ is defined for a general class of operators), and it has been connected to the entropy of certain algebraic actions in the work of Bowen, [Bow], Deninger [Den2] and Kerr and Li [KLi]. We point out the paper [Den1], where the determinant

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is calculated for certain operators in the von Neumann algebra generated by the action of the group $\mathbf{Z}$ on a probability space. In this paper we will extend the setting in which such computations can be carried out. Thus, instead of a $\mathbf{Z}$ action, we will consider a standard preserving equivalence relation and the von Neumann algebra it generates by the Feldman-Moore construction ([FMII]). In order to make sure we deal with a finite von Neumann algebra, we will require the equivalence relation to be ergodic, which together with (finite) measure preservation will bring in more, namely a II_1 factor. We set out to calculate the determinant for operators of the form $\sum_{i=1}^{N} M_{f_i} L_{g_i}$, where the g_i are Borel partial isomorphisms that (partly) generate the equivalence relation and the f_i are bounded. A more detailed setup follows.

Let $(X, \mathcal{B}, \mu)$ be a Borel standard probability space without atoms, $\{A\}_{i\in I}$ and $\{B\}_{i\in I}$ two families of measurable subsets of X, and $\Lambda = \{g_i : A_i \to B_i \mid i \in I\}$ a family of measure preserving bijections. Assuming further that the index set I is at most countable, let $\mathcal{R}_\Lambda$ be the equivalence relation generated by the g_i; i.e., $(x, y) \in \mathcal{R}_\Lambda$ if and only if $x = y$ or there exists a map $\omega = g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_k}^{\epsilon_k}$ such that the domain of ω contains x and $\omega x = y$, where all exponents $\epsilon_t = \pm 1$. For simplicity we will denote the fact that two elements x and y are equivalent by $x \sim y$. An equivalence relation $\mathcal{R}$ is (SP1) if it satisfies the properties listed below:

(S) Almost each orbit $\mathcal{R}[x] = \{y \in X,\, y \sim x\}$ is at most countable and $\mathcal{R}$ is a Borel set of $X \times X$.

(P) For any $T \in Aut(X, \mu)$ such that $\operatorname{graph} T \subset \mathcal{R}$, we have that T preserves the measure μ.

It can be shown that an equivalence relation $\mathcal{R}_\Lambda$ with Λ as above is (SP1). This is needed in Proposition 2.2 and Theorem 2.3.

If A is measurable we will denote its saturation with respect to $\mathcal{R}$ by $\mathcal{R}[A] = \{y \in X, y \sim x \text{ for some } x \in A\}$.

Definition 1.1. The equivalence relation $\mathcal{R}$ is called ergodic if for any measurable set $A \in \mathcal{B}$ with $A = \mathcal{R}(A)$ we have $\mu(A) = 0$ or $\mu(A) = 1$.

Definition 1.2. A family of measure preserving bijections $\Lambda = \{g_i : A_i \to B_i \mid i \in I\}$ is a treeing if $\mu\{x \in Dom(\omega) \mid \omega(x) = x\} = 0$ for every non-trivial reduced word ω (i.e., $\omega = g_1^{\varepsilon_1} g_2^{\varepsilon_2} \dots g_k^{\varepsilon_k}$ with $\varepsilon_i = \pm 1$, and if $g_i = g_{i+1}$, then $\varepsilon_i = \varepsilon_{i+1}$) with domain $Dom(\omega)$. An equivalence relation $\mathcal{R}$ is called treeable if there exists a treeing Λ such that $\mathcal{R} = \mathcal{R}_\Lambda$.

The Feldman-Moore construction is based on the following Hilbert space (see [FMII]):

$$L^2(\mathcal{R}) = \{\Psi : \mathcal{R} \to \mathbb{C} \quad \text{such that} \quad ||\Psi||_2 < \infty\}$$

where the norm $||\cdot||_2$ comes from the scalar product:

$$\langle \Psi, \Phi \rangle = \int_X \sum_{z\sim x} \Psi(x, z) \overline{\Phi(x, z)} d\mu(x).$$

For ω a reduced word and $f \in L^\infty(X)$ we consider the operators $L_\omega, M_f : L^2(\mathcal{R}) \to L^2(\mathcal{R})$:

$$(L_\omega \Psi)(x, y) = \chi_{D_\omega}(x) \Psi(\omega^{-1} x, y), \quad D_\omega \text{ the domain of } \omega^{-1}, \tag{1.1}$$

$$(M_f \Psi)(x, y) = f(x)\Psi(x, y). \tag{1.2}$$

If ω is defined on almost all of X, then L_ω is unitary. Direct computations show the following basic properties:

$$L_\omega^* = L_{\omega^{-1}} \quad \text{and} \quad L_{\omega^{-1}} \circ L_\omega = M_{\chi_{D_\omega}}. \tag{1.3}$$

Also, if $\omega = g_1 g_2 \dots g_k$, then $L_\omega = L_{g_1} L_{g_2} \dots L_{g_k}$. These imply that

$$||L_\omega|| \leqslant 1. \tag{1.4}$$

Clearly M_f is a multiplication operator and

$$||M_f|| = ||f||_\infty. \tag{1.5}$$

The closure in the weak topology of the linear span of all products (compositions) made with the operators L_g and M_f is a von Neumann algebra, denoted by $\mathcal{M}(\mathcal{R})$. When $\mathcal{R}$ is ergodic and (SP1), this algebra becomes a II_1-factor. Its trace is given by

$$\tau(T) = \langle T\delta_0, \delta_0 \rangle,$$

where δ_0 is the characteristic function of the diagonal of $\mathcal{R}$.

Examples 1.3. If Γ is a countable group, then any free, ergodic action on a standard probability space gives rise to an (SP1) equivalence relation on that space. Such a relation is generated for example by $g : X \to X$, $\forall g \in \Gamma$; however, it may be done in a more efficient way in case Γ possesses generators. In turn, there is always an action around: any countable group acts freely and ergodically on $X = \{0,1\}^\Gamma$ equipped with the product measure by means of the Bernoulli shifts. Many countable groups gives rise to treeable, ergodic equivalence relations such as those coming from actions of the free groups or the amenable ones. Moreover, it is not necessary that the domains of the generators be all of X: there are groups of non-integer costs; hence some of their generators must be defined on "chopped off" measurable pieces. For a detailed discussion on costs of groups and treeability and for more examples of (SP1) equivalence relations, we refer the reader to [Gab]. Finally, we note that if the equivalence relation comes from the action of a countable group $\alpha : \Gamma \to \mathrm{Aut}(X, \mu)$, then the Feldman-Moore construction yields the usual cross-product von Neumann algebra $L^\infty(X) \rtimes_\alpha \Gamma$.

Going back to the determinant let us recall the following familiar properties:

$$\text{For } n \times n \text{ matrices } \tau = \mathrm{Tr}/n : \ \Delta(T) = \sqrt[n]{|\det T|},$$

$$\Delta(ST) = \Delta(S)\Delta(T),$$

$$\Delta(S) = \Delta(|S|) = \Delta(S^*),$$

$$\Delta(U) = 1, \text{ where } U \text{ is unitary},$$

$$\Delta(\lambda I) = |\lambda|.$$

Although the determinant is not necessarily continuous, it is upper-semicontinuous, both in the strong operator topology and the norm topology.

The following formula can be proven using the description of the spectral resolution of a multiplication operator and functional calculus (see the proof of Theorem 2.5 below):

$$\log \Delta(M_f) = \int_X \log |f| \, d\mu(x). \tag{1.6}$$

The goal of the present work is to compute the Fuglede-Kadison determinant of the operator $T \in \mathcal{M}(\mathcal{R})$:

$$T = \sum_{i=1}^{N} M_{f_i} L_{g_i}. \tag{1.7}$$

We will accomplish this under some restrictions. First we will recall the following proposition due to Deninger (see [Den1]):

Proposition 1.4 (Deninger)**.** *For an operator Φ in a finite von Neumann algebra with a trace $\tau(1) = 1$, the formula*

$$\Delta(z - \Phi) = |z| \tag{1.8}$$

holds if the following two conditions are satisfied:

$$r(\Phi) < |z| \tag{1.9}$$

and

$$\tau(\Phi^n) = 0, \quad \forall\ n \geq 1, \tag{1.10}$$

where $r(\Phi)$ stands for the the spectral radius of the operator Φ.

Remark 1.5. The case $n = 2$ in (1.7) can be dealt with by following the methods in [Den1] provided that g_1 and g_2 are full Borel isomorphisms, that is, $g_i : X \to X$. Assuming further that $g_1^{-1} g_2$ or $g_2^{-1} g_1$ is ergodic, then one can embed the calculation of the determinant in the hyperfinite II_1-factor generated by the $\mathbf{Z}$-action of $g_1^{-1} g_2$ or $g_2^{-1} g_1$ on the probability space X (notice that the ergodicity of an equivalence relation does not guarantee that of a subrelation). Thus our results will be relevant for the case $n \geqslant 3$. Moreover, we will allow all of the g_i but one to be Borel partial isomorphisms. However, this level of generality has a slight drawback in the fact that we will impose some requirements on the domains of either the f_i or the g_i.

2. Main results

We will first need the following simple but important observation:

Lemma 2.1. *Let $g : A \to B$ be a Borel partial isomorphism and denote by fg^{-1} the function $(fg^{-1})(x) = f(g^{-1}x)$. Then we have*

$$L_g M_f = M_{fg^{-1}} L_g, \tag{2.1}$$

$$(M_f L_g)^n = M_{f \cdot fg^{-1} \cdot \ldots \cdot fg^{-(n-1)}} L_{g^n}. \tag{2.2}$$

Proof. This follows from a direct computation:

$$(L_g M_f)(\Psi)(x, y) = L_g(f(\cdot)\Psi(\cdot, \cdot))(x, y) = \chi_B(x) f(g^{-1}x)\Psi(g^{-1}x, y).$$

On the other hand,

$$\begin{aligned} (M_{fg^{-1}} L_g)(\Psi)(x, y) &= M_{fg^{-1}}(\chi_B(\cdot)\Psi(g^{-1}\cdot, \cdot))(x, y) \\ &= f(g^{-1}x)\chi_B(x)\Psi(g^{-1}x, y). \end{aligned}$$

Next, we can write $(M_f L_g)^n = M_f (L_g M_f)^{n-1} L_g$. Now (2.2) follows by applying (2.1) repeatedly. $\square$

Proposition 2.2. *Let Λ be a treeing and assume that the equivalence relation $\mathcal{R}_\Lambda$ is ergodic. If N is a positive integer and $g_i \in \Lambda$, $f_i \in L^\infty(X)$ $\forall i \in \{1, ..., N\}$, then for $\Phi = \sum_{i=1}^N M_{f_i} L_{g_i}$ we have*

$$\tau(\Phi^n) = 0, \quad \forall n \geqslant 1.$$

Proof. Notice that Φ^n will be a sum of terms having the format $M_{f_{i_1}} \circ L_{g_{i_1}} \circ \cdots \circ M_{f_{i_n}} \circ L_{g_{i_n}}$, where $i_1, \ldots, i_n$ belong to the index set $\{1, 2, \ldots, N\}$. Applying (2.1) successively we obtain that these terms are equal to an operator having the form $M_h L_{g_1 g_2 \ldots g_k}$, where $h = f_{i_1} \cdot f_{i_2}(g_{i_1}^{-1}) \cdot f_{i_3}(g_{i_2}^{-1} g_{i_1}^{-1}) \cdots f_{i_k}(g_{i_{k-1}}^{-1} \cdots g_{i_1}^{-1})$. This shows that it suffices to calculate the trace of the operators $M_h L_\omega$, where $\omega = g_{i_1} g_{i_2} \ldots g_{i_k}$. We have

$$\begin{aligned}
\tau(M_h L_\omega) = \langle M_h L_\omega \delta_0, \delta_0 \rangle &= \int_X \sum_{z \sim x} \left(M_h (L_\omega \delta_0)(x, z) \overline{\delta_0(x, z)} \right) d\mu(x) \\
&= \int_X M_h (L_\omega \delta_0)(x, x) d\mu(x) = \int_{\{x | \omega(x) = x\}} h(x) d\mu(x) = 0,
\end{aligned}$$

since the equivalence relation is treeable. □

Theorem 2.3. *Let Λ be a treeing such that the equivalence relation $\mathcal{R}_\Lambda$ is ergodic. For $n \geq 1$ and $i \in \{1, \ldots, n\}$ let $g_i \in \Lambda$, $f_i \in L^\infty(X)$, and $T = \sum_{i=1}^n M_{f_i} L_{g_i}$ in $\mathcal{M}(\mathcal{R}_\Lambda)$. Assume that there is an index i_0 such that*

$$\sum_{i \neq i_0} ||f_i / f_{i_0}||_\infty < 1, \tag{2.3}$$

f_{i_0} is non-vanishing on sets of positive measure and

$$g_{i_0} : A_{i_0} = X \to B_{i_0} = X. \tag{2.4}$$

Then

$$\log \Delta(T) = \int_X \log |f_{i_0}| d\mu(x). \tag{2.5}$$

Proof. Using (2.1) we can rewrite the operator T as

$$T = M_{f_{i_0}} L_{g_{i_0}} \left[I + \sum_{i=1}^n M_{f_i / f_{i_0} g_{i_0}} L_{g_{i_0}^{-1} g_i} \right], \tag{2.6}$$

where condition (2.4) was used to ensure that $L_{g_{i_0}} L_{g_{i_0}^{-1}} = I$.

Notice that the operator $\Phi = \sum_{i \neq i_0}^n M_{f_i / f_{i_0} g_{i_0}} L_{g_{i_0}^{-1} g_i}$ satisfies (1.10) by Proposition 2.2. Notice also that the spectral radius of Φ satisfies

$$\begin{aligned}
r(\Phi) &\leqslant \sum_{i \neq i_0} ||M_{f_i / f_{i_0} g_{i_0}}|| \cdot ||L_{g_{i_0}}|| \cdot ||L_{g_i}|| \\
&\leqslant \sum_{i \neq i_0} ||M_{f_i / f_{i_0} g_{i_0}}|| \\
&\leqslant \sum_{i \neq i_0} ||f_i / f_{i_0}||_\infty < 1
\end{aligned}$$

where we used (1.4) and (1.5). So (1.9) is satisfied and we can apply Proposition 1.4 with $z = 1$ to conclude that

$$\Delta(I + \Phi) = 1.$$

Taking the determinant on both sides of (2.6) we obtain (2.5). □

Remark 2.4. In [Den1] the equivalence relation $\mathcal{R}$ comes from the action of a single ergodic automorphism $\gamma : X \to X$, and among other things a formula for $\Delta(T)$ is found when $n = 2$. This is done without the requirement (2.3). It would be interesting to extend the calculation of $\Delta(T)$ beyond the restriction (2.3).

For our next result we will consider Borel partial isomorphisms $g_i : A_i \to B_i$ that do not generate unitaries in $\mathcal{M}(\mathcal{R})$. In particular, we will not be able to proceed as in Theorem 2.3. However under some restrictions on the domains of the partial isomorphisms, a calculation is still possible, even without the treeability assumption.

Theorem 2.5. *Let $f_i \in L^\infty(X)$ and $g_i : A_i \to B_i$, $i = 1, ..., n$ be Borel partial isomorphisms in the standard probability space $(X, \mathcal{B}, \mu)$ such that the g_i are among the generators of an (SP1) equivalence relation. If the following conditions are satisfied:*

$$\mu(A_1 \cup A_2 \ldots \cup A_n) = 1, \tag{2.7}$$

$$\mu(B_i \cap B_j) = 0 \text{ if } i \neq j, \tag{2.8}$$

then the Fuglede-Kadison determinant of the operator $T = \sum_{i=1}^n M_{f_i} L_{g_i}$ is given by

$$\log \Delta(T) = \sum_{i=1}^n \int_{B_i} \log |f_i| d\mu(x). \tag{2.9}$$

Proof. First we will prove

$$\mu(A_i \cap A_j) = 0 \text{ if } i \neq j. \tag{2.10}$$

Let $A := \bigcup_{i=2}^n A_i$. We have

$$\begin{aligned} 1 &\geqslant \mu(\bigcup_{i=1}^n B_i) = \sum_{i=1}^n \mu(B_i) = \sum_{i=1}^n \mu(A_i) \\ &\geqslant \mu(A_1) + \mu(A) = \mu(A_1 \cup A) + \mu(A_1 \cap A) \\ &= 1 + \mu(A_1 \cap A). \end{aligned}$$

Hence $\mu(A_1 \cap A) = 0$ and (2.10) follows for $i = 1$. Analogously we obtain (2.10) for all i.

Notice that the hypothesis $\mu(B_i \cap B_j) = 0$ insures $L_{g_i}^* L_{g_j} = 0$ for $i \neq j$. Indeed, for $\psi \in L^2(\mathcal{R})$,

$$L_{g_i}^* L_{g_j} \psi(\cdot, \cdot) = \chi_{D_{g_j^{-1} g_i}} \psi(g_j^{-1} g_i \cdot, \cdot) = 0$$

because the maximum domain of the word $g_j^{-1} g_i$ is $D_{g_j^{-1} g_i} = g_i^{-1}(B_i \cap B_j) \cap A_i$.

Using this last observation and (2.1) we show that T^*T is a multiplication operator:

$$\begin{aligned} T^*T &= \sum_{i,j} L_{g_i}^* M_{\overline{f_i} f_j} L_{g_j} = \sum_{i,j} M_{\overline{f_i} f_j g_i} L_{g_i}^* L_{g_j} \\ &= \sum_i M_{|f_i|^2 g_i} M_{\chi_{A_i}} = \sum_i M_{|f_i|^2 g_i \cdot \chi_{A_i}} \\ &= M_{\sum_i |f_i|^2 g_i \cdot \chi_{A_i}}. \end{aligned}$$

Let's denote $h := \sum_i |f_i|^2 g_i \cdot \chi_{A_i}$. From the spectral theorem

$$\tau(\log T^*T) = \int \log \lambda d\tau(E_\lambda),$$

where (E_λ) is the spectral resolution of the multiplication operator M_h. The spectral resolution has an explicit form in this case (see for example [Co]): for any Borel set $A \subset \sigma(M_h)$, $E(A) = M_{\chi_{h^{-1}A}}$ and $\tau(M_{\chi_{h^{-1}A}}) = \mu(h^{-1}A)$. Similarly to the classic multiplication operators on $L^2(X)$ and the fact that on $\mathcal{M}(\mathcal{R})$ the trace is a vector-trace, we have

$$\int \log \lambda d\tau(E_\lambda) = \int_X \log h(x) d\mu(x).$$

Combining this with the definition of the determinant,

$$2 \log \Delta(T) = \tau(\log T^*T) = \int_X \log h(x) d\mu(x).$$

Because of (2.7) and (2.10) we can continue with

$$\begin{aligned} 2\log\Delta(T) &= \sum_i \int_{A_i} \log h(x) d\mu(x) = \sum_i \int_{A_i} \log |f_i g_i(x)|^2 d\mu(x) \\ &= 2 \sum_i \int_{B_i} \log |f_i(x)| d\mu(x). \end{aligned}$$

Now (2.9) follows. □

Remark 2.6. The theorem above is true if we switch (2.7) with (2.10). Similar arguments can be used to show that (2.10) and (2.7) imply (2.8).

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