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INEQUALITIES FOR THE SECOND COHOMOLOGY OF FINITE DIMENSIONAL LIE ALGEBRAS

ALI REZA SALEMKAR, BEHROUZ EDALATZADEH, AND HAMID MOHAMMADZADEH

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Abstract. We will extend the Hochschild-Serre spectral sequence for cohomology of Lie algebras a step further. Also, some inequalities and upper bounds for the dimension of the second cohomology of finite dimensional Lie algebras will be given.

1. Introduction

All Lie algebras are considered over a fixed field Λ and $[\ ,\ ]$ denotes the Lie bracket. Let L be a Lie algebra, K an ideal of L and M an L-module. Let $H_n(L,M)$ and $H^n(L,M)$ denote the nth *homology* and nth *cohomology* of L with coefficients in M, respectively. Then the Hochshild-Serre spectral sequence for homology and cohomology of Lie algebras yields the following exact sequences:

(1)
$$\mathrm{H}_2(L,M)\longrightarrow \mathrm{H}_2(\frac{L}{K},M_K)\longrightarrow \mathrm{H}_1(K,M)_{L/K}\longrightarrow \mathrm{H}_1(L,M)\longrightarrow \mathrm{H}_1(\frac{L}{K},M_K)\longrightarrow 0,$$

(2)
$$0\longrightarrow \mathrm{H}^1(\frac{L}{K},M^K)\longrightarrow \mathrm{H}^1(L,M)\longrightarrow \mathrm{H}^1(K,M)^{L/K}\longrightarrow \mathrm{H}^2(\frac{L}{K},M^K)\longrightarrow \mathrm{H}^2(L,M).$$

Here M^K and M_K are the invariant and the coinvariant submodules of M, respectively, in which M is regarded as an L/K-module [10].

Consider the field Λ as a trivial L-module. In 1991, Ellis [4] showed that the exact sequence (1) can be extended to an infinite long exact sequence. In particular, he obtained the following exact sequence:

$$\mathrm{H}_2(L;K)\longrightarrow \mathrm{H}_2(L,\Lambda)\longrightarrow \mathrm{H}_2(\frac{L}{K},\Lambda)\longrightarrow \mathrm{H}_1(L;K)\longrightarrow \mathrm{H}_1(L,\Lambda)\longrightarrow \mathrm{H}_1(\frac{L}{K},\Lambda)\longrightarrow 0,$$

where $\mathrm{H}_1(L;K)\cong L/[L,K]$ and $\mathrm{H}_2(L;K)$ is the kernel of the commutator map $L\wedge K\longrightarrow L$ (here $L\wedge K$ denotes the non-abelian exterior product of Lie algebras [3]).

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On the other hand, assuming that L is a finite dimensional nilpotent Lie algebra with central ideal K, Yankosky [11] defined a homomorphism $\delta : \mathrm{H}^2(L,\Lambda)_K \longrightarrow (L/(L^2+K))\otimes K$ and generalized the exact sequence (2) as follows:

$$
\begin{aligned}
(3)\qquad & 0 \longrightarrow \mathrm{Hom}(\frac{L}{K},\Lambda) \xrightarrow{\mathrm{Inf}} \mathrm{Hom}(L,\Lambda) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(K,\Lambda) \xrightarrow{\mathrm{Trans}} \mathrm{H}^2(\frac{L}{K},\Lambda)\\
& \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(L,\Lambda)_K \xrightarrow{\delta} \frac{L}{L^2+K}\otimes K,
\end{aligned}
$$

in which $H^2(L,\Lambda)_K$ is the kernel of the restriction homomorphism $H^2(L,\Lambda) \longrightarrow H^2(K,\Lambda)$.

In this article, for any Lie algebra L with an ideal K, we define a homomorphism $\delta : \mathrm{H}^2(L,\Lambda)_K \longrightarrow \mathrm{H}^1(L/K, \mathrm{Hom}(K,\Lambda))$ in which the action of L/K on $\mathrm{Hom}(K,\Lambda)$ is given by

$$
{}^{(l+K)}f(k) = -f([l,k]) \text{ for all } l\in L,\ k\in K,\ f\in \mathrm{Hom}(K,\Lambda).
$$

Using this homomorphism, we establish that the following sequence is exact:

$$
\begin{aligned}
(4)\qquad & 0 \longrightarrow \mathrm{Hom}(\frac{L}{K},\Lambda) \xrightarrow{\mathrm{Inf}} \mathrm{Hom}(L,\Lambda) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(K,\Lambda) \xrightarrow{\mathrm{Trans}} \mathrm{H}^2(\frac{L}{K},\Lambda)\\
& \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(L,\Lambda)_K \xrightarrow{\delta} \mathrm{H}^1(\frac{L}{K}, \mathrm{Hom}(K,\Lambda)),
\end{aligned}
$$

which is the vast generalization of the sequence (3). In addition, some inequalities for the dimension of the second cohomology of finite dimensional Lie algebras will be given.

2. The main results

Let L be a Lie algebra with an ideal K, and let the field Λ be regarded as a trivial L-module. We choose an arbitrary cocycle $\alpha \in Z^2(L,\Lambda)$ such that the restriction of α to $K\times K$ belongs to $B^2(K,\Lambda)$. Then there exists a linear map $f_\alpha : K \to \Lambda$ such that $\alpha|_{K\times K}(k,k') = -f_\alpha([k,k'])$, for all $k,k'\in K$. Clearly, f_α can be extended to a linear map $g_\alpha : L\to\Lambda$. We consider the map $\gamma : L\times L\to \Lambda$ given by $\gamma(l,l') = -g_\alpha([l,l'])$. It is obvious that $\gamma\in B^2(L,\Lambda)$. If we put $\beta = \alpha-\gamma$, then $\beta\in Z^2(L,\Lambda)$, β is cohomologous to α and $\beta|_{K\times K} = 0$. Therefore, without loss of generality, we may assume that the restriction of α to $K\times K$ is zero and define the following mapping:

$$
\delta : \mathrm{H}^2(L,\Lambda)_K \to \mathrm{H}^1(L/K,\mathrm{Hom}(K,\Lambda)),\ \alpha + B^2(L,\Lambda) \longmapsto \bar{\alpha} + B^1(L/K,\mathrm{Hom}(K,\Lambda)),
$$

where $\bar{\alpha} : L/K \to \mathrm{Hom}(K,\Lambda)$ is a linear map given by $\bar{\alpha}(l+K)(k) = \alpha(l,k)$, for all $l\in L, k\in K$. Note that $\bar{\alpha}\in Z^1(L/K,\mathrm{Hom}(K,\Lambda))$ because the action of L/K on $Z^1(L/K,\mathrm{Hom}(K,\Lambda))$ yields that

$$
\begin{aligned}
&{}^{(l+K)}\bar{\alpha}(l'+K)(k) - {}^{(l'+K)}\bar{\alpha}(l+K)(k) - \bar{\alpha}([l,l']+K)(k)\\
&= -\alpha(l',[l,k]) + \alpha(l,[l',k]) - \alpha([l,l'],k) = 0.
\end{aligned}
$$

If $\alpha\in B^2(L,\Lambda)$, then there exists a linear map $h : L\to\Lambda$ such that $\alpha(l,l') = -h([l,l'])$, and consequently

$$
\bar{\alpha}(l+K)(k) = -h([l,k]) = {}^{(l+K)}h(k).
$$

Therefore, δ is a well-defined homomorphism. Now to prove the exactness of the sequence (4), it is sufficient to show the following.

Theorem 2.1. *Using the above assumptions and notation, the following sequence is exact:*

$$\mathrm{H}^2(\frac{L}{K},\Lambda) \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(L,\Lambda)_K \xrightarrow{\delta} \mathrm{H}^1(\frac{L}{K},\mathrm{Hom}(K,\Lambda)).$$

Proof. Let $\alpha + B^2(L,\Lambda) = \mathrm{Inf}(\beta + B^2(L/K,\Lambda))$. Then, by the definition of an inflation map, we must have $\alpha(l,l') = \beta(l+K,l'+K)$, for all $l,l' \in L$, and hence

$$\bar{\alpha}(l+K)(k) = \alpha(l,k) = \beta(l+K,k+K) = \beta(l+K,0) = 0.$$

It therefore follows that $\mathrm{Im}(\mathrm{Inf}) \subseteq \ker(\delta)$.

Conversely, let $\alpha + B^2(L,\Lambda) \in \ker(\delta)$. We first show that there exists a cocycle $\beta \in Z^2(L,\Lambda)$ cohomologous to α such that $\beta|_{L\times K} = 0$. As $\bar{\alpha} \in B^1(L/K,\mathrm{Hom}(K,\Lambda))$, there exists a linear map $f : K \to \Lambda$ such that $\bar{\alpha}(l+K) = {}^{(l+K)}f$, for all $l \in L$. Assume $g : L \to \Lambda$ is a linear map with $g|_K = f$ and put $\alpha'(l,l') = g([l,l'])$. Clearly $\alpha' \in B^2(L,\Lambda)$ and the cocycle $\beta' = \alpha - \alpha'$ satisfies $\beta'|_{L\times K} = 0$. So, we may assume that $\alpha|_{L\times K} = 0$ for any $\alpha + B^2(L,\Lambda) \in \ker(\delta)$. By the anti-symmetric property of cochains, we have $\alpha|_{K\times L} = 0$. We now define the mapping $\beta : L/K \times L/K \to \Lambda$ given by $\beta(l+K,l'+K) = \alpha(l,l')$. β is well-defined, for if $l+K = l_1+K$ and $l'+K = l'_1+K$, then $l_1 = l+k_1$ and $l'_1 = l'+k_2$ for some $k_1,k_2 \in K$. But this implies that

$$\begin{aligned}\beta(l_1+K,l'_1+K) &= \alpha(l_1,l'_1) = \alpha(l+k_1,l'+k_2)\\ &= \alpha(l,l') + \alpha(k_1,l') + \alpha(l,k_2) + \alpha(k_1,k_2)\\ &= \alpha(l,l') = \beta(l+K,l'+K).\end{aligned}$$

It is readily verified that $\beta \in Z^2(L/K,\Lambda)$ and $\alpha + B^2(L,\Lambda) = \mathrm{Inf}(\beta + \mathrm{B}^2(\mathrm{L/K},\Lambda))$. Thus, $\ker(\delta) \subseteq \mathrm{Im}(\mathrm{Inf})$, and hence the required assertion follows. $\square$

In the above theorem, if L is finite dimensional and K is central, then L/K acts trivially on $\mathrm{Hom}(K,\Lambda)$, and consequently

$$\mathrm{H}^1(\frac{L}{K},\mathrm{Hom}(K,\Lambda)) \cong \mathrm{Hom}(\frac{L}{K},K) \cong \mathrm{Hom}(\frac{L}{L^2+K},K) \cong \frac{L}{L^2+K} \otimes K.$$

The result shows that the sequence (3) may be inferred from the sequence (4) .

In the following, using the exact sequence (4), we give some inequalities for the dimension of the second cohomology of finite dimensional Lie algebras, which are analogous to Vermani's results in group theory [9].

Proposition 2.2. *Let L be a finite dimensional Lie algebra with ideal K such that the restriction homomorphism* $\mathrm{Res} : \mathrm{H}^2(L,\Lambda) \longrightarrow \mathrm{H}^2(K,\Lambda)$ *is zero. Then*

$$\begin{aligned}\dim(\mathrm{H}^2(L,\Lambda)) + \dim(K\cap L^2) \le\ & \dim(\mathrm{H}^2(\frac{L}{K},\Lambda)) + \dim(H^1(\frac{L}{K},\mathrm{Hom}(\frac{K}{K^2},\Lambda)))\\ & + \dim(K^2).\end{aligned}$$

In particular, if K is central, then

$$\dim(\mathrm{H}^2(L,\Lambda)) + \dim(K\cap L^2) \le \dim(\mathrm{H}^2(\frac{L}{K},\Lambda)) + \dim(\frac{L}{L^2+K} \otimes K).$$

Proof. It follows from the sequence (4) that

$$\begin{aligned}\dim(\mathrm{H}^2(L,\Lambda)) \leq\ & \dim(\mathrm{Im}(\mathrm{Inf}:\mathrm{H}^2(\frac{L}{K},\Lambda)\to \mathrm{H}^2(L,\Lambda)))\\ &+\dim(H^1(\frac{L}{K},\mathrm{Hom}(\frac{K}{K^2},\Lambda))),\end{aligned}$$

and $\dim(\mathrm{Im}(\mathrm{Inf}:\mathrm{H}^2(\frac{L}{K},\Lambda)\to \mathrm{H}^2(L,\Lambda)))$ is equal to

$$\begin{aligned}&\dim(H^2(\frac{L}{K},\Lambda)) - \dim(\mathrm{Hom}(K,\Lambda)) + \dim(\mathrm{Im}(\mathrm{Res}:\mathrm{Hom}(L,\Lambda)\to \mathrm{Hom}(K,\Lambda)))\\ &= \dim(H^2(\frac{L}{K},\Lambda)) - \dim(\mathrm{Hom}(K,\Lambda)) + \dim(\mathrm{Hom}(L,\Lambda)) - \dim(\mathrm{Hom}(\frac{L}{K},\Lambda))\\ &= \dim(\mathrm{H}^2(\frac{L}{K},\Lambda)) - \dim(\frac{K}{K^2}) + \dim(\frac{L}{L^2}) - \dim(\frac{L}{L^2+K})\\ &= \dim(H^2(\frac{L}{K},\Lambda)) - \dim(\frac{K}{K^2}) + \dim(\frac{L^2+K}{L^2})\\ &= \dim(\mathrm{H}^2(\frac{L}{K},\Lambda)) + \dim(K^2) - \dim(K\cap L^2).\end{aligned}$$

Hence the results hold. □

Note that if K is a 1-dimensional ideal of a finite dimensional Lie algebra, then the restriction homomorphism $\mathrm{Res}: H^2(L,\Lambda)\to H^2(K,\Lambda)$ is trivially zero. In this case, when $K\subseteq Z(L)\cap L^2$, the above proposition reduces to [11, Corollary 3.2].

In view of Proposition 2.2, we have the following corollary.

Corollary 2.3. *Let L and K be as in Proposition 2.2. If L/K is a nilpotent Lie algebra of dimension m and $\dim((L^2+K)/K)=r$, then*

$$\dim(\mathrm{H}^2(L,\Lambda)) + \dim(\frac{L^2}{K^2}) \leq \dim(H^1(N,\mathrm{Hom}(\frac{K}{K^2},\Lambda))) + \frac{1}{2}(m-r)(m+r-1).$$

In particular, if $K\subseteq Z(L)$, then

$$\dim(\mathrm{H}^2(L,\Lambda)) + \dim(L^2) \leq \dim(\frac{L}{L^2+K}\otimes K) + \frac{1}{2}(m-r)(m+r-1).$$

Proof. By [8, Corollary 3.3] and [7, Lemma 22], we have

$$\begin{aligned}\dim(\mathrm{H}^2(\frac{L}{K},\Lambda))+\dim(\frac{L^2+K}{K}) &\leq \dim(\mathrm{H}^2(\frac{L}{L^2+K},\Lambda))+\dim(\frac{L^2+K}{K})d(\frac{L/K}{Z(L/K)})\\ &\leq \frac{1}{2}(m-r)(m-r-1)+r(m-r)\\ &= \frac{1}{2}(m-r)(m+r-1).\end{aligned}$$

It now follows from Proposition 2.2 that

$$\begin{aligned}\dim(\mathrm{H}^2(L,\Lambda)) + \dim(\frac{L^2}{K^2}) \leq\ & \dim(H^1(\frac{L}{K},\mathrm{Hom}(\frac{K}{K^2},\Lambda)))\\ &+\dim(\mathrm{H}^2(\frac{L}{K},\Lambda)) + \dim(\frac{L^2+K}{K})\\ \leq\ & \dim(H^1(\frac{L}{K},\mathrm{Hom}(\frac{K}{K^2},\Lambda))) + \frac{1}{2}(m-r)(m+r-1),\end{aligned}$$

as required. □

To give some other corollaries of the exact sequence (4), we first need the following proposition.

Proposition 2.4. *Let $0 \to K \to L \to N \to 0$ be an exact sequence of finite dimensional Lie algebras. Then the following sequence is exact:*

$$0 \to \frac{K}{[L,K]} \to \frac{K}{K^2} \to \frac{N}{N^2} \otimes \frac{K}{K^2} \to H^1(N, \mathrm{Hom}(\frac{K}{K^2}, \Lambda)) \to 0.$$

Proof. Let UN and IN denote the universal enveloping algebra and the augmentation ideal of N, respectively. Then the UN-free presentation $0 \to IN \to UN \to \Lambda \to 0$ of Λ gives rise to an exact sequence

$$0 \to \mathrm{Hom}(\frac{K}{K^2}, \Lambda) \to \mathrm{Hom}(UL, \mathrm{Hom}(\frac{K}{K^2}, \Lambda)) \to \mathrm{Hom}(IN, \mathrm{Hom}(\frac{K}{K^2}, \Lambda)) \to 0.$$

Applying the functor $\mathrm{Hom}_N(\Lambda, -)$ to the above sequence, we get an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_N(\Lambda, \mathrm{Hom}(K/K^2, \Lambda)) \to \mathrm{Hom}_N(\Lambda, \mathrm{Hom}(UN, \mathrm{Hom}(K/K^2, \Lambda))) \\ &\to \mathrm{Hom}_N(\Lambda, \mathrm{Hom}(IN, \mathrm{Hom}(K/K^2, \Lambda))) \to \mathrm{H}^1(N, \mathrm{Hom}(K/K^2, \Lambda)) \\ &\to \mathrm{H}^1(N, \mathrm{Hom}(UN, \mathrm{Hom}(K/K^2, \Lambda))) \to ... \end{aligned}$$

or, equivalently,

$$\begin{aligned} 0 &\to \mathrm{Hom}_N(K/K^2, \Lambda) \to \mathrm{Hom}_N(UN, \mathrm{Hom}(K/K^2, \Lambda)) \\ &\to \mathrm{Hom}_N(IN, \mathrm{Hom}(K/K^2, \Lambda)) \\ &\to \mathrm{H}^1(N, \mathrm{Hom}(K/K^2, \Lambda)) \to \mathrm{H}^1(N, \mathrm{Hom}(UN, \mathrm{Hom}(K/K^2, \Lambda))) \to \dots . \end{aligned} \tag{5}$$

We show that $\mathrm{H}^1(N, \mathrm{Hom}(UN, \mathrm{Hom}(K/K^2, \Lambda))) = 0$. Assume $\mathbf{P}$ is a UN-projective resolution of Λ. As

$$\begin{aligned} \mathrm{Hom}_N(\mathbf{P}, \mathrm{Hom}(UN, \mathrm{Hom}(K/K^2, \Lambda))) &\cong \mathrm{Hom}(\mathbf{P} \otimes_N UN, \mathrm{Hom}(K/K^2, \Lambda)) \\ &\cong \mathrm{Hom}(\mathbf{P}, \mathrm{Hom}(K/K^2, \Lambda)), \end{aligned}$$

we conclude that $\mathrm{H}^1(N, \mathrm{Hom}(UN, \mathrm{Hom}(K/K^2, \Lambda))) \cong \mathrm{Ext}^1_\Lambda(\Lambda, \mathrm{Hom}(K/K^2, \Lambda)) = 0$. Also, according to some results obtained in Sections 4 and 5 of Chapter 7 in [10], we have

$$\begin{aligned} \mathrm{Hom}_N(K/K^2, \Lambda) &\cong \mathrm{Hom}_N(K/K^2, \mathrm{Hom}(\Lambda, \Lambda)) \cong \mathrm{Hom}(K/K^2 \otimes_N \Lambda, \Lambda) \\ &\cong K/K^2 \otimes_N \Lambda \cong K/[K, L], \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_N(IN, \mathrm{Hom}(K/K^2, \Lambda)) &\cong \mathrm{Hom}(IN \otimes_N K/K^2, \Lambda) \\ &\cong \mathrm{Hom}((IN \otimes_N UN/IN) \otimes K/K^2, \Lambda) \\ &\cong \mathrm{Hom}((IN)/(IN)^2 \otimes K/K^2, \Lambda) \\ &\cong \mathrm{Hom}(N/N^2 \otimes K/K^2, \Lambda) \cong N/N^2 \otimes K/K^2, \end{aligned}$$

and $\mathrm{Hom}_N(UN, \mathrm{Hom}(K/K^2, \Lambda)) \cong K/K^2$. The result therefore follows by replacing these isomorphisms in the sequence (5). $\square$

Now we are ready to state the following corollary, which is analogous to the work of T. Gokulchandra Singh [5] for the group case.

Corollary 2.5. *Let L be a finite dimensional Lie algebra with ideal K, and $N = L/K$. If the inflation homomorphism* $\mathrm{Inf} : \mathrm{H}^2(N,\Lambda) \to \mathrm{H}^2(L,\Lambda)$ *is zero, then*

$$\dim(\mathrm{H}^2(L,\Lambda)) \leq \dim(\mathrm{H}^2(K,\Lambda)) + \dim(\frac{N}{N^2} \otimes \frac{K}{K^2}) + \dim(K^2) - \dim([L,K]).$$

In particular, if $\dim(N) = 1$, *then* $\dim(\mathrm{H}^2(L,\Lambda)) \leq \dim(\mathrm{H}^2(K,\Lambda)) + \dim(K/[L,K])$.

Proof. The natural exact sequence $0 \to \mathrm{H}^2(L,\Lambda)_K \to \mathrm{H}^2(L,\Lambda) \overset{\mathrm{Res}}{\to} \mathrm{H}^2(K,\Lambda)$ shows that

$$\dim(\mathrm{H}^2(L,\Lambda)) \leq \dim(\mathrm{H}^2(K,\Lambda)) + \dim(\mathrm{H}^2(L,\Lambda)_K).$$

Also, by the assumption and Theorem 2.1,

$$\dim(\mathrm{H}^2(L,\Lambda)_K) \leq \dim(\mathrm{H}^1(N, \mathrm{Hom}(K/K^2,\Lambda))).$$

We therefore conclude from Proposition 2.4 that

$$\begin{aligned}\dim(\mathrm{H}^2(L,\Lambda)) \leq\ & \dim(\mathrm{H}^2(K,\Lambda)) + \dim(\mathrm{H}^1(N, \mathrm{Hom}(K/K^2,\Lambda))) \\ =\ & \dim(\mathrm{H}^2(K,\Lambda)) + \dim(N/N^2 \otimes K/K^2) \\ & + \dim(K/[L,K]) - \dim(K/K^2) \\ =\ & \dim(\mathrm{H}^2(K,\Lambda)) + \dim(N/N^2 \otimes K/K^2) + \dim(K^2) - \dim([L,K]).\end{aligned}$$

This proves the corollary. □

It was established by Vermani [9] that if G is a p-group of order p^n with the centre of exponent p^k, then $|G'||H^2(G,T)|$ is no more than $p^{\frac{1}{2}(n-k)(n+k-1)}$, where T is the additive group of rationals mod 1 regarded as a trivial G-module (see also [6, Corollary 2.3]). In particular, if G is non-abelian, then $|H^2(G,T)| \leq p^{\frac{1}{2}(n-2)(n+1)}$. We now get a similar result for the case of Lie algebras, which generalizes Lemma 22 of Moneyhun [7] for finite dimensional non-abelian nilpotent Lie algebras.

Corollary 2.6. *Let L be a non-abelian nilpotent Lie algebra of dimension n. Then*

$$\dim(\mathrm{H}^2(L,\Lambda)) \leq \frac{1}{2}(n-2)(n+1).$$

Proof. Let K be a maximal subalgebra of L. Then K is an ideal of codimension 1, and hence Corollary 2.5 implies that $\dim(\mathrm{H}^2(L,\Lambda)) \leq \dim(\mathrm{H}^2(K,\Lambda)) + \dim(K/[L,K]) \leq \frac{1}{2}(n-1)(n-2) + (n-2) = \frac{1}{2}(n-2)(n-1)$, as desired. □

Note that if L is a Lie algebra of maximal class with $\dim L > 3$, Bosko [2] showed that the dimension of the second cohomology of L is bounded above by $n-2$, which is sharper than the one obtained in the above corollary.

The following example shows that the upper bound given in Corollary 2.5 is the best possible.

Example. Let $L = H(1) \oplus A(n)$, where $H(1)$ denotes the Heisenberg algebra of dimension 3 and $A(n)$ is an n-dimensional abelian Lie algebra. Then by virtue of [7, Theorem 3] and [1, Theorem 1], we have

$$\begin{aligned}\dim(H^2(L,\Lambda)) &= \dim(H^2(H(1),\Lambda)) + \dim(H^2(A(n),\Lambda)) + \dim(\frac{H(1)}{(H(1))^2} \otimes A(n)) \\ &= \frac{1}{2}(n^2+3n+4).\end{aligned}$$

On the other hand, if we put $K = H(1) \oplus A(n-1)$, then K is an ideal of L of codimension 1, and $\dim(H^2(K,\Lambda)) + \dim(K/[L,K]) = \frac{1}{2}(n^2+n+2)+n+1 = \frac{1}{2}(n^2+3n+4)$.

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Faculty of Mathematical Sciences, Shahid Beheshti University, G.C., Tehran, Iran
E-mail address: `salemkar@sbu.ac.ir`

Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah, Iran

School of Mathematics, Iran University of Science and Technology, Tehran, Iran