ON IRREDUCIBLE MODULES OVER $q$-SKEW POLYNOMIAL RINGS AND SMASH PRODUCTS

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Abstract. Let $M$ be an irreducible left module over a $q$-skew polynomial ring $R[x; \sigma, \delta]$. We give sufficient conditions for the complete reducibility of $M$ considered as a module over the coefficient ring $R$. We apply it to irreducible modules over smash product $R\#H$, where $H$ is a Hopf algebra generated by skew primitive elements.

1. Introduction

For a given extension $R \subseteq S$ of associative rings (with the same unity), it is natural to ask whether (or when) irreducible left $S$-modules are completely reducible as $R$-modules. This question has a positive answer for several classes of "finite type" extensions: for example,

(i) finite normalizing extensions $R \subseteq \sum_{i=1}^{n} R s_i$ (2),
(ii) fixed rings of a finite group actions $R^G \subseteq R$, with $|G|^{-1} \in R$ (8),
(iii) rings graded by finite groups $R_1 \subseteq \bigoplus_{g \in G} R_g$ (4).

In this paper we study some extensions of "infinite type"; namely, we consider modules over $q$-skew polynomial rings. We show that, under certain conditions, for a given left $R[x; \sigma, \delta]$-module $M$ its socle $\text{Soc}(R M)$ over $R$ is also a module over the ring $R[x; \sigma, \delta]$. Our conditions imply in particular that if $q$ is not a root of 1, then:

1. finite dimensional irreducible $R[x; \sigma, \delta]$-modules are completely reducible over $R$;
2. if $R$ is left socular (e.g., left artinian or right perfect), then irreducible left $R[x; \sigma, \delta]$-modules are completely reducible over $R$.

As a consequence of our results on modules over $q$-skew polynomial rings, we obtain a description of certain modules over smash products $R\#H$, where $H$ is a Hopf algebra generated by skew primitive elements. Namely, we show that if $H$ is a character Hopf algebra (see [5]) over the field $k$ of characteristic 0, and $\chi^h(g)$ is not an $n^{th}$ primitive root of 1 ($n > 1$) for any character skew $g$-primitive element $h \in H$, then

3. every finite dimensional irreducible left $R\#H$-module is completely reducible as a left $R$-module;

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4. if $R$ is left socular, then irreducible left $R\#H$-modules are completely reducible as left $R$-modules. Thus $\mathcal{J}(R) \subseteq \mathcal{J}(R\#H)$, where $\mathcal{J}$ is the Jacobson radical.

On the other hand, we should also point out that in the case where $H$ is finite dimensional and pointed, there is a strong relationship between the Jacobson radicals of $R$ and the crossed product $R\#H$; namely, it is proved in \cite{7} that $\mathcal{J}(R\#H)^{\dim_{\mathbb{H}} H} \subseteq \mathcal{J}(R) \cdot (R\#H)$.

We will now introduce the terminology and notation that will be used throughout the paper. Let $R$ be an associative ring and $\sigma$ be an automorphism of $R$. Then the additive map $\delta: R \to R$ is a $\sigma$-derivation if

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$$

for all $a, b \in R$. Suppose that $q$ is a nonzero central $(\sigma, \delta)$-constant in $R$, i.e., $\sigma(q) = q$ and $\delta(q) = 0$. If $\delta\sigma = q\sigma\delta$, then $\delta$ is called a $q$-skew $\sigma$-derivation. If in addition $R$ is a $k$-algebra, we assume that $q \in k^\times$. The following $q$-Leibniz Rules hold in $R$ and $R[x; \sigma, \delta]$:

$$\delta(ab) = \sum_{i=0}^{n} \binom{n}{i}_q \sigma^{n-i}\delta^i(a)\delta^{n-i}(b) \text{ and } x^n a = \sum_{i=1}^{n} \binom{n}{i}_q \sigma^{n-i}\delta^i(a)x^{n-i}$$

for all $a, b \in R$ and $n \geq 0$. The Gaussian $q$-binomial coefficient $\binom{n}{i}_q$ is defined as the evaluation at $t = q$ of the polynomial function

$$(1) \binom{n}{i}_q = \frac{(t^n - 1)(t^{n-1} - 1)\ldots(t^{1} - 1)}{(t^i - 1)(t^{i-1} - 1)\ldots(t - 1)}.$$  

We will use the following $q$-Pascal identity:

$$\binom{n}{i}_q = \binom{n-1}{i-1}_q + q^{n-i}\binom{n-1}{i-1}_q = \binom{n-1}{i-1}_q + q^i\binom{n-1}{i}_q$$

for $n > i > 0$ (cf. \cite{3}).

We will say that the ring $R$ has $q$-characteristic zero if $1 + q + \ldots + q^{m}$ is invertible in $R$, for any integer $m \geq 1$. If in addition $R$ is a $k$-algebra, then either $q$ is not a root of unity or $q = 1$ and char $k = 0$.

If $r \in R$, then a left $R$-module $M$ is said to be $r$-torsion free if $rm \neq 0$ for all nonzero $m \in M$. If for any $m \in M$ there exists an integer $n = n(m)$ such that $r^nm = 0$, then $M$ is called an $r$-torsion module.

A submodule $E$ of an $R$-module $M$ is said to be essential if $E \cap X \neq 0$ for any nonzero submodule $X \subseteq M$. It is well known that the intersection of all essential submodules of an $R$-module $M$ is equal to the sum of all irreducible submodules of $M$ and is called the socle of $M$, denoted by $	ext{Soc}(M)$. Finally, $\text{Sing}(M)$ will be the singular submodule of $M$; that is, $\text{Sing}(M) = \{m \in M \mid \text{ann}_R(m) \text{ is essential in } R\}$.  

2. $m$-SEQUENCES AND ESSENTIAL SUBMODULES

Let $R[x; \sigma, \delta]$ be a $q$-skew polynomial ring and $M$ a left $R[x; \sigma, \delta]$-module. Let $E$ be an essential $R$-submodule of $M$ and $0 \neq m \in E$. By an $m$-sequence we mean a sequence $r = \{r_n\}_{n \geq 0}$ of elements of $R$ satisfying the following properties:

1° $\sigma^n(r_n)x^nm \in E$ for all $n \geq 0$ and $\sigma^s(r_s)x^sm \neq 0$ for some $s$;
2° if \( r_{n+1} x^{n+1} m \in E \), then \( r_{n+1} = r_n \);  
3° if \( r_{n+1} x^{n+1} m \notin E \), then \( r_{n+1} \in R r_n \) and \( r_{n+1} \) is not in special form.

The smallest integer \( s \) such that \( \sigma^s (r_s) x^s m \neq 0 \) we denote by \( \deg r \) and call the degree of \( r \).

**Lemma 1.** If \( a \in R \) and \( \sigma^s (a) x^s m \neq 0 \) for some \( s \geq 1 \), then there exists an \( m \)-sequence \( r = \{ r_n \}_{n \geq 0} \) such that \( r_0 = a \) and \( \deg r \leq s \).

**Proof.** The sequence \( r \) we define inductively starting with \( r_0 = \cdots = r_{i-1} = a \), where \( i \) is the smallest integer such that \( \sigma^i (a) x^i m \notin E \). If such an \( i \) does not exist, the constant sequence \( r = \{ a \} \) satisfies the desired property. Next suppose that \( j \geq i \) and \( r_0, \ldots, r_j \) are given. If \( r_{j+1} x^{j+1} m \in E \), then put \( r_{j+1} = r_j \). If \( r_{j+1} x^{j+1} m \notin E \), then by essentiality of \( E \) there exists \( 0 \neq c = \sigma^{j+1} (r_j) \in R \) such that

\[
0 \neq c \sigma^{j+1} (r_j) x^{j+1} m = \sigma^{j+1} (r_j r_j) x^{j+1} m \in E.
\]

In this situation we put \( r_{j+1} = r_j r_j \). Clearly the sequence \( r \) satisfies conditions 1° – 3°, and from the construction it follows immediately that \( \deg r \leq s \). \( \square \)

An \( m \)-sequence \( r = \{ r_n \}_{n \geq 0} \) is said to be **weak** if \( r_j = r_{j+1} \) for some \( j \geq \deg r \). If \( r_j \neq r_{j+1} \) for all \( j \geq \deg r \), we call \( r \) a **strict** \( m \)-sequence. Note that if \( r \) is strict and \( j \geq \deg r \), then \( \sigma^j (r_j) x^j m \neq 0 \). Indeed, if \( \sigma^j (r_j) x^j m = 0 \), then \( \sigma^j (r_{j-1}) x^j m \) must equal 0, and hence \( r_j = r_{j-1} \).

**Lemma 2.** Suppose that every \( m \)-sequence in \( R \) is strict. Then:

1. if \( a \in R \) is such that \( 0 \neq ax^l m \in E \), then \( \sigma(a) x^{l+1} m \notin E \);
2. if \( r = \{ r_n \}_{n \geq 0} \) is an \( m \)-sequence and \( l \geq \deg r \), then \( \sigma^l (r_l) x^l m = 0 \) for all \( j < l \);  
3. \( \text{ann}(x^{j+1} m) \subseteq \sigma^{-1} (\text{ann}(x^j m)) \) for all \( j \geq 0 \).

**Proof.** 1. Suppose that \( 0 \neq ax^l m \in E \) and \( \sigma(a) x^{l+1} m \notin E \). By Lemma 1 we can take an \( m \)-sequence \( r \) such that \( r_0 = \sigma^{-l} (a) \) and \( \deg r \leq l \). Then \( r_1 = br_0 = b \sigma^{-1} (a) \), where \( b \in R \). Notice that

\[
\sigma^{l+1} (r_l) x^{l+1} m = \sigma^{l+1} (b) \sigma(a) x^{l+1} m \in E.
\]

Hence \( r_l = r_{l+1} \), contradicting our assumption that every \( m \)-sequence in \( R \) is strict.

2. Suppose that \( \sigma^j (r_j) x^j m \neq 0 \) for some \( j < l \). From the definition of an \( m \)-sequence it follows that we can choose \( a, b \in R \) such that \( r_1 = ar_j = br_{j+1} \). Then \( 0 \neq \sigma^j (r_j) x^j m = \sigma^{j} (a) \sigma^{j} (r_j) x^j m \in E \). On the other hand, \( \sigma^{j+1} (r_l) x^{j+1} m = \sigma^{j+1} (b) \sigma^{j+1} (r_{j+1}) x^{j+1} m \in E \), which is impossible by 1.

3. Suppose \( a \in R \) is such that \( \sigma(a) x^{j+1} m = 0 \). By item 1, it follows that either \( ax^j m = 0 \) or \( ax^j m \notin E \). If \( ax^j m \notin E \), then there exists \( r \in R \) such that \( 0 \neq r ax^j m \in E \). But in this situation \( 0 = \sigma(r a) x^{j+1} m \in E \). By item 1 we obtain that \( ax^j m \) must be equal to 0; thus \( \text{ann}(x^{j+1} m) \subseteq \sigma^{-1} (\text{ann}(x^j m)) \). \( \square \)

**Corollary 3.** If every \( m \)-sequence in \( R \) is strict, then \( R \) contains an infinite strictly descending chain of left ideals

\[
\text{ann}(m) \supseteq \sigma^{-1} (\text{ann}(x m)) \supseteq \cdots \supseteq \sigma^{-l} (\text{ann}(x^l m)) \supseteq \cdots.
\]

**Proof.** Lemma 2(3) implies that \( \sigma^{-1} (\text{ann}(x^l m)) \subseteq \sigma^{-(l-1)} (\text{ann}(x^{l-1} m)) \) for any \( l > 0 \). To see that the inclusion is strict, it is enough to consider an \( m \)-sequence \( r \) of degree \( \leq l - 1 \). Then Lemma 2(2) yields that \( r_l \in \sigma^{-(l-1)} (\text{ann}(x^{l-1} m)) \), but clearly \( r_l \notin \sigma^{-l} (\text{ann}(x^l m)) \). \( \square \)
Lemma 4. If $R$ contains a weak $m$-sequence, then there exists an element $r \in R$ and a nonnegative integer $n$ such that

1. $0 \neq \sigma^n(r)x^nm \in E$ and $\sigma^{n+1}(r)x^{n+1}m \in E$,
2. $rm = \sigma(r)xrm = \cdots = \sigma^{n-1}(r)x^{n-1}m = 0$.

Proof. Let $l \geq \deg(r)$ be the smallest integer with respect to the equality $r_i = r_{i+1}$. Then $\sigma^i(r_i)x^im \neq 0$. Otherwise, if $\sigma^i(r_i)x^im = 0$, then from the definition it follows that $\sigma^i(r_{i-1})x^im \in E$, and hence $r_{i-1} = r_i$. Next consider the smallest integer $n$ with respect to $\sigma^n(r)x^nm \neq 0$. It is clear that $n \leq l$. Note that if $j \leq l$, then $r_i = s_jr_j$ for some $s_j \in R$. Thus $\sigma^j(r_i)x^jm = \sigma^j(s_j)\sigma^j(r_j)x^jm \in E$. Therefore $r = r_i$ and $n$ satisfy the lemma.

Lemma 5. Let $M$ be a $q$-torsion free left $R[x; \sigma, \delta]$-module and $r \in R$, $m \in M$ be such that

$$rm = \sigma(r)xrm = \cdots = \sigma^{n-1}(r)x^{n-1}m = 0.$$ 

Then $\sigma^i\delta^j(r)x^im = 0$ if $i + j \leq n - 1$, and $\sigma^n(r)x^nm = (-1)^nq^{\frac{n(n-1)}{2}}\delta^n(r)m$.

Proof. First we show that if $i,j$ are nonnegative integers and $i + j \leq n - 1$, then $\sigma^i\delta^j(r)x^im = 0$.

Suppose that $\sigma^i\delta^j(r)x^im \neq 0$ and take $i,j$ such that the sum $i + j$ is possibly minimal. Next take $j$ possibly minimal. By assumption it follows that $j > 0$, so

$$\sigma^{i+1}\delta^{j-1}(r)x^{i+1}m = 0 \quad \text{and} \quad \sigma^i\delta^{j-1}(r)x^im = 0.$$ 

Thus

$$0 = x(\sigma^i\delta^{j-1}(r)x^im) = \sigma^{i+1}\delta^{j-1}(r)x^{i+1}m + \delta\sigma^i\delta^{j-1}(r)x^im$$

$$= q^i\sigma^i\delta^j(r)x^im,$$

a contradiction. The above implies, in particular, that if $i + j = n - 1$, then

$$0 = x(\sigma^i\delta^j(r)x^im) = \sigma^{i+1}\delta^j(r)x^{i+1}m + q^i\sigma^i\delta^{j+1}(r)x^im.$$ 

Hence

$$\sigma^n(r)x^nm = -q^{n-1}\sigma^n\delta(r)x^{n-1}m = q^{-n-1}q^{-2}\sigma^{n-2}\delta^2(r)x^{n-2}m$$

$$= \cdots = (-1)^nq^{-n-2}\cdots q\delta^n(r)m = (-1)^nq^{\frac{n(n-1)}{2}}\delta^n(r)m.$$ 

Lemma 6. $D_n = q^{\frac{n^2-n}{2}}(1 + q + \cdots + q^n)$.
Proof. Notice that by using the $q$-Pascal identity,
\[
a_{i+1,j} = \binom{i+2}{j} q^{n-i-1}j = \binom{i+1}{j-1} q^{n-i-1}j + \binom{i+1}{j} q^j q^{n-i-1}j
\]
\[
= \binom{i+1}{j-1} q^{n-i-1}j + a_{ij}.
\]
The above implies that
\[
D_n = \det \begin{pmatrix}
\frac{2}{1}q^{n-1} & \frac{2}{1}q^{2(n-1)} & 0 & \ldots & 0 \\
\frac{2}{1}q^{n-2} & \frac{2}{1}q^{2(n-2)} & \frac{2}{1}q^{3(n-2)} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{2}{1}q & \frac{2}{1}q^{n-1} & \frac{2}{1}q^2 & \frac{2}{1}q^3 & \ldots & (n-1)q^n \\
\frac{2}{1}q & \frac{2}{1}q^2 & \frac{2}{1}q & \frac{2}{1}q & \ldots & (n-1)q \\
\frac{2}{1}q & \frac{2}{1}q & \frac{2}{1}q & \frac{2}{1}q & \ldots & (n-1)q
\end{pmatrix}
\]
\[
= \frac{2}{1}q^{n-1} q^{-2} \ldots q \cdot D_{n-1} - \frac{2}{1} q^{2(n-1)} W_{n-1},
\]
where
\[
W_{n-1} = \det \begin{pmatrix}
q^{n-2} & \frac{2}{1}q^{3(n-2)} & 0 & \ldots & 0 \\
\frac{2}{1}q^{n-3} & \frac{2}{1}q^{3(n-3)} & \frac{3}{1}q^{4(n-3)} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
q & \frac{2}{1}q^{n-1} & \frac{2}{1}q & \frac{2}{1}q^3 & \ldots & (n-1)q^n \\
1 & \frac{2}{1}q & \frac{2}{1}q & \frac{2}{1}q & \ldots & (n-1)q \\
1 & \frac{2}{1}q & \frac{2}{1}q & \frac{2}{1}q & \ldots & (n-1)q
\end{pmatrix}.
\]
Again applying the $q$-Pascal identity, one immediately obtains that
\[
W_{n-1} = \det \begin{pmatrix}
q^{n-2} & \frac{2}{1}q^{3(n-2)} & 0 & \ldots & 0 \\
0 & \frac{2}{1}q^{3(n-3)} & \frac{2}{1}q^{4(n-3)} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \frac{2}{1}q^{n-2} & \frac{2}{1}q^{n-1} & \frac{2}{1}q^n & \ldots & (n-2)q^n \\
0 & \frac{2}{1}q^{n-2} & \frac{2}{1}q^{n-1} & \frac{2}{1}q^n & \ldots & (n-2)q^n \\
0 & \frac{2}{1}q^{n-2} & \frac{2}{1}q^{n-1} & \frac{2}{1}q^n & \ldots & (n-2)q^n
\end{pmatrix}
\]
\[
= q^{n-2} q^{2(n-3)} q^{2(n-4)} \ldots q^2 \cdot D_{n-2} = q^{n^2-4n+4} D_{n-2}.
\]
Thus
\[
D_n = (1 + q) q^{n(n-1)} D_{n-1} - q^{n^2-2n+2} D_{n-2}
\]
with $D_1 = 1 + q$ and $D_2 = q(1 + q + q^2)$. The lemma now follows by an easy induction. \qed

**Proposition 7.** Let $M$ be a left $R[x; \sigma, \delta]$-module which is $D_n$-torsion free for all $n \geq 1$. Let $E$ be an essential $R$-submodule of $M$ such that for every $m \in E$, the ring $R$ contains a weak $m$-sequence. Then
\[
E \cap x^{-1} E = \{m \in E \mid xm \in E\}
\]
is also essential as an $R$-submodule.
proof. Notice that if \( e \in E \) and \( xe \in E \), then for every \( r \in R \),
\[
xe = \sigma(r)xe + \delta(r)e \in E.
\]
Thus \( E \cap x^{-1}E \) is an \( R \)-submodule of \( M \).

Suppose that \( E \cap x^{-1}E \) is not essential. Then there exists a nonzero element \( m \in E \) such that \( (E \cap x^{-1}E) \cap Rm = 0 \). Since \( R \) contains a weak \( m \)-sequence, by Lemma 4 we can take \( r \in R \) and \( n \geq 0 \) such that
\[
rm = \sigma(r)xm = \cdots = \sigma^{n-1}(r)x^{n-1}m = 0,
\]
\[
0 \neq \sigma^n(r)x^nm \in E \quad \text{and} \quad \sigma^{n+1}(r)x^{n+1}m \in E.
\]
For \( 1 \leq i, j \leq n \), let \( a_{ij} = \binom{i+1}{j} q^{(n-i)j} \) and \( x_j = \sigma^{n+1-j}\delta(r)x^{n+1-j}m \). Applying the \( q \)-Leibniz rule for \( i = 1, 2, \ldots, n-1 \), we obtain
\[
0 = x^{i+1}(\sigma^{n-i}(r)x^{n-i}m) = \sum_{j=0}^{i+1} \binom{i+1}{j} q^{(n-i)j}\sigma^{n+1-j}\delta(r)x^{n+1-j}m
\]
\[
= \sum_{j=0}^{i+1} \binom{i+1}{j} q^{(n-i)j}\sigma^{n+1-j}\delta(r)x^{n+1-j}m
\]
\[
= \sigma^{n+1}(r)x^{n+1}m + \sum_{j=1}^{i+1} a_{ij}x_j.
\]
Thus \( \sum_{j=1}^{i+1} a_{ij}x_j = -\sigma^{n+1}(r)x^{n+1}m \in E \). Moreover, for \( i = n \) we have
\[
0 = x^{n+1}rm = \sigma^{n+1}(r)x^{n+1}m + \sum_{j=1}^{n} a_{nj}x_j + \delta^{n+1}(r)m,
\]
so \( \sum_{j=1}^{n} a_{nj}x_j \in E \). Now it is clear that for any \( j = 1, 2, \ldots, n \) the element \( D_nx_j \in E \), where \( D_n \) is the determinant from Lemma 6. We note that \( D_nx_1 = D_n\sigma^n\delta(r)x^nm \in E \), so
\[
x(D_n\sigma^n(r)x^nm) = D_n\sigma^{n+1}(r)x^{n+1}m + D_n\delta\sigma^n(r)x^nm
\]
\[
= D_n\sigma^{n+1}(r)x^{n+1}m + D_nq^n\sigma^n\delta(r)x^nm \in E.
\]
On the other hand, by Lemma 5 \( \sigma^n(r)x^nm = (-1)^n q^\frac{n(n-1)}{2} \delta^n(r)m \) and \( M \) is \( D_n \)-torsion free; thus
\[
0 \neq D_n\sigma^n(r)x^nm \in E \cap x^{-1}E \cap Rm,
\]
a contradiction. Therefore \( E \cap x^{-1}E \) is an essential submodule of \( M \).

**Corollary 8.** Let \( M \) be a left \( R[x; \sigma, \delta] \)-module which is \( D_n \)-torsion free for all \( n \geq 1 \). Suppose that for every essential \( R \)-submodule \( E \) of \( M \) and \( 0 \neq m \in E \), the ring \( R \) contains a weak \( m \)-sequence. Then \( \text{Soc}(R \cap M) \) is an \( R[x; \sigma, \delta] \)-module. In particular, if \( M \) is simple as an \( R[x; \sigma, \delta] \)-module, then either \( \text{Soc}(R \cap M) = 0 \) or \( R \cap M \) is completely reducible.

**Proof.** Let \( m \in \text{Soc}(R \cap M) \). If \( E \) is an essential submodule of \( R \cap M \), then by Proposition 7 \( E \cap x^{-1}E \) is also essential, so \( m \in E \cap x^{-1}E \). Hence \( xm \in E \). Therefore \( \text{Soc}(R \cap M) \) is an \( R[x; \sigma, \delta] \)-module. \( \square \)
3. Applications

In this section we describe situations in which our condition on the existence of weak \( m \)-sequences is automatically satisfied.

Let \( \Lambda \) be a well ordered set of ordinal numbers with the least element 0. For a ring \( R \) one can define a chain of ideals \( \{ S_\alpha \}_{\alpha \in \Lambda} \) as follows: \( S_0 = 0 \); if \( \alpha \in \Lambda \), then \( S_{\alpha+1}/S_\alpha = \text{Soc}(R/S_\alpha) \), the left socle of \( R/S_\alpha \). If \( \beta \in \Lambda \) is a limit number, set \( S_\beta = \bigcup_{\alpha < \beta} S_\alpha \). Recall that a ring \( R \) is said to be left socular (cf. \( \text{[1]} \)) if every nonzero left \( R \)-module contains a simple submodule. If \( R \) is left socular, the set \( \Lambda \) can be chosen such that \( R = S_\alpha \) for some \( \alpha \in \Lambda \). Note that the class of socular rings contains left artinian rings and right perfect rings.

If \( A \) is a \( k \)-algebra, then \( A \)-module \( M \) is locally finite dimensional if every finitely generated submodule of \( M \) is finite dimensional.

**Proposition 9.** Let \( M \) be a left \( R[x; \sigma, \delta] \)-module and \( E \) its essential \( R \)-submodule. Suppose that one of the following conditions is fulfilled:

1. \( R \) is left socular;
2. \( R \) is a left noetherian \( k \)-algebra and \( M \) is locally finite dimensional as a \( k[x] \)-module;
3. \( \dim_k M < \infty \);
4. there exists an integer \( N \) such that \( d^{N+1}(r) \in \bigoplus_{j=0}^{N} \sigma^{j}(r) \) for all \( r \in R \);
5. \( M \) is \( x \)-torsion; i.e., for any \( m \in M \) there exists \( n = n(m) \) such that \( x^nm = 0 \);
6. \( R \) is a \( k \)-algebra, \( \sigma = \text{id}_R \) and \( M \) is locally finite dimensional as a \( k[x] \)-module.

Then for any nonzero \( m \in E \) the ring \( R \) contains a weak \( m \)-sequence.

**Proof.**

1. Suppose that \( R \) is left socular. Let \( \gamma \) be the smallest ordinal such that \( S_\gamma \) contains an \( m \)-sequence \( \{ r_\ell \}_{\ell \geq 0} \). It is clear that \( \gamma \) is not a limit ordinal. Note that if \( a \in S_{\gamma-1} \), then \( \sigma^l(a)x^lm = 0 \). Otherwise, we have an \( m \)-sequence \( \{ r'_{\ell} \}_{\ell \geq 0} \) with \( r'_0 = a \in S_{\gamma-1} \). Since \( Rr'_{\ell} \supseteq Rr'_{\ell+1} \), one obtains that \( r'_\ell \in S_{\gamma-1} \) for all \( \ell \). This contradicts minimality of \( \gamma \).

Let \( \varphi: R \rightarrow R/S_{\gamma-1} \) be the canonical homomorphism. Since \( Rr_0 \supseteq Rr_1 \supseteq \cdots \supseteq Rr_l \supseteq \cdots \), we have a chain

\[ \varphi(Rr_0) \supseteq \varphi(Rr_1) \supseteq \cdots \supseteq \varphi(Rr_l) \supseteq \cdots \]

of cyclic submodules of a semisimple module \( S_{\gamma}/S_{\gamma-1} \). Since \( \varphi(Rr_0) \) is contained in a finite direct sum of simple modules, this chain terminates. On the other hand, if \( \varphi(Rr_l) = \varphi(Rr_{l+1}) \), then there exist \( r' \in R \) and \( a \in S_{\gamma-1} \) such that \( r_l = r'r_{l+1} + a \).

By the above, \( \sigma^{l+1}(a)x^{l+1}m = 0 \), so

\[ \sigma^{l+1}(r_l)x^{l+1}m = \sigma^{l+1}(r')\sigma^{l+1}(r_{l+1})x^{l+1}m \in E. \]

From the definition of an \( m \)-sequence it follows that \( r_l = r_{l+1} \). Therefore the sequence \( r \) is weak.

2. Suppose that every \( m \)-sequence in \( R \) is strict. Corollary \( \text{[3]} \) tells us that the chain of left ideals

\[ \text{ann}(m) \supseteq \sigma^{-1}(\text{ann}(xrm)) \supseteq \cdots \supseteq \sigma^{-l}(\text{ann}(x'm)) \supseteq \cdots \]
is strict. Since \( \dim \text{span}_F(m, xm, x^2m, \ldots) < \infty \), there exists an integer \( t \) such that \( x^n m \in \text{span}_F(m, xm, x^2m, \ldots, x^t m) \) for all \( n \geq t \). Then

\[
\text{ann}(m, xm, x^2m, \ldots, x^t m) \subseteq \text{ann}(x^n m)
\]

for \( n \geq t \), and consequently \( \bigcap_{l=0}^{\infty} \text{ann}(x^l m) = \bigcap_{l=0}^{t} \text{ann}(x^l m) \). Set \( I = \bigcap_{l=0}^{t} \text{ann}(x^l m) \) and take \( r \in I \). For any \( l \geq 1 \), \( r \in \text{ann}(x^l m) \), so

\[
\sigma^{-l}(r) \in \sigma^{-l}(\text{ann}(x^l m)) \subseteq \sigma^{-(l-1)}(\text{ann}(x^{l-1} m));
\]

hence \( \sigma^{-1}(r) \in \text{ann}(x^{l-1} m) \). Then it follows that \( \sigma^{-1}(I) \subseteq I \), and so \( I \subseteq \sigma(I) \). The ring \( R \) is left noetherian, so the chain \( I \subseteq \sigma(I) \subseteq \sigma^2(I) \ldots \) must stop. It implies immediately that \( \sigma(I) = I \).

Next we claim that there exists an increasing sequence \( \{f(n)\}_{n \geq 0} \) of nonnegative integers such that

\[
\sigma \left( \bigcap_{l=0}^{f(n)} \text{ann}(x^l m) \right) \not\subseteq \bigcap_{j > f(n)} \text{ann}(x^j m).
\]

We proceed by induction. By Corollary \( 3 \) we can put \( f(0) = 0 \). Assume \( n > 0 \) and let \( a \in \bigcap_{l=0}^{f(n)} \text{ann}(x^l m) \) be such that \( \sigma(a)x^i m \neq 0 \) for some \( i > f(n) \). Since \( I \) is \( \sigma \)-stable, \( a \not\in I \), so there exists \( s > f(n) \) such that \( a \in \bigcap_{l=0}^{s-1} \text{ann}(x^l m) \) and \( ax^s m \neq 0 \). Take \( b \in R \) such that \( 0 \neq bax^s m \in E \). If every \( m \)-sequence is strict, then by Lemma \( 1 \), \( \sigma(ba)x^{s+1} m \not\in E \). Since \( E \) is essential, one can choose \( c \in R \) such that \( 0 \neq \sigma(cba)x^{s+1} m \in E \). Again by Lemma \( 1 \), \( cbax^s m = 0 \), so \( cb \in \bigcap_{l=0}^{s} \text{ann}(x^l m) \). Since \( \sigma(cba)x^{s+1} m \neq 0 \), we have \( \sigma \left( \bigcap_{l=0}^{s} \text{ann}(x^l m) \right) \not\subseteq \bigcap_{j > s} \text{ann}(x^j m) \).

Thus it suffices to put \( f(n+1) = s \). This proves the claim.

But now, if \( f(n) > t \), then \( I = \bigcap_{l=0}^{f(n)} \text{ann}(x^l m) = \bigcap_{l=0}^{\infty} \text{ann}(x^l m) \). Since \( I \) is \( \sigma \)-stable,

\[
\sigma \left( \bigcap_{l=0}^{f(n)} \text{ann}(x^l m) \right) \subseteq \bigcap_{l=0}^{\infty} \text{ann}(x^l m) \subseteq \bigcap_{j > f(n)} \text{ann}(x^j m),
\]

contradicting the definition of \( f(n) \). Thus \( R \) contains a weak \( m \)-sequence.

3. Let \( P = \text{ann}(M) \). Then \( \dim_F(R/P) < \infty \) and \( P \subseteq \text{ann}(x^l m) \) for any \( l \). Note that the mapping \( a + \text{ann}(x^n m) \mapsto \sigma^{-1}(a) + \sigma^{-l}(\text{ann}(x^l m)) \) provides an isomorphism of vector spaces \( R/\text{ann}(x^l m) \approx R/\sigma^{-1}(\text{ann}(x^l m)) \). Thus

\[
\dim_F R/\sigma^{-1}(\text{ann}(x^l m)) \leq \dim_F(R/P).
\]

From Corollary \( 3 \) it follows that \( R \) contains a weak \( m \)-sequence.

4. Let \( r = \{r_n\}_{n \geq 0} \) be a strict \( m \)-sequence with \( \deg r \leq N \). Then \( \sigma^j(r_{N+1})x^j m = 0 \) for all \( j \leq N \) and \( \sigma^{N+1}(r_{N+1})x^{N+1} m \neq 0 \). By Lemma \( 3 \)

\[
0 = \sigma^j(r_{N+1})x^j m = (-1)^j q^{\ell(j-1)+1} \delta^j(r) m
\]
for all \( j \leq N \). Thus
\[
\sigma^{N+1}(r_{N+1})x^{N+1} = (-1)^{N+1} \frac{N(N+1)}{2} \delta^{N+1}(r_{N+1})m
\]
\[
\in \sum_{j=0}^{N} R\delta^j(r_{N+1})m = 0,
\]
a contradiction. Consequently, in this situation, every \( m \)-sequence is weak.

5. This follows directly from Corollary \( 3 \).

6. Suppose \( \sigma = \text{id}_R \). If every \( m \)-sequence in \( R \) is strict, Corollary \( 2 \) says that the chain \( \text{ann}(m) \supset \text{ann}(xm) \supset \cdots \supset \text{ann}(x^nm) \supset \cdots \) is strict. But this contradicts our assumption that \( \text{span}_F \{m, xm, \ldots, x^l m \ldots\} \) is finite dimensional. \( \square \)

Recall that an automorphism \( \sigma \) of the ring \( R \) is said to be of locally finite order if for every \( r \in R \), there exists an integer \( n = n(r) > 0 \) such that \( \sigma^n(r) = r \). If the ring \( R \) is left socular, then nonzero left \( R \)-modules contain simple submodules. Therefore Proposition \( 9 \) condition 1, and Corollary \( 8 \) give us

**Corollary 10.** If \( R \) is a left socular ring of \( q \)-characteristic zero, then simple left \( R[x; \sigma, \delta] \)-modules are completely reducible as left \( R \)-modules. Thus the Jacobson radical \( \mathcal{J}(R) \) is contained in the Jacobson radical \( \mathcal{J}(R[x; \sigma, \delta]) \). Moreover, if the automorphism \( \sigma \) has locally finite order, then

\[
\mathcal{J}(R[x; \sigma, \delta]) = \mathcal{J}(R)[x; \sigma, \delta].
\]

**Proof.** Since simple \( R[x; \sigma, \delta] \)-modules are completely reducible as \( R \)-modules, we have \( \mathcal{J}(R) \subseteq \mathcal{J}(R[x; \sigma, \delta]) \). Suppose that \( \sigma \) has locally finite order. We know that \( \mathcal{J}(R[x; \sigma, \delta]) \cap R \) is a quasi-regular ideal of \( R \), so \( \mathcal{J}(R[x; \sigma, \delta]) \cap R \subseteq \mathcal{J}(R) \) and consequently \( \mathcal{J}(R[x; \sigma, \delta]) \cap R = \mathcal{J}(R) \). This implies that \( \mathcal{J}(R) \) is \( \delta \)-stable and

\[
R[x; \sigma, \delta]/\mathcal{J}(R)[x; \sigma, \delta] \approx (R/\mathcal{J}(R))[x; \hat{\sigma}, \hat{\delta}],
\]
where \( \hat{\sigma} \) is an induced automorphism and \( \hat{\delta} \) is a \( q \)-skew \( \hat{\sigma} \)-derivation of \( R/\mathcal{J}(R) \), respectively. Now it remains to prove that if \( R \) is semiprimitive and socular, then \( S = R[x; \sigma, \delta] \) is semiprimitive. To this end, suppose that \( \mathcal{J}(S) \neq 0 \) and let \( n \) be the minimum of degrees of nonzero polynomials from \( \mathcal{J}(S) \). The set \( \{0\} \cup \{a \mid ax^n + g(x) \in \mathcal{J}(S), \text{where } \deg g(x) < n \} \) is a nonzero ideal of \( R \). In particular, it contains a minimal left ideal of the form \( I = Re \), where \( e \) is a nonzero idempotent.

Let \( f(x) = ex^n + g(x) \in \mathcal{J}(S) \) and \( m > 0 \) be such that \( \sigma^m(e) = e \). By eventually replacing \( f(x) \) by \( f(x)x^k \), where \( k \) is such that \( \deg f(x)x^k \) is divisible by \( m \), we have in the Jacobson radical of \( S \) a nonzero polynomial \( f(x) = ex^l + h(x) \) such that \( e \) is a nonzero idempotent, \( \sigma^l(e) = e \), and \( \deg h(x) < l \). It is well known that \( \mathcal{J}(eSe) = e\mathcal{J}(S)e \). Therefore
\[
ef(x)e = ex^l + eh(x)e = ex^l + \bar{h}(x) \in \mathcal{J}(eSe),
\]
where \( \bar{h}(x) \in eSe \). Let \( ef(x)e \in eSe \) be a quasi-inverse for \( ef(x)e \). Then \( ef(x)e \) has a positive degree \( s \) in \( x \) and
\[
ef(x)e + eg(x)e = ef(x)eg(x)e.
\]
Since \( e \) is the identity element in \( eSe \), the right-hand side of the above equality has degree \( n + s > \max\{n, s\} \geq \deg(ef(x)e + eg(x)e) \). Thus \( \mathcal{J}(S) = 0 \). \( \square \)
In [6] the authors considered the so-called “finite Jacobson radical” $\mathcal{J}_{\text{fin}}(R)$ of a $k$-algebra $R$, defined as the intersection of all the annihilators of all finite dimensional irreducible (left) $R$-modules. Thus by Proposition [9] condition 3, and Corollary [8] we have

**Corollary 11.** Let $R$ be a $k$-algebra with a $q$-skew $\sigma$-derivation $\delta$. If $R$ has $q$-characteristic zero, then every finite dimensional irreducible left $R[x; \sigma, \delta]$-module is completely reducible as a left $R$-module. Thus

$$\mathcal{J}_{\text{fin}}(R) \subseteq \mathcal{J}_{\text{fin}}(R[x; \sigma, \delta]).$$

We note that $R$ can be viewed as a left $R[x; \sigma, \delta]$-module with the action defined as

$$\left(\sum_i a_i x^i\right) r = \sum_i a_i \delta^i(r).$$

The $R[x; \sigma, \delta]$-submodules of $R$ are precisely the left ideals of $R$ which are stable under $\delta$. Recall that $\delta$ is said to be locally algebraic if $R$ is locally finite dimensional as a left $k[x]$-module. Moreover in this case, if $m \in R$, then $\sigma^{-1}(\text{ann}_R(x^j m)) = \text{ann}_R(\delta^j(\sigma^{-1}(m)))$. Thus if $R$ satisfies descending chain condition on left annihilators, then Corollary [8] guarantees that for any essential left ideal $E$ and a nonzero element $m \in E$, the ring $R$ contains a weak $m$-sequence. Therefore we can apply Propositions [7] [9] and Corollary [8] to obtain the following.

**Corollary 12.** Let $R$ be a $k$-algebra of $q$-characteristic zero, with a $q$-skew $\sigma$-derivation $\delta$. Suppose that one of the following conditions is fulfilled:

1. $R$ satisfies dcc on left annihilators;
2. $R$ is left noetherian and $\delta$ is locally algebraic;
3. $\delta$ is locally nilpotent;
4. there exists an integer $N$ such that for any $r \in R$, $\delta^{N+1}(r) \in \sum_{j=0}^N R\delta^j(r)$;
5. $\sigma = \text{id}_R$, $q = 1$ and the derivation $\delta$ is locally algebraic.

If $M$ is a left $R[x; \sigma, \delta]$-module, then the singular submodule $\text{Sing}(R M)$ over $R$ is also an $R[x; \sigma, \delta]$-submodule. The left socle $\text{soc}(R)$ of $R$ and left singular ideal $\text{Sing}(R R)$ are $\delta$-invariant. In addition, if $R$ contains a minimal left ideal and $R$ does not contain proper $\delta$-stable two-sided ideals, then $R$ is a semisimple artinian ring.

**Proof.** Let $m \in \text{Sing}(R M)$ and $L = \text{ann}_R(m)$. If $L$ is an essential left ideal of $R$, then by Proposition [7] $\hat{L} = L \cap \delta^{-1}(L) = \{r \in L \mid \delta(r) \in L\}$ is essential. It is also clear that $\sigma(\hat{L})$ is essential, and for every $r \in \hat{L}$,

$$\sigma(r) xm = xrm - \delta(r)m = 0.$$

Hence $\sigma(\hat{L}) \subseteq \text{ann}_R(xm)$ and $xm \in \text{Sing}(R M)$. Consequently, $\text{Sing}(R M)$ is an $R[x; \sigma, \delta]$-submodule of $M$.

If $R$ contains a minimal ideal, then $\text{soc}(R R)$ is a nonzero and $\delta$-stable ideal of $R$. Therefore if $R$ is $\delta$-simple, then $R = \text{soc}(R R)$. Since $R$ has unity, $R$ is a finite direct sum of minimal left ideals. \hfill \square

Let $H$ be a Hopf algebra with comultiplication $\Delta$ and with the group $G$ of group-like elements, i.e., $G = \{g \in H \mid \Delta(g) = g \otimes g\}$. For $g \in G$, let

$$L_g = \{h \in H \mid \Delta(h) = h \otimes 1 + g \otimes h\}$$
be the subspace of $g$-primitive (skew primitive) elements. It is clear that the group $G$ acts on $H$ by the conjugations $h^g = g^{-1}hg$ and that the subspace $L = \bigoplus_{g \in G} L_g$ is $G$-stable under this action. Following [5], recall that an element $h \in H$ is said to be a character element if there exists a character $\chi: G \to k^\times$ such that for all $g \in G$,

$$g^{-1}hg = \chi(g)h.$$ 

If $h$ is a nonzero character element, then the character $\chi$ is uniquely determined by the above equality, and $\chi = \chi^h$ is called a weight of $h$. A Hopf algebra $H$ is called a character if the group $G$ is abelian and $H$ is generated as an algebra with unity by character skew primitive elements. This is a large class of Hopf algebras containing, among others, quantum planes, Drinfeld-Jimbo quantized enveloping algebras $U_q(g)$, and $G$-universal enveloping algebras of Lie color algebras.

If $R$ is an associative algebra acted on by a character Hopf algebra $H$, then any character skew primitive element $h \in L_g$ acts on $R$ as a $h^*(g)$-skew $g$-derivation. In this situation, any left module $M$ over the smash product $R\#H$ is a module over the skew polynomial ring $R[x; g, h]$, where the action of $x$ coincides with the action of $h$, i.e., $x.m = hm$. Therefore, we are in a position to apply Propositions 7, 9 and Corollary 3 to actions of character Hopf algebras.

**Theorem 13.** Let $H$ be a character Hopf algebra over the field $k$ of characteristic 0 and suppose that $\chi^h(g)$ is not an $n^{th}$ primitive root of unity ($n > 1$) for any character skew primitive element $h \in L_g$ and $g \in G$. Let $R$ be an associative $H$-module algebra. Then:

1. Every finite dimensional irreducible left $R\#H$-module is completely reducible as a left $R$-module. In particular, $J_{\text{fin}}(R) \subseteq J_{\text{fin}}(R\#H)$.
2. If $R$ is left socular, then irreducible left $R\#H$-modules are completely reducible as left $R$-modules. Thus $J(R) \subseteq J(R\#H)$.

**References**


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