

DIVISIBILITY OF AN EIGENFORM BY AN EIGENFORM

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ABSTRACT. It has been shown in several settings that the product of two eigenforms is rarely an eigenform. In this paper we consider the more general question of when the product of an eigenform with any modular form is again an eigenform. We prove that this can occur only in very special situations. We then relate the divisibility of eigenforms to linear independence of vectors of Rankin-Selberg L -values.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

There have been several works regarding the factorization of eigenforms for the full modular group $\Gamma = SL_2(\mathbb{Z})$. In particular, Rankin [15] considered products of Eisenstein series. Independently, Duke [6] and Ghate [10] show that the product of two eigenforms is an eigenform in only finitely many cases. More generally, Emmons and Lanphier [7] show that the product of any number of eigenforms is an eigenform only finitely many times. The present paper will consider a factorization that allows one factor to be any modular form. It is shown in Sections 2, 3, and 4 that given some technical conditions, the only eigenforms that can divide other eigenforms come from one dimensional spaces. This is a corollary of Theorems 1.3, 1.4, and 1.5.

It is well known that there is a basis of eigenforms for the space S_k of cuspforms of weight k on $SL_2(\mathbb{Z})$. Together S_k and the Eisenstein series E_k generate the full space M_k of modular forms of weight k . Further, every noncuspidal eigenform is an Eisenstein series. Additionally, a basis of eigenforms is necessarily an orthogonal basis under the Petersson inner product [5, p. 163]. For more information on these topics, see any basic text on modular forms, such as [14] or [5]. We define Eisenstein series as follows.

Definition 1.1. The weight k Eisenstein series is the modular form given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where B_k is the k th Bernoulli number, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, and $q = e^{2\pi iz}$.

In this paper we investigate an eigenform h divided by an eigenform f with quotient g which is a modular form, that is,

$$(1.2) \quad h = fg.$$

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Without loss of generality we assume that all eigenforms considered are normalized so that the first nonzero coefficient is one. The dividend h could be either cuspidal or an Eisenstein series. Likewise the divisor f could be either cuspidal or an Eisenstein series. It is impossible to divide an Eisenstein series by a cuspidal eigenform and obtain a quotient which is again a modular form, so our problem naturally breaks into three cases to consider:

- Case 1. Both the dividend h and divisor f are cuspidal eigenforms.
- Case 2. The dividend h is a cuspidal eigenform, but the divisor f is an Eisenstein series.
- Case 3. Both the dividend h and divisor f are Eisenstein series.

Each of these cases leads to a theorem related to the factorization of some polynomials. In Cases 1 and 2 these polynomials are the characteristic polynomials $T_{n,k}(x)$ of the n^{th} Hecke operator of weight k . In the third case this polynomial is the Eisenstein polynomial $\varphi_k(x)$, whose roots are the j -zeroes of the weight k Eisenstein series E_k (see Definition 4.1).

In Case 1, both dividend h and divisor f are cuspidal eigenforms. In this case the quotient g cannot be cuspidal. The following theorem gives a comparison of the dimension of $S_{wt(h)}$ and $M_{wt(g)}$, the spaces which contain the dividend h and quotient g , respectively.

Theorem 1.3. *Suppose a cuspidal eigenform f divides another cuspidal eigenform h with quotient g a modular form. Then either $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$ or, for every $n \geq 2$, $T_{n,wt(h)}(x)$ is reducible over the field \mathbb{F}_f (see Definition 2.1 for \mathbb{F}_f).*

In Case 2, the divisor f is an Eisenstein series, but the dividend h is still a cuspidal modular form. Hence the quotient g must be cuspidal. In this case our result is as follows.

Theorem 1.4. *Suppose an Eisenstein series f divides a cuspidal eigenform h with quotient g a modular form. Then either $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$ or, for every $n \geq 2$, $T_{n,wt(h)}(x)$ is reducible over \mathbb{Q} .*

In Case 3, the dividend h is an Eisenstein series, and so the quotient g must be noncuspidal. In this case in place of the Hecke polynomial we are led to consider the Eisenstein polynomial $\varphi_k(x)$ of weight k (see Definition 4.1). Our result is as follows.

Theorem 1.5. *Suppose an Eisenstein series f divides another Eisenstein series h with quotient g a modular form. Then either $\dim(M_{wt(h)}) = \dim(M_{wt(g)})$ or the polynomial $\varphi_{wt(h)}(x)$ is reducible over \mathbb{Q} .*

In each of the above theorems there is either an equality of the dimensions of the appropriate spaces or information about the factorization of a certain polynomial, $T_{n,wt(h)}(x)$ or $\varphi_{wt(h)}(x)$. For small weights it is known that these polynomials do not factor, and so the dividend h and quotient g must come from spaces of the same dimension. For higher weights it is conjectured that this is still the case. See Section 6 for details.

2. PROOF OF THEOREM 1.3

Theorem 1.3 tells us that if we write $h = fg$ where h and f are cuspidal eigenforms, then either $T_{n,wt(h)}(x)$ is reducible over \mathbb{F}_f or $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$.

We now present the formal definition of \mathbb{F}_f .

Definition 2.1. Given a normalized eigenform f , let \mathbb{F}_f denote the field generated over \mathbb{Q} by its Fourier coefficients. That is, if $f = \sum_{n \geq 0} a_n q^n$, then $\mathbb{F}_f = \mathbb{Q}(a_0, a_1, a_2, \dots)$.

Recall that \mathbb{F}_f/\mathbb{Q} is a finite extension and $\dim(\mathbb{F}_f) \leq \dim(S_{wt(f)})$ [16, ch. 3].

The following special subspaces of S_k will play an important role in our proofs.

Definition 2.2. Let $\mathbb{F} \subseteq \mathbb{C}$ be a field. A subspace $S \subseteq S_k$ is said to be \mathbb{F} -rational if it is stable under the action of $Gal(\mathbb{C}/\mathbb{F})$; i.e., $\sigma(S) = S$ for all $\sigma \in Gal(\mathbb{C}/\mathbb{F})$. Here an automorphism σ acts on modular forms through their Fourier coefficients.

We consider such spaces to obtain information about the Hecke polynomials. The following crucial lemma gives a condition guaranteeing that all of the Hecke polynomials for a certain weight are reducible.

Lemma 2.3. *If S is a proper \mathbb{F} -rational subspace of S_k and S contains an eigenform, then all the Hecke polynomials of weight k are reducible over \mathbb{F} .*

In all known cases the Hecke polynomials $T_{n,k}(x)$ are irreducible. Hence the contrapositive is more practical.

Corollary 2.4. *If for some n , $T_{n,k}(x)$ is irreducible over \mathbb{F} , then no proper \mathbb{F} -rational subspace of S_k contains an eigenform.*

We now prove Lemma 2.3.

Proof of Lemma 2.3. Let $S \subset S_k$ be a proper \mathbb{F} -rational subspace containing an eigenform h . Then define

$$R := \langle \sigma(h) \mid \sigma \in Gal(\mathbb{C}/\mathbb{F}) \rangle_{\mathbb{C}} \leq S,$$

which is also a proper \mathbb{F} -rational subspace of S_k . Then $S_k = R \oplus R^\perp$, both of which are proper and stable under the action of the Hecke operators because they have eigenform bases. Denote by $T_{n,k}|_R(x)$ the characteristic polynomial of $T_{n,k}$ restricted to R . Note that $T_{n,k}(x) = T_{n,k}|_R(x) \cdot T_{n,k}|_{R^\perp}(x)$. Since R is \mathbb{F} -rational, $T_{n,k}|_R(x) \in \mathbb{F}[x]$. Also, $T_{n,k}(x) \in \mathbb{F}[x]$ (actually $T_{n,k}(x) \in \mathbb{Z}[x]$). So $T_{n,k}|_{R^\perp}(x) \in \mathbb{F}[x]$. Therefore $T_{n,k}(x)$ is reducible over \mathbb{F} for all n . \square

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose we have the factorization $h = fg$ where f and h are cuspidal eigenforms. Then by dimension considerations

$$\dim(S_{wt(h)}) \geq \dim(M_{wt(g)}).$$

If $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$, the proof is complete. So we assume $\dim(S_{wt(h)}) > \dim(M_{wt(g)})$. Let $\{g_1, \dots, g_b\}$ be a rational basis of $S_{wt(g)}$. Then the space $fM_{wt(g)} = \langle fE_{wt(g)}, fg_1, fg_2, \dots, fg_b \rangle$ is an \mathbb{F}_f -rational subspace of $S_{wt(h)}$ of dimension $\dim(M_{wt(g)})$. Because $\dim(S_{wt(h)}) > \dim(M_{wt(g)})$, it is a proper \mathbb{F}_f -rational subspace of $S_{wt(h)}$. On the other hand, this space contains an eigenform $h = fg$. Hence by Lemma 2.3 we know that $T_{n,wt(h)}(x)$ is reducible over \mathbb{F}_f for all n . \square

From the dimension formula [5] for spaces of modular forms, we find that

$$\dim(S_{wt(h)}) = \dim(M_{wt(g)})$$

occurs only as in the following cases.

Lemma 2.5. *Write $h = fg$ where h and f are both cuspidal eigenforms. Then $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$ in and only in the following cases:*

$$wt(f) = 12, wt(g) \equiv 4, 6, 8, 10, 12, 14 \pmod{12}$$

$$wt(f) = 16, wt(g) \equiv 4, 6, 10, 12 \pmod{12}$$

$$wt(f) = 18, wt(g) \equiv 4, 8, 12 \pmod{12}$$

$$wt(f) = 20, wt(g) \equiv 6, 12 \pmod{12}$$

$$wt(f) = 22, wt(g) \equiv 4, 12 \pmod{12}$$

$$wt(f) = 26, wt(g) \equiv 12 \pmod{12}.$$

On the other hand, if $\dim(S_{wt(h)}) = \dim(M_{wt(g)})$, then Lemma 2.5 implies that $wt(f)$ is one of 12, 16, 18, 20, 22, or 26. Thus $\dim(S_{wt(f)}) = 1$. In these cases we can use linear algebra to construct a factorization $h = fg$. In particular, the basis $\{E_{wt(g)}, g_1, \dots, g_b\}$ of $M_{wt(g)}$ maps to the basis $\{fE_{wt(g)}, fg_1, \dots, fg_b\}$ of $S_{wt(h)}$ so that everything, including the eigenforms in $S_{wt(h)}$, has a factor of f . Also note that if $\dim(S_{wt(h)}) = 1$, then the above reduces to the cases that are treated in [6] and [10].

Corollary 2.6. *If $h = fg$ with h and f cuspidal eigenforms and for some n , $T_{n, wt(h)}(x)$ is irreducible over every field \mathbb{F} of degree less than $\dim(S_{wt(h)})$, then f comes from a one dimensional space, i.e., $wt(f) = 12, 16, 18, 20, 22, 26$.*

We note that while part of the hypothesis regarding $T_{n,k}(x)$ used in the above corollary may appear strange, it follows from Maeda's Conjecture [11]; see also Section 6.

3. PROOF OF THEOREM 1.4

Theorem 1.4 tells us that if we write $h = fg$ where h is a cuspidal eigenform and f is an Eisenstein series, then either $T_{n, wt(h)}(x)$ is reducible over \mathbb{Q} or $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$.

Proof of Theorem 1.4. Suppose we have a factorization $h = fg$ where h is a cuspidal eigenform and f is an Eisenstein series. Then by dimension considerations $\dim(S_{wt(h)}) \geq \dim(S_{wt(g)})$. If $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$, the proof is complete. So we assume $\dim(S_{wt(h)}) > \dim(S_{wt(g)})$. Let $\{g_1, \dots, g_b\}$ be a rational basis of $S_{wt(g)}$. Then the space $fS_{wt(g)} = \langle fg_1, \dots, fg_b \rangle$ is a rational subspace of $S_{wt(h)}$ of dimension $\dim(S_{wt(g)})$. Because $\dim(S_{wt(h)}) > \dim(S_{wt(g)})$, it is a proper rational subspace of $S_{wt(h)}$. On the other hand, this space contains an eigenform $h = fg$. Hence by Lemma 2.3 we know that $T_{n, wt(h)}(x)$ is reducible over \mathbb{Q} for all n . \square

From the dimension formula [5] for spaces of modular forms, we find that

$$\dim(S_{wt(h)}) = \dim(S_{wt(g)})$$

occurs only in the following cases.

Lemma 3.1. *Write $h = fg$ where h is a cuspidal eigenform and f is an Eisenstein series. Then $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$ in and only in the following cases:*

$$wt(f) = 4, wt(g) \equiv 0, 4, 6, 10 \pmod{12}$$

$$wt(f) = 6, wt(g) \equiv 0, 4, 8 \pmod{12}$$

$$wt(f) = 8, wt(g) \equiv 0, 6 \pmod{12}$$

$$wt(f) = 10, wt(g) \equiv 0, 4 \pmod{12}$$

$$wt(f) = 14, wt(g) \equiv 0 \pmod{12}.$$

On the other hand, if $\dim(S_{wt(h)}) = \dim(S_{wt(g)})$, then Lemma 3.1 implies that $wt(f)$ is one of 4, 6, 8, 10, or 14. Thus $\dim(M_{wt(f)}) = 1$. In these cases we can use linear algebra to construct a factorization $h = fg$. In particular, the basis $\{g_1, \dots, g_b\}$ of $S_{wt(g)}$ maps to the basis $\{fg_1, \dots, fg_b\}$ of $S_{wt(h)}$ so that everything, including the eigenforms in $S_{wt(h)}$, has a factor of f . Also note that if $\dim(S_{wt(h)}) = 1$, then the above reduces to the cases that are treated in [6] and [10].

Corollary 3.2. *If $h = fg$ with h a cuspidal eigenform, f an Eisenstein series and for some n , $T_{n, wt(h)}(x)$ is irreducible over \mathbb{Q} , then f comes from a one dimensional space. i.e., $wt(f) = 4, 6, 8, 10, 14$.*

Again we note the connection of our hypothesis to Maeda's Conjecture [11]; see also Section 6.

4. PROOF OF THEOREM 1.5

Theorem 1.5 tells us that if we write $h = fg$ where h and f are both Eisenstein series, then either the Eisenstein polynomial $\varphi_{wt(h)}(x)$ of weight k is reducible over \mathbb{Q} or $\dim(M_{wt(h)}) = \dim(M_{wt(g)})$.

We now define the Eisenstein polynomial $\varphi_k(x)$ of weight k .

Definition 4.1. Let $\varphi_k = \prod (x - j(z_l))$, where the product runs over all the j -zeros of E_k except for 0 and 1728. (Under the j -mapping, ρ and i correspond to 0 and 1728, respectively.)

Note that $\varphi_k(x)$ is monic with rational coefficients. See [9] or [4] for more information on this function.

Proof of Theorem 1.5. Suppose $h = fg$ where both h and f are Eisenstein series. Then $\varphi_{wt(f)}(x)$ divides $\varphi_{wt(h)}(x)$. Hence either $\varphi_{wt(f)}(x)$ is a constant, a constant multiple of $\varphi_{wt(h)}(x)$ or $\varphi_{wt(h)}(x)$ is reducible.

If $\varphi_{wt(f)}(x)$ is a constant, then f must be one of E_4, E_6, E_8, E_{10} , or E_{14} . Thus by dimension considerations $\dim(M_{wt(h)}) = \dim(M_{wt(g)})$.

If $\varphi_{wt(f)}(x)$ is a constant multiple of $\varphi_{wt(h)}(x)$, then $\dim(M_{wt(f)}) = \dim(M_{wt(h)})$, so that by dimension considerations g must be one of E_4, E_6, E_8, E_{10} , or E_{14} . However, then f, g , and h are all Eisenstein series, which by [6] and [10] can occur only if $\dim(M_{wt(f)}) = \dim(M_{wt(g)}) = \dim(M_{wt(h)}) = 1$. \square

From the dimension formula [5] for spaces of modular forms, we find that

$$\dim(M_{wt(h)}) = \dim(M_{wt(g)})$$

occurs only in the following cases.

Lemma 4.2. *Write $h = fg$ where h and f are both Eisenstein series. Then $\dim(M_{wt(h)}) = \dim(M_{wt(g)})$ in and only in the following cases:*

$$wt(f) = 4, wt(g) \equiv 0, 4, 6, 10 \pmod{12}$$

$$wt(f) = 6, wt(g) \equiv 0, 4, 8 \pmod{12}$$

$$wt(f) = 8, wt(g) \equiv 0, 6 \pmod{12}$$

$$wt(f) = 10, wt(g) \equiv 0, 4 \pmod{12}$$

$$wt(f) = 14, wt(g) \equiv 0 \pmod{12}.$$

On the other hand, if $\dim(M_{wt(h)}) = \dim(M_{wt(g)})$, Lemma 4.2 implies that $wt(f)$ is one of 4, 6, 8, 10, or 14. Thus $\dim(M_{wt(f)}) = 1$. In these cases we can construct a factorization $h = fg$ as in the previous section. Again note that if $\dim(S_{wt(h)}) = 1$, then the above reduces to the cases that are treated in [6] and [10].

Corollary 4.3. *If $h = fg$ with h and f Eisenstein series and $\varphi_{wt(h)}(x)$ is irreducible over \mathbb{Q} , then f comes from a one dimensional space, i.e., $wt(f) = 4, 6, 8, 10, 14$.*

We note that the part of the hypothesis regarding $\varphi_{wt(h)}(x)$ used in the above is conjectured to always hold [4, 9].

5. RELATIONSHIP TO L -VALUES

In this section we investigate the relationship between the divisibility properties discussed in Section 3 and Rankin Selberg L -values. As in (1.2) we write $h = fg$ to denote the eigenform f dividing the eigenform h . Here the dividend h is a cuspform, and the divisor $f = E_r$ is an Eisenstein series. Thus the quotient g is cuspidal. Let $\{h_1, \dots, h_d\}$ and $\{g_1, \dots, g_b\}$ be normalized eigenform bases for $S_{wt(h)}$ and $S_{wt(g)}$ respectively.

Write $g = \sum_{n \geq 1} a_n q^n$ and $h = \sum_{n \geq 1} b_n q^n$. The Rankin-Selberg convolution of g and h is defined by

$$L(g \times h, s) = \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^s}.$$

With this notation the Rankin-Selberg method [2] yields

$$(5.1) \quad \langle E_r g, h \rangle = (4\pi)^{-(s+wt(h)-1)} \Gamma(s+wt(h)-1) L(g \times h, s+wt(h)-1).$$

We are particularly interested in the Rankin-Selberg L -function value at $s = wt(h) - 1$; hence we use the following notation:

$$L(g, h) := L(g \times h, wt(h) - 1).$$

We will employ Theorem 1.4 to give insight into the question of linear independence of certain vectors of Rankin-Selberg L -values. Recall that eigenforms are orthogonal under the Petersson inner product ($\langle h_j, h_i \rangle = 0$ for $j \neq i$). Let $h_1 = h = E_r g$ and express g in terms of its eigenform bases, $g = c_1 g_1 + \dots + c_d g_d$. Then for each $i \neq 1$, we have

$$c_1 \langle E_r g_1, h_i \rangle + \dots + c_b \langle E_r g_b, h_i \rangle = \langle h_1, h_i \rangle = 0.$$

Setting $s = 0$ and dividing by $(4\pi)^{-wt(h)+1} \Gamma(wt(h) - 1)$, (5.1) yields for each $i \neq 1$,

$$(5.2) \quad c_1 L(g_1, h_i) + \dots + c_b L(g_b, h_i) = 0.$$

We express the coefficients in (5.2) as a set of vectors,

$$(5.3) \quad \left\{ \begin{bmatrix} L(g_1, h_2) \\ \vdots \\ L(g_1, h_d) \end{bmatrix}, \dots, \begin{bmatrix} L(g_b, h_2) \\ \vdots \\ L(g_b, h_d) \end{bmatrix} \right\}.$$

Proposition 5.4. *Let $\{h_1, \dots, h_d\}$ and $\{g_1, \dots, g_b\}$ be normalized eigenform bases for the spaces $S_{wt(h)}$ and $S_{wt(g)}$ respectively, with $wt(h) \geq wt(g) + 4$. If there is an n such that $T_{n, wt(h)}(x)$ is irreducible over \mathbb{Q} and $d > b$, then the vectors of L values given in (5.3) are linearly independent over \mathbb{C} . If there is an n such that $T_{n, wt(h)}(x)$ is irreducible over \mathbb{Q} and $d = b$, there is precisely one dependence relation.*

Proposition 5.4 can be restated in terms of the matrix $M = M(g \times h)$ whose columns are the vectors in (5.3).

Proposition 5.4'. *Let $\{h_1, \dots, h_d\}$ and $\{g_1, \dots, g_b\}$ be normalized eigenform bases for the spaces $S_{wt(h)}$ and $S_{wt(g)}$ respectively, with $wt(h) \geq wt(g) + 4$. If there is an n such that $T_{n, wt(h)}(x)$ is irreducible over \mathbb{Q} , then the matrix $M(g \times h)$ is of full rank.*

Proof. Suppose $T_{n, wt(h)}(x)$ is irreducible for some n . There are two cases to consider.

Case 1: $d > b$. Suppose there is a solution $[c_1, \dots, c_b]^T$ to the matrix equation $M\vec{x} = \vec{0}$. We must show that $[c_1, \dots, c_b]^T = \vec{0}$. We have, for each $i = 2, 3, \dots, d$,

$$c_1 L(g_1, h_i) + \dots + c_b L(g_b, h_i) = 0.$$

By using the Rankin-Selberg method and denoting $G := c_1 g_1 + \dots + c_b g_b$, we have $\langle G \cdot E_r, h_i \rangle = 0$ for $i = 2, \dots, d$. Hence $G \cdot E_r$ is orthogonal to each of h_2, h_3, \dots, h_d , and so $G \cdot E_r = c h_1$ for some $c \in \mathbb{C}$. Theorem 1.4 implies that $g = 0$ and $c = 0$, which further implies that $c_1 = \dots = c_b = 0$.

Case 2: $d = b$. Because M is underdetermined there clearly are nonzero solutions to the matrix equation $M\vec{x} = \vec{0}$. We must show that M has nullity 1. Suppose there are two nonzero solutions $[c_1, \dots, c_b]^T$ and $[c'_1, \dots, c'_b]^T$ to the matrix equation $M\vec{x} = \vec{0}$. Similarly to the above, we construct $G := c_1 g_1 + \dots + c_b g_b$ and $G' := c'_1 g_1 + \dots + c'_b g_b$, which satisfy, respectively, $E_r G = c h_1$, $E_r G' = c' h_1$ for some $c, c' \in \mathbb{C}$. Thus G and G' are scalar multiples of each other. Thus any two solutions are dependent. \square

6. CONCLUSIONS AND MAEDA'S CONJECTURE

The main results of this paper state that if there are eigenforms h and f and a modular form g such that $h = fg$, then either the modular spaces containing g and h must have the same dimension or all of the Hecke polynomials for $S_{wt(h)}$ or the Eisenstein polynomial of weight $wt(h)$ are reducible, depending on whether or not h is cuspidal. In this section we discuss the unlikeliness that these polynomials are reducible and we discuss the cases where the modular spaces containing g and h do in fact have the same dimension. First, we state the following partial converse of Theorems 1.3, 1.4, and 1.5.

Proposition 6.1. *Let h and f be eigenforms.*

Case 1. *Both h and f are cuspidal eigenforms. If*

$$\dim(S_{wt(h)}) = \dim(M_{wt(h)-wt(f)}),$$

then there is a modular form g such that $fg = h$.

Case 2. *Only h is a cuspidal eigenform; f is an Eisenstein series. If*

$$\dim(S_{wt(h)}) = \dim(S_{wt(h)-wt(f)}),$$

then there is a cuspidal modular form g such that $fg = h$.

Case 3. *Both h and f are Eisenstein series. If*

$$\dim(M_{wt(h)}) = \dim(M_{wt(h)-wt(f)}),$$

then there is a modular form g such that $fg = h$.

In each of Cases 1, 2, and 3 there are infinitely many examples of eigenforms f and h such that f divides h as in equation (1.2).

Proof. Here, we only consider one specific instance of Case 2. The other instances and cases follow similarly. From Lemma 3.1 we see that there are twelve infinite classes such as $wt(f) = 4$, $wt(h) \equiv 4$ modulo 12. In each of these instances we can divide any cuspidal eigenform h of weight $wt(h)$ by E_4 . This is because $\dim(S_{wt(g)}) = \dim(S_{wt(h)})$, and so $E_4 S_{wt(g)} = S_{wt(h)}$. \square

Example 6.2. We now present an explicit example of a factorization in which g is not an eigenform. Let $\{h_1, h_2\}$ be an eigenform basis for S_{28} . Note that $\{E_{16}\Delta, E_4\Delta^2\}$ is another, more explicit, basis. Hence we can write h_1 and h_2 in terms of these functions, one of which is

$$E_{16}\Delta + \left(-\frac{14903892}{3617} - 108\sqrt{18209} \right) E_4\Delta^2.$$

Factoring E_4 out of the above form gives the following equation expressed in terms of the basis $\{E_{12}\Delta, \Delta^2\}$ of S_{24} :

$$\begin{aligned} E_4 \left(E_{12}\Delta + \left(-\frac{3075516}{691} - 108\sqrt{18209} \right) \Delta^2 \right) \\ = E_{16}\Delta + \left(-\frac{14903892}{3617} - 108\sqrt{18209} \right) E_4\Delta^2. \end{aligned}$$

Note in particular that the quotient $E_{12}\Delta + \left(-\frac{3075516}{691} - 108\sqrt{18209} \right) \Delta^2$ is not an eigenform and recall that $E_4 \cdot E_{12} \neq E_{16}$.

Call a factorization not counted by Proposition 6.1 *exceptional*; such a factorization would involve a quotient g and a dividend h that come from modular spaces of different dimensions. In light of the following conjectures, we believe there are no exceptional factorizations. If this is true, then Proposition 6.1 is a full converse of Theorems 1.3, 1.4, and 1.5.

Conjecture 6.3 (Maeda, [11]). *The Hecke algebra over \mathbb{Q} of $S_k(SL_2(\mathbb{Z}))$ is a simple (that is, a single) number field whose Galois closure over \mathbb{Q} has Galois group isomorphic to the symmetric group \mathcal{S}_n (with $n = \dim S_k(SL_2(\mathbb{Z}))$).*

Maeda's conjecture significantly restricts the factorization of the Hecke polynomial $T_{n,k}(x)$. Proposition 6.4 below tells us that if $T_{n,k}(x)$ has a full Galois group, then $T_{n,k}(x)$ is irreducible over all fields \mathbb{F} with $[\mathbb{F} : \mathbb{Q}] < \dim(S_k)$. This is significant because \mathbb{F}_f used in Section 2 satisfies this condition.

This conjecture appeared in [11], and in the same paper was verified for weights less than 469. Buzzard [3] showed that $T_{2,k}(x)$ is irreducible up to weight 2000. The fact that $T_{p,k}(x)$ has a full Galois group was verified for $p \leq 2000$ up to weight 2000 by Farmer and James [8]. Kleinerman [13] showed that $T_{2,k}(x)$ is irreducible up to weight 3000. Alhgren [1] showed for all weights k that if for some n , $T_{n,k}(x)$ is irreducible and has a full Galois group, then $T_{p,k}(x)$ does as well for all $p \leq 4,000,000$. Finally from correspondence between Stein and Ghitza it is known that $T_{2,k}(x)$ is irreducible up to weight 4096. In particular, for weights less than 2000, Case 1 in Proposition 6.1 is a full converse of Theorem 1.3, and for weights less than 4096, Case 2 in Proposition 6.1 is a full converse of Theorem 1.4.

Proposition 6.4. *Let $P(x) \in \mathbb{Q}[x]$ be a degree d polynomial. Let K_P be its splitting field. Assume $[K_P : \mathbb{Q}] = d!$. If P factors over K , then $[K : \mathbb{Q}] \geq d$.*

Proof. Suppose P is reducible over K and $[K : \mathbb{Q}] < d$. Write $P = QR$, where $Q, R \in K[x]$ are polynomials of degrees d_1, d_2 and have splitting fields K_Q, K_R respectively. Then $d_1 + d_2 = d$, and so

$$d_1!d_2! \geq [K_Q : K] \cdot [K_R : K] \geq [K_Q K_R : K] \geq [K_P : K] > (d-1)!,$$

which occurs if and only if $d_1 = 0$ or $d_2 = 0$. Hence one of Q or R is a constant so that P is irreducible over K . \square

Concerning the Eisenstein polynomials, $\varphi_k(x)$, we have the following.

Conjecture 6.5 (Cornelissen [4] and Gekeler [9]). *The Eisenstein polynomials $\varphi_k(x)$ have full Galois group \mathcal{S}_n (with $n = \dim(S_k)$); in particular, they are irreducible over \mathbb{Q} .*

This question was first raised by Cornelissen [4] and Gekeler [9], who found that $\varphi_k(x)$ has a full Galois group for all weights $k \leq 172$. We have verified the irreducibility of $\varphi_k(x)$ for weights up to 2500.

We computed $\varphi_k(x)$ modulo small primes p for weights through 2500 to verify that it is irreducible. An equation presented in [12] and [4] gives the equation

$$\frac{E_k}{E_4^a E_6^b \Delta^c} = \varphi_k(j(\tau)),$$

where $4a + 6b + 12c = r$, with $0 \leq a \leq 2, 0 \leq b \leq 1$. For each weight computed there is a small prime p such that $\varphi_k(x)$ is indeed irreducible modulo p , and so $\varphi_k(x)$ is irreducible over \mathbb{Q} . There is no reason other than runtime that the highest weight computed was 2500. In these weights Case 3 of Proposition 6.1 is a full converse of Theorem 1.5.

As a final remark we note that if Proposition 6.4 and Conjecture 6.5 are true, then Proposition 6.1 is a full converse to all the main theorems. This means that an eigenform is divisible by another eigenform precisely in the cases listed in Lemmas 2.5, 3.1, and 4.2.

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