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EXTENSION OF LYAPUNOV'S CONVEXITY THEOREM TO SUBRANGES

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Abstract. Consider a measurable space with a finite vector measure. This measure defines a mapping of the σ-field into a Euclidean space. According to Lyapunov's convexity theorem, the range of this mapping is compact and, if the measure is atomless, this range is convex. Similar ranges are also defined for measurable subsets of the space. We show that the union of the ranges of all subsets having the same given vector measure is also compact and, if the measure is atomless, it is convex. We further provide a geometrically constructed convex compactum in the Euclidean space that contains this union. The equality of these two sets, which holds for two-dimensional measures, can be violated in higher dimensions.

1. Introduction

Let $(X, \mathcal{F})$ be a measurable space and $\mu = (\mu_1, ..., \mu_m)$, $m = 1, 2, \ldots$, be a finite vector measure on it. For each $Y \in \mathcal{F}$ consider the range $R_\mu(Y) = \{\mu(Z) : Z \in \mathcal{F}, Z \subset Y\} \subset \mathbb{R}^m$ of the vector measures of all its measurable subsets Z. Lyapunov's convexity theorem [11] states that the range $R_\mu(X)$ is compact and furthermore, if μ is atomless, this range is convex. Of course, this is also true for any $Y \in \mathcal{F}$.

Let $\mathcal{S}_\mu^p(X)$ be the set of all measurable subsets of X with the vector measure $p \in R_\mu(X)$,

$$\mathcal{S}_\mu^p(X) = \{Y \in \mathcal{F} : \mu(Y) = p\}.$$

Of course, $\mathcal{S}_\mu^p(X) = \emptyset$ if $p \notin R_\mu(X)$. For $p \in \mathbb{R}^m$ consider the union of the ranges of all subsets of X with the vector measure p,

$$R_\mu^p(X) = \bigcup_{Y \in \mathcal{S}_\mu^p(X)} R_\mu(Y).$$

In particular, $R_\mu^p(X) = \emptyset$ if $p \notin R_\mu(X)$, and $R_\mu^{\mu(X)}(X) = R_\mu(X)$.

Since the relation $Y_1 = Y_2$ (μ-everywhere) is an equivalence relation on $\mathcal{F}$, it partitions any subset of $\mathcal{F}$ into equivalence classes. For an atomless μ, Lyapunov [11, Theorem III] proved that: (i) $\mathcal{S}_\mu^p(X)$ consists of one equivalence class if and only if p is an extreme point of $R_\mu(X)$, and (ii) if $p \in R_\mu(X)$ is not an extreme

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point of $R^p_\mu(X)$, then the set of equivalence classes in $\mathcal{S}^p_\mu(X)$ has cardinality of the continuum.

In general, a union of an infinite number of compact convex sets may be neither closed nor convex. As follows from Dai and Feinberg [1], the set $R^p_\mu(X)$ is a convex compactum if $m = 2$ and μ is atomless. This fact follows from stronger results that hold for $m = 2$. For $m = 2$ and atomless μ, Dai and Feinberg [1, Theorem 2.3] showed that there exists a set $Z^* \in \mathcal{S}^p_\mu(X)$, called a maximal subset, such that

$$R_\mu(Z^*) = R^p_\mu(X), \tag{1.1}$$

and, in addition, the following equality holds:

$$R^p_\mu(X) = Q^p_\mu(X), \tag{1.2}$$

where $Q^p_\mu(X)$ is the intersection of $R_\mu(X)$ with its shift by a vector $-(\mu(X) - p)$,

$$Q^p_\mu(X) = (R_\mu(X) - \{\mu(X) - p\}) \cap R_\mu(X), \tag{1.3}$$

with $S_1 - S_2 = \{q - r : q \in S_1, r \in S_2\}$ for $S_1, S_2 \subset \mathbb{R}^m$. In particular, $R_\mu(X) - \{r\}$ is a parallel shift of $R_\mu(X)$ by $-r \in \mathbb{R}^m$. Examples 3.3 and 3.4 below demonstrate that equalities (1.1) and (1.2) may not hold when μ is not atomless, even for $m = 1$.

Each of equalities (1.1) and (1.2) implies that $R^p_\mu(X)$ is convex and compact. However, [1, Example 4.2] demonstrates that a maximal set Z^* may not exist for an atomless vector measure μ when $m > 2$.

In this paper, we prove (Theorem 2.1) that for any natural number m the set $R^p_\mu(X)$ is compact and, if μ is atomless, this set is convex. This is a generalization of Lyapunov's convexity theorem, which is a particular case of this statement for $p = \mu(X)$. We also prove that $R^p_\mu(X) \subset Q^p_\mu(X)$ (Theorem 2.2). Example 3.1 demonstrates that it is possible that equality (1.2) may not hold when $m > 2$ and μ is atomless.

For a countable set A, consider a set $\{p^a : a \in A\}$ of m-dimensional vectors. The conditions

$$\text{(i)} \sum_{a \in A} p^a = \mu(X) \text{ and (ii)} \sum_{a \in B} p^a \in R_\mu(X) \text{ for any finite subset } B \subset A \tag{1.4}$$

are obviously necessary for the existence of a partition $\{X^a : X^a \in \mathcal{F}, a \in A\}$ of the set X such that $\mu(X^a) = p^a$ for all $a \in A$. According to Dai and Feinberg [1, Theorem 2.5], (1.4) is a necessary and sufficient condition for the existence of such a partition when $m = 2$, μ is atomless, and A is countable. Example 3.2 below demonstrates that this condition is not sufficient for the existence of such a partition for an atomless μ when $m > 2$ and A consists of more than two points.

By using Lyapunov's theorem, Dvoretzky, Wald, and Wolfowitz [2], [3] proved the existence of the described partition when μ is atomless, A is finite, and

$$p^a = \int_X \pi(a|x)\mu(dx), \qquad a \in A, \tag{1.5}$$

for some transition probability π from X to A. Edwards [5, Theorem 4.5] generalized this result to countable A. Khan and Rath [8, Theorem 2] gave another proof of this generalization. The existence of such a partition is equivalent to the existence of a measurable mapping $\varphi : (X, \mathcal{F}) \to A$ such that for any $B \subseteq A$,

$$\int_X I\{\varphi(x) \in B\}\mu_i(dx) = \int_X \pi(B|x)\,\mu_i(dx), \qquad i = 1, \ldots, m. \tag{1.6}$$

Consider a measurable space $(A, \mathcal{A})$. According to the contemporary terminology, a transition probability π can be purified if there exists a measurable function $\varphi : (X, \mathcal{F}) \to (A, \mathcal{A})$ satisfying (1.6) for all $B \in \mathcal{A}$.

Loeb and Sun [9, Example 2.7] constructed an elegant example when a transition probability cannot be purified for $m = 2$, $X = [0, 1]$, $A = [-1, 1]$, and atomless μ. However, purification holds for nowhere countably generated measurable spaces $(X, \mathcal{F}, \mu)$, also called saturated. Loeb and Sun [9] discovered this for nonatomic Loeb measure spaces, Podczeck [12] extended this result to saturated spaces by using specially developed functional analysis techniques, and Loeb and Sun [10] showed that by using the methods from Hoover and Keisler [6], purification can be easily extended from Loeb's measure spaces to saturated spaces; see also Keisler and Sun [7] for properties of saturated spaces.

2. Main results

Theorem 2.1. *For any vector $p \in R_\mu(X)$, the set $R^p_\mu(X)$ is compact and, in addition, if the vector measure μ is atomless, this set is convex.*

Proof. We say that a partition is measurable if all its elements are measurable sets. Consider the set

$$V_{\mu,3}(X) = \{(\mu(S_1), \mu(S_2), \mu(S_3)) : \{S_1, S_2, S_3\} \text{ is a measurable partition of } X\}.$$

According to Dvoretzky, Wald, and Wolfowitz [4, Theorems 1 and 4], $V_{\mu,3}(X)$ is compact and, if μ is atomless, this set is convex. Now let

$$W^p_\mu(X) = \{(s_1, s_2, s_3) : (s_1, s_2, s_3) \in V_{\mu,3}(X),\ s_3 = \mu(X) - p,\ s_1 + s_2 = p\}.$$

This set is compact and, if μ is atomless, it is convex. This is true, because $W^p_\mu(X)$ is an intersection of $V_{\mu,3}(X)$ and two planes in $\mathbb{R}^{3m}$. These planes are defined by the equations $s_3 = \mu(X) - p$ and $s_1 + s_2 = p$ respectively. We further define

$$T^p_\mu(X) = \left\{s_1 : (s_1, s_2, s_3) \in W^p_\mu(X)\right\}.$$

Since $T^p_\mu(X)$ is a projection of $W^p_\mu(X)$, the set $T^p_\mu(X)$ is compact and, if μ is atomless, it is convex.

The last step of the proof is to show that $T^p_\mu(X) = R^p_\mu(X)$ by establishing that (i) $T^p_\mu(X) \subset R^p_\mu(X)$ and (ii) $T^p_\mu(X) \supset R^p_\mu(X)$. Indeed, for (i), for any $s_1 \in T^p_\mu(X)$, there exists $(s_1, s_2, s_3) \in W^p_\mu(X)$ or equivalently there exists a measurable partition $\{S_1, S_2, S_3\}$ of X such that $\mu(S_3) = \mu(X) - p$ and $\mu(S_1) + \mu(S_2) = p$. Let $Z = S_1 \cup S_2$. Then $\mu(Z) = p$, $s_1 \in R_\mu(Z)$, and thus $s_1 \in R^p_\mu(X)$. For (ii), for any $s_1 \in R^p_\mu(X)$, there exists a set $Z \in \mathcal{F}$ such that $\mu(Z) = p$ and $s_1 \in R_\mu(Z)$, which further implies that there exists a measurable subset S_1 of Z such that $\mu(S_1) = s_1$. Let $S_2 = Z \setminus S_1$ and $S_3 = X \setminus Z$. Then $\mu(S_1) + \mu(S_2) = p$ and $\mu(S_3) = \mu(X) - p$, which further implies that $(s_1, \mu(S_2), \mu(S_3)) \in W^p_\mu(X)$. Thus $s_1 \in T^p_\mu(X)$. □

Theorem 2.2. *$R^p_\mu(X) \subset Q^p_\mu(X)$ for any vector $p \in R_\mu(X)$.*

Recall that $R^p_\mu(X) = Q^p_\mu(X)$ when $m = 2$ and μ is atomless; see (1.2). The proof of Theorem 2.2 uses the following lemma.

Lemma 2.3 ([1, Lemma 3.3])**.** *For any vector $p \in R(X)$, each of the sets $R^p_\mu(X)$ and $Q^p_\mu(X)$ is centrally symmetric with the center $\frac{1}{2}p$.*

Though it is assumed in [1] that the measure μ is atomless, this assumption is not used in the proofs of Lemmas 3.1-3.3 in [1].

Proof of Theorem 2.2. Let $q \in R^p_\mu(X)$. Since $R^p_\mu(X) \subset R_\mu(X)$, then $q \in R_\mu(X)$. Furthermore, in view of Lemma 2.3, $p-q \in R^p_\mu(X)$. Therefore, $p-q \in R_\mu(X)$. Since $R_\mu(X)$ is centrally symmetric with the center $\frac{1}{2}\mu(X), R_\mu(X) = \{\mu(X)\} - R_\mu(X)$ and $p - q \in R_\mu(X) = \{\mu(X)\} - R_\mu(X)$. Therefore, $q \in R_\mu(X) - \{\mu(X) - p\}$. As follows from the definition of $Q^p_\mu(X)$ in (1.3), $q \in Q^p_\mu(X)$. $\square$

3. Counterexamples

The first example shows that equality (1.2) may not hold when μ is atomless and $m > 2$. In particular, the inclusion in Theorem 2.2 cannot be substituted with the equality.

Example 3.1. Consider the measure space $(X, \mathcal{B}, \mu)$, where $X = [0, 6]$, $\mathcal{B}$ is the Borel σ-field on X, and $\mu(dx) = (\mu_1, \mu_2, \mu_3)(dx) = (f_1(x), f_2(x), f_3(x))\,dx$, where

$$f_1(x) = \begin{cases} 30 & x \in [0,1), \\ 40 & x \in [1,2), \\ 10 & x \in [2,4), \\ 15 & x \in [4,5), \\ 5 & x \in [5,6]; \end{cases} \quad f_2(x) = \begin{cases} 40 & x \in [0,1), \\ 10 & x \in [1,2), \\ 20 & x \in [2,4), \\ 10 & x \in [4,5), \\ 30 & x \in [5,6]; \end{cases} \quad f_3(x) = \begin{cases} 10 & x \in [0,1), \\ 20 & x \in [1,3), \\ 30 & x \in [3,4), \\ 20 & x \in [4,5), \\ 25 & x \in [5,6]. \end{cases}$$

These density functions are plotted in Figure 1. Note that $\mu(X) = (110, 130, 125)$

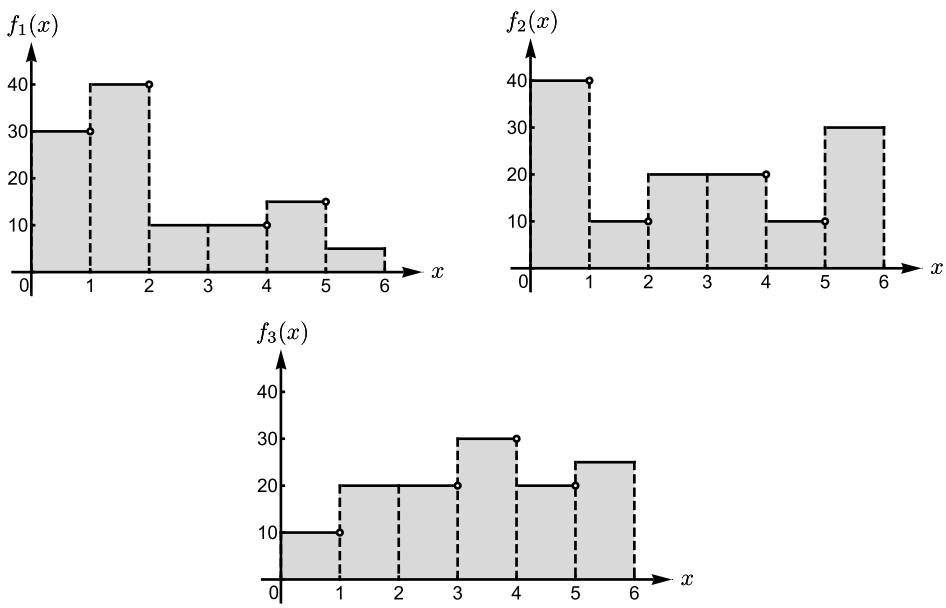

Figure 1. Density functions of the vector measure in Example 3.1.

and

$$R_\mu(X) = \left\{ \sum_{i=1}^{6} \alpha_i p^i \,:\, \alpha_i \in [0,1],\ i = 1, \ldots, 6 \right\}$$

is a zonotope, where $p^1 = \mu([0,1)) = (30, 40, 10)$, $p^2 = \mu([1,2)) = (40, 10, 20)$, $p^3 = \mu([2,3)) = (10, 20, 20)$, $p^4 = \mu([3,4)) = (10, 20, 30)$, $p^5 = \mu([4,5)) = (15, 10, 20)$, and $p^6 = \mu([5,6)) = (5, 30, 25)$.

Let $p = p^1 + p^2 + p^3 = (80, 70, 50)$. Observe that p is an extreme point of $R_\mu(X)$. Indeed, consider the vector $d = \left(\frac{7}{5}, 1, -\frac{8}{5}\right)$ and the linear function $l_d(\alpha)$ defined for all $\alpha = (\alpha_1, \ldots, \alpha_6) \in \mathbb{R}^6$ by the scalar product

$$\begin{aligned} l_d(\alpha) &= d \cdot \left(\sum_{i=1}^{6} \alpha_i p^i \right) = \sum_{i=1}^{6} \alpha_i (d \cdot p^i) \\ &= 66\alpha_1 + 34\alpha_2 + 2\alpha_3 - 14\alpha_4 - \alpha_5 - 3\alpha_6. \end{aligned}$$

For $\alpha \in [0,1]^6$, this function achieves its maximum at the unique point $\alpha^* = (1, 1, 1, 0, 0, 0)$, and $l_d(\alpha^*) = 66 + 34 + 2 = 102$. In addition, $\sum_{i=1}^{6} \alpha_i^* p^i = p$. So, $d \cdot r - 102 \leq 0$ for all $r \in R_\mu(X)$ and the equality holds if and only if $r = p$. Thus, $d \cdot r - 102 = 0$ is a supporting hyperplane of the convex polytope $R_\mu(X)$, and the intersection of the polytope and hyperplane consists of the single point p. This implies that p is an extreme point of $R_\mu(X)$.

According to the definition of $R_\mu(X)$, for $p \in R_\mu(X)$ there exists a measurable subset $Z \in \mathcal{F}$ such that $\mu(Z) = p$ and, according to [11, Theorem III] described in Section 1, since p is extreme, such Z is unique up to null sets. In particular, $p = \mu(Z)$ for $Z = [0,3]$. Thus,

$$R_\mu^p(X) = R_\mu(Z) = \left\{ \sum_{i=1}^{3} \alpha_i p^i \,:\, \alpha_i \in [0,1],\ i = 1, 2, 3 \right\}.$$

Choose $q = (56, 29, 31)$ and observe that $q \notin R_\mu^p(X)$. Indeed, $q \in R_\mu^p(X)$ if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$ such that $\sum_{i=1}^{3} \alpha_i p^i = q$, which is equivalent to

$$\alpha_1(30, 40, 10) + \alpha_2(40, 10, 20) + \alpha_3(10, 20, 20) = (56, 29, 31), \tag{3.1}$$

but the only solution to the linear system of equations (3.1) is

$$\alpha_1 = \frac{3}{10},\ \alpha_2 = \frac{11}{10},\ \alpha_3 = \frac{3}{10},$$

where $\alpha_2 \notin [0,1]$.

On the other hand, $q \in Q_\mu^p(X)$, because: (i) $q \in R_\mu(X)$ and (ii) $q \in R_\mu(X) - \{\mu(X) - p\}$. Indeed, (i) holds since, for $Z_1 = \left[0, \frac{42}{115}\right) \cup \left[1, 1\frac{229}{230}\right) \cup \left[2, 2\frac{33}{460}\right) \cup \left[4, 4\frac{3}{10}\right)$,

$$\begin{aligned} \mu(Z_1) &= \frac{42}{115} \times (30, 40, 10) + \frac{229}{230} \times (40, 10, 20) \\ &+ \frac{33}{460} \times (10, 20, 20) + \frac{3}{10} \times (15, 10, 20) \\ &= (56, 29, 31) = q. \end{aligned}$$

Notice that (ii) is equivalent to $q+\mu(X)-p \in R_\mu(X)$, where $q+\mu(X)-p = (56,29,31)+(110,130,125)-(80,70,50)=(86,89,106)$. Let $Z_2=\left[0,\frac{15}{46}\right)\cup\left[1,1\frac{45}{46}\right)\cup\left[2,2\frac{209}{230}\right)\cup[3,5)\cup\left[5,5\frac{3}{5}\right)$. Then

$$\begin{aligned}\mu(Z_2) &= \frac{15}{46}\times(30,40,10)+\frac{45}{46}\times(40,10,20)+\frac{209}{230}\times(10,20,20)\\ &+ 1\times(10,20,30)+1\times(15,10,20)+\frac{3}{5}\times(5,30,25)\\ &= (86,89,106)=q+\mu(X)-p.\end{aligned}$$

Thus (ii) holds too, and $R^p_\mu(X)\neq Q^p_\mu(X)$. □

The following example demonstrates that the necessary condition (1.4) for the existence of a measurable partition $\{X^a : a\in A\}$ with $\mu(X^a)=p^a$, $a\in A$, is not sufficient for an atomless measure μ when $m>2$. In this example, A consists of three points. According to [1, Theorem 2.5], this condition is necessary and sufficient when $m=2$, A is countable, and μ is atomless. If A consists of two points, say a and b, and $p^a\in R_\mu(X)$, $p^b=\mu(X)-p^a$, then the partition $\{X^a,X^b\}$ always exists with X^a selected as any $X^a\in\mathcal{F}$ satisfying $\mu(X^a)=p^a$ and with $X^b=X\setminus X^a$.

Example 3.2. Consider the measure space $(X,\mathcal{B},\mu)$ defined in Example 3.1. Let $p^1=(56,29,31)$, $p^2=(24,41,19)$, $p^3=(30,60,75)$, and $A=\{1,2,3\}$. Then $p^1+p^2+p^3=\mu(X)$. We further observe that: (i) p^1 is the vector q from Example 3.1, so $p^1\in R_\mu(X)$ and therefore $p^2+p^3=\mu(X)-p^1\in R_\mu(X)$; (ii) p^1+p^3 is the vector $q+\mu(X)-p$ from Example 3.1, so $p^1+p^3\in R_\mu(X)$ and therefore $p^2=\mu(X)-p^1-p^3\in R_\mu(X)$; (iii) p^1+p^2 is the vector p from Example 3.1, so $p^1+p^2\in R_\mu(X)$ and therefore $p^3=\mu(X)-p^1-p^2\in R_\mu(X)$. Thus, the vectors p^a, $a\in A$, satisfy (1.4).

If there exists a partition $\{X^a\in\mathcal{B}: a\in A\}$ of X with $\mu(X^a)=p^a$ for all $a\in A$, let $Y=X^1\cup X^2$. Since $X^1\cap X^2=\emptyset$, $\mu(X^1)=p^1=q$, and $\mu(Y)=p^1+p^2=p$, then $q\in R^p_\mu(X)$. However, according to Example 3.1, $q\notin R^p_\mu(X)$. This contradiction implies that a partition $\{X^a\in\mathcal{B}: a\in A\}$ of X, with $\mu(X^a)=p^a$ for all $a\in A$, does not exist. □

In conclusion, we provide two simple examples showing that if μ is not atomless, then even for $m=1$ (and, therefore, for any natural number m) a maximal subset Z^*, defined in (1.1), may not exist and equality (1.2) may not hold.

Example 3.3. Consider the probability space $(X,2^X,\mu)$, where $X=\{1,2,3,4\}$ and

$$\mu(\{1\})=0.1,\ \mu(\{2\})=0.4,\ \mu(\{3\})=0.2,\ \mu(\{3\})=0.3.$$

Let $p=0.5$. Then $\mathcal{S}^p_\mu=\{\{1,2\},\{3,4\}\}$. In other words, the only subsets that have the measure 0.5 are $Z^1=\{1,2\}$ and $Z^2=\{3,4\}$. However, $R_\mu(Z^1)$ is not a subset of $R_\mu(Z^2)$ and vice versa. Therefore, a maximal subset does not exist for $p=0.5$. □

Example 3.4. Consider the probability space $(X,2^X,\mu)$, where $X=\{1,2,3\}$ and

$$\mu(\{1\})=0.1,\ \mu(\{2\})=0.55,\ \mu(\{3\})=0.35.$$

The range of μ on X is $R_\mu(X)=\{0,0.1,0.35,0.45,0.55,0.65,0.9,1\}$. Let $p=0.55$. Then $Q^p_\mu=\{0,0.1,0.45,0.55\}$ and $R^p_\mu=\{0.55\}$. Thus $R^p_\mu\subset Q^p_\mu$, but $R^p_\mu\neq Q^p_\mu$. □

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