AN OVERDETERMINED PROBLEM FOR THE HELMHOLTZ EQUATION

ROBERT DALMASSO

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Abstract. We give a partial answer to a conjecture concerning an overdetermined problem for the Helmholtz equation.

1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be a smooth bounded connected open set. Let \( a, b \in \mathbb{R} \) and \( \lambda \in \mathbb{C} \) be constants. We consider solutions of the overdetermined elliptic boundary value problem for the Helmholtz equation:

\[
\begin{align*}
\Delta u + \lambda u &= 0 \quad \text{in } \Omega, \\
u &= a \quad \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \nu} &= b \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \partial/\partial \nu \) is the outward normal derivative.

If \( b = 0 \) and if we consider nonconstant solutions, we get as a special case Schiffer’s problem (Yau [13], p. 688, problem 80). If \( a = 0 \) and \( b \neq 0 \), the problem was posed by Berenstein [1] when \( n = 2 \) (see also Berenstein and Yang [2] when \( n \geq 3 \)).

In 1981 Williams [10] proved that if \( \partial \Omega \) is Lipschitz and (1)-(3) has a nonconstant solution for \( b = 0 \), then \( \partial \Omega \) is real analytic. In 2002 Williams [11] proved that if \( \partial \Omega \) is \( C^1 \) and (1)-(3) has a nonconstant solution \( u \in C^2(\overline{\Omega}) \), then \( \partial \Omega \) is real analytic.

The following conjecture is stated in [11] (see also [12]):

Conjecture. Let \( \Omega \subset \mathbb{R}^n \) be a Lipschitz bounded connected open set. Assume that \( \mathbb{R}^n \setminus \Omega \) is connected. If (1)-(3) has a nonconstant solution for some constants \( \lambda \in \mathbb{C} \) and \( a, b \in \mathbb{R} \), then \( \Omega \) is a ball.

When \( \partial \Omega \) is of class \( C^{2,\alpha} \) with \( \alpha \in (0, 1] \) we may assume that \( \lambda > 0 \) in trying to prove the conjecture; see [11] Remark 1, p. 299.

We shall prove the following theorem.

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Theorem 1. Assume that $\partial \Omega$ is of class $C^1$. Let $\lambda_1$ (resp. $\lambda_2$) denote the first (resp. second) eigenvalue of $-\Delta$ for the Dirichlet problem in $\Omega$.

1) If there is a nonconstant solution $u \in C^2(\Omega)$ of (1)-(3) with $\lambda \leq \lambda_1$, then $\Omega$ is a ball.

2) Assume moreover that $\Omega$ is convex and symmetric about a hyperplane. If there is a nonconstant solution $u \in C^2(\Omega)$ of (1)-(3) with $\lambda \leq \lambda_2$, then $\Omega$ is a ball.

Recently Deng [5] has obtained some results when $n = 2$ and $b = 0$. More precisely, let $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$ denote the eigenvalues of $-\Delta$ for the Neumann problem in $\Omega$. If $\lambda < \mu_8$, then $\Omega$ is a disk. If $\Omega$ is strictly convex and centrally symmetric and $\lambda < \mu_{13}$, then $\Omega$ is a disk.

We shall need the following particular case of [9].

Theorem 2. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded connected open set with $C^2$ boundary. Let $\lambda > 0$, $\epsilon = 0$ or $\epsilon = \pm 1$. Suppose that there exists $u \in C^2(\Omega)$ such that

$$\Delta u + \lambda u + \epsilon = 0 \quad \text{in} \quad \Omega,$$

$$u > 0 \quad \text{in} \quad \Omega$$

$$u = 0, \quad \frac{\partial u}{\partial \nu} = c \quad \text{on} \quad \partial \Omega,$$

where $c$ denotes some constant. Then $\Omega$ is a ball and $u$ is radially symmetric.

2. Preliminaries

For $\nu \in \mathbb{C}$ and $z \in \mathbb{C}\setminus(-\infty,0]$, the $\nu$-th Bessel function is defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n \geq 0} \frac{(-1)^n(z^2/4)^n}{n! \Gamma(\nu + n + 1)}.$$

Here is a collection of some basic properties of the Bessel function; see [3].

Lemma 1. 1) $z^{-\nu} J_\nu(z)$ is an entire function of $z \in \mathbb{C}$ with $z^{-\nu} J_\nu(z) \big|_{z=0} = 1/2\nu \Gamma(\nu + 1)$.

2) For real $\nu > -1$, $J_\nu$ has only real zeros $j_{\nu,k}$, $k \in \mathbb{N}$, such that $\lim_{k \to \infty} j_{\nu,k} = \infty$.

3) For $r > 0$ we have

$$\frac{d}{dr} (r^\nu J_\nu(r)) = r^\nu J_{\nu-1}(r) \quad \text{and} \quad \frac{d}{dr} (r^{-\nu} J_{\nu}(r)) = -r^{-\nu} J_{\nu+1}(r).$$

4) For real $\nu$ the zeros of $J_\nu$ and $J_{\nu+1}$ separate each other. If $j(\nu,k)$ denotes the $k$-th positive zero of $J_\nu$, we have

$$j(\nu,k) < j(\nu+1,k) < j(\nu,k+1).$$

We shall repeatedly use the following function:

$$z(x) = |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x|), \quad x \in \mathbb{R}^n.$$  

With the help of Lemma 1 (3) we easily verify that $\Delta z + z = 0$. 

Proposition 1. Assume that $\Omega$ is the unit ball $B(0, 1)$ of $\mathbb{R}^n$. If $u$ is a nonconstant solution of (1)-(3), then $ab < 0$ if $0 < \lambda < \lambda_1$ and $ab > 0$ if $\lambda_1 < \lambda < \lambda_2$.

Proof. $\sqrt{\lambda} = 1$ is the first positive zero of $J_{\frac{n}{2}}$ and $\sqrt{\lambda} = 2$ is the first positive zero of $J_{\frac{n}{2}}$. For $\lambda \in (0, \lambda_1) \cup (\lambda_1, \lambda_2)$ the solution of (1)-(3) is unique and is given by

$$u(x) = \frac{a\lambda^\frac{n}{2} - \frac{1}{2}}{J_{\frac{n}{2} - 1}(\sqrt{\lambda})} z(\sqrt{\lambda}|x|), \quad |x| < 1,$$

with

$$b = -\frac{a\lambda J_{\frac{n}{2}}(\sqrt{\lambda})}{J_{\frac{n}{2} - 1}(\sqrt{\lambda})}.$$

Moreover, $a \neq 0$ if $u$ is nonconstant. The proposition follows using Lemma 1. □

Proposition 2. Retain the setting of Proposition 1. If $u$ is a nonconstant solution of (1)-(3) with $\lambda = \lambda_2$, then $b = 0$.

Proof. The eigenspace corresponding to $\lambda_2$ is the $n$-dimensional space generated by the linearly independent eigenfunctions

$$v_j(x) = \frac{\partial z}{\partial x_j} (\sqrt{\lambda} x), \quad |x| < 1, \quad j = 1, \cdots, n.$$

We first show that $a \neq 0$. Indeed, if $a = 0$, there exist $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ not all zero such that

$$u = \sum_{j=1}^{n} \alpha_j v_j.$$

We have

$$\frac{\partial u}{\partial \nu}(x) = \lambda_2^{1 - \frac{n}{2}} J_{\frac{n}{2} + 1}(\sqrt{\lambda} x) \sum_{j=1}^{n} \alpha_j x_j, \quad x \in \partial B(0, 1),$$

which is not constant since by Lemma 1 $J_{\frac{n}{2} + 1}(\sqrt{\lambda}) \neq 0$, and we get a contradiction. Now the only nonconstant solution of (1)-(3) is $u(x) = \alpha z(\sqrt{\lambda_2}|x|), \quad x \in B(0, 1)$, where $\alpha = a/z(\sqrt{\lambda_2})$. Therefore $b = 0$. □

Proposition 3. Retain the setting of the Conjecture. If (1)-(3) has a nonconstant solution $u$ when $b = 0$, then $\lambda \geq \lambda_2$.

Proof. Since $u$ is nonconstant, $\lambda \neq 0$ and $a \neq 0$. If $w = (u - a)/\lambda a$ we have

$$\Delta w + \lambda w + 1 = 0 \quad \text{in } \Omega, \quad w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial \Omega.$$

By (1), Lemma 2, we have

$$\int_{\Omega} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_k} \, dx = 0 \quad \text{for } j \neq k.$$

Note that $\partial u/\partial x_k \neq 0$ for $1 \leq k \leq n$. Indeed, assume that $\partial u/\partial x_k \equiv 0$ for some $k \in \{1, \cdots, n\}$. For all $x \in \Omega$ there exists $y \in \partial \Omega$ such that $y_j = x_j$ for $j \neq k$ and $tx + (1 - t)y \in \Omega$ for $0 < t \leq 1$. Then $u(x) = u(y) = a$ and we reach a contradiction since $u$ is nonconstant. Suppose that there exist $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ such that

$$\sum_{j=1}^{n} \alpha_j \frac{\partial u}{\partial x_j} = 0.$$
Multiplying (6) by $\partial u/\partial x_k$, integrating over $\Omega$ and using (5) we get

$$\alpha_k \int_{\Omega} \left( \frac{\partial u}{\partial x_k} \right)^2 \, dx = 0,$$

which implies that $\alpha_k = 0$. Therefore the functions $\partial u/\partial x_j$ are linearly independent eigenfunctions for the Dirichlet problem with eigenvalue $\lambda$. Since $\lambda_1$ is a simple eigenvalue, we deduce that $\lambda \geq \lambda_2$. \hfill \Box

**Remark 1.** Assume that $n = 2$, $\lambda = \lambda_2$ and $b = 0$. If (1)-(3) has a nonconstant solution, then $\Omega$ is a ball; see [1]. Moreover, when $n = 2$, Proposition 3 is established in [1].

### 3. Proof of Theorem 1

Let $u \in C^2(\overline{\Omega})$ be a nonconstant solution of (1)-(3) with $\lambda \leq \lambda_2$. By [11] $\partial \Omega$ is real analytic. Then we can assume that $\lambda > 0$.

We denote by $\varphi$ an eigenfunction corresponding to $\lambda_1$; $\varphi$ can be taken positive by [3].

Suppose that $a \neq 0$. If $w = (u - a)/\lambda a$ we have

$$\Delta w + \lambda w + 1 = 0 \quad \text{in} \quad \Omega, \quad w = 0, \quad \frac{\partial w}{\partial \nu} = \frac{b}{\lambda a} \quad \text{on} \quad \partial \Omega.$$  

Let $\Omega_1 = \{ x \in \Omega; \ w(x) < -1/\lambda \}$.

**Lemma 2.** Suppose that $\Omega_1 \neq \emptyset$. Then $\Omega_1$ is connected.

**Proof.** Assume the contrary. Let $C_1$ and $C_2$ denote two connected components of $\Omega_1$. Define

$$v_j(x) = \begin{cases} w(x) + \frac{1}{\lambda}, & x \in C_j, \\ 0, & x \in \Omega \setminus C_j, \end{cases} \quad j = 1, 2.$$ 

Then we have

$$\lambda_2 \int_{\Omega} \psi^2 \, dx \leq \int_{\Omega} |\nabla \psi|^2 \, dx = \alpha^2 \int_{\Omega} |\nabla v_1|^2 \, dx + \int_{\Omega} |\nabla v_2|^2 \, dx.$$ 

Let $j = 1$ or 2. For all $\rho$ in the closure of $C_0^\infty(C_j)$ in the topology of $H^1_0(C_j)$ we have

$$\int_{\Omega} \nabla v_j \nabla \rho \, dx = \lambda \int_{\Omega} v_j \rho \, dx,$$

$j = 1, 2$. Hence from (8) we get

$$\lambda_2 \int_{\Omega} \psi^2 \, dx \leq \lambda \int_{\Omega} \psi^2 \, dx,$$

which implies that $\lambda \geq \lambda_2$, and we reach a contradiction when $\lambda < \lambda_2$. When $\lambda = \lambda_2$ we have

$$\lambda_2 \int_{\Omega} \psi^2 \, dx = \int_{\Omega} |\nabla \psi|^2 \, dx.$$

We deduce that $\psi$ is an eigenfunction corresponding to $\lambda_2$ (see [6]). Owing to the regularity theory $\psi$ is analytic in $\Omega$. Since $\psi$ vanishes in $\Omega \setminus (C_1 \cup C_2)$, we obtain a contradiction. \hfill \Box
Now we can give a proof of Theorem 1.

1) Suppose first that \( \lambda = \lambda_1 \). Multiplying (1) by \( \varphi \) and integrating over \( \Omega \) we get

\[
\lambda_1 \int_{\Omega} \varphi u \, dx = - \int_{\Omega} \varphi \Delta u \, dx = - \int_{\Omega} u \Delta \varphi \, dx + a \int_{\partial \Omega} \frac{\partial \varphi}{\partial \nu} \, ds
\]

\[
= a \int_{\partial \Omega} \frac{\partial \varphi}{\partial \nu} \, ds + \lambda_1 \int_{\Omega} \varphi u \, dx.
\]

Since \( \partial \varphi / \partial \nu < 0 \) on \( \partial \Omega \) (see [4], p. 212), we deduce that \( a = 0 \). Since \( u \) is nonconstant, \( u = C \varphi \), where \( C \in \mathbb{R} \backslash \{0\} \). Hence from Theorem 2, it follows that \( \Omega \) is a ball.

Now let \( 0 < \lambda < \lambda_1 \). \( \Omega \) is strictly contained in \( \Omega \), \( \lambda > \lambda_1 \) (see [3]) and we have a contradiction. Therefore \( \Omega_1 = \emptyset \). Then \( \Delta w \leq 0 \) in \( \Omega \) and the maximum principle implies that \( w > 0 \) in \( \Omega \). As above we conclude with the help of Theorem 2.

2) Since the problem is invariant under translations and rotations we can assume that \( \Omega \) is symmetric about the plane \( T = \{x \in \mathbb{R}^n ; x_n = 0\} \). If \( E \subset \mathbb{R}^n \) we denote by \( E' \) its reflection in \( T \). By 1) we can assume that \( \lambda_1 < \lambda \leq \lambda_2 \).

Case 1: \( \lambda_1 < \lambda < \lambda_2 \), \( a \neq 0 \) since \( u \) is nonconstant. Then \( w = (u - a) / \lambda a \) satisfies (7). \( w \) is symmetric in \( x_n \). Indeed let

\[
v(x) = w(x', x_n) - w(x', -x_n), \quad x = (x', x_n) \in \Omega.
\]

Since

\[
\Delta v + \lambda v = 0 \quad \text{in} \quad \Omega, \quad v = 0, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega,
\]

we deduce that \( v = 0 \). Therefore \( \partial w / \partial x_n = 0 \) on \( \Omega \cap T \).

Let \( \mathbb{R}_+^n = \{x \in \mathbb{R}^n ; x_n > 0\} \) and \( A = \Omega \cap \mathbb{R}_+^n \).

i) Assume that \( ab > 0 \). \( \partial w / \partial x_n = (b / \lambda a) \nu_n \) on \( \partial \Omega \). Since \( \Omega \) is convex and symmetric about \( T \) and \( \partial \Omega \) is analytic, \( \nu_n(x) > 0 \) for all \( x \in \partial \Omega \cap \mathbb{R}_+^n \). Therefore, for all \( x \in \partial \Omega \cap \mathbb{R}_+^n \), there exists \( r > 0 \) such that \( \partial w / \partial x_n > 0 \) in \( B(x, r) \cap \overline{\Omega} \).

Suppose first that \( B = \{x \in A ; \partial w / \partial x_n(x) < 0\} \neq \emptyset \). Then \( \partial w / \partial x_n = 0 \) on \( \partial B \) and \( \partial w / \partial x_n > 0 \) in \( B' \). There exists \( \alpha > 0 \) so that

\[
\psi = \alpha \frac{\partial w}{\partial x_n} \chi_{B'} + \frac{\partial w}{\partial x_n} \chi_B
\]

(with \( \chi_E \) denoting the characteristic function of \( E \subset \mathbb{R}^n \)) satisfies

\[
\int_{\Omega} \varphi \psi \, dx = 0.
\]

Then, using the same arguments as in the proof of Lemma 2, we get \( \lambda > \lambda_2 \) and we have a contradiction. Therefore \( \partial w / \partial x_n \geq 0 \) in \( A \) and the maximum principle implies that \( \partial w / \partial x_n > 0 \) in \( A \). We deduce that \( \partial w / \partial x_n < 0 \) in \( A' \). We claim that \( u < 0 \) in \( \Omega \). Let \( x = (x', x_n) \in \Omega \). Since \( \Omega \) is convex and symmetric about \( T \) there...
exist \( t_1 < 0 < t_2 \) such that \((x',t_j) \in \partial \Omega \) for \( j = 1,2 \) and \((x',t) \in \Omega \) for \( t_1 < t < t_2 \). We have \( \partial w/\partial x_n(x',t) < 0 \) for \( t_1 < t < 0 \) and \( \partial w/\partial x_n(x',t) > 0 \) for \( 0 < t < t_2 \). Since \( w(x',t_j) = 0 \) for \( j = 1,2 \) we deduce that \( w(x) < 0 \), and our claim is proved. Then Theorem 2 implies that \( \Omega \) is a ball.

ii) Assume that \( ab < 0 \). Now \( \partial w/\partial x_n < 0 \) on \( \partial \Omega \cap \mathbb{R}^n_+ \). If \( \{ x \in A : \partial w/\partial x_n(x) > 0 \} \neq \emptyset \), we obtain a contradiction using the same argument as in i). Therefore \( \partial w/\partial x_n \leq 0 \) in \( A \), and arguing as in i) we obtain that \( w > 0 \) in \( \Omega \). Then Theorem 2 implies that \( \Omega \) is a ball, a contradiction to Proposition 1. We conclude that \( ab < 0 \) cannot occur.

iii) Suppose that \( b = 0 \).

**Lemma 3.** We have

\[
\frac{\partial^2 w}{\partial x_j \partial x_k} = -\nu_j \nu_k \quad \text{on} \quad \partial \Omega, \quad 1 \leq j, k \leq n.
\]

**Proof.** \( \partial \Omega \) is locally the graph of an analytic function of \( n - 1 \) variables \( \rho \); that is, \( \partial \Omega \) is locally defined by \( x_n = \rho(x_1, \ldots, x_{n-1}) \) and \( x = (x_1, \ldots, x_n) = (x', x_n) \in \Omega \) if and only if \( x_n < \rho(x_1, \ldots, x_{n-1}) = \rho(x') \). Then the outward unit normal \( \nu \) is given by

\[
\nu(x) = \frac{1}{\sqrt{1 + |\nabla \rho(x')|^2}} (-\nabla \rho(x'), 1).
\]

We have

\[
\frac{\partial w}{\partial x_j} = 0 \quad \text{on} \quad \partial \Omega, \quad 1 \leq j \leq n.
\]

Differentiating \((9)\) with respect to \( x_k \) for \( k \in \{1, \ldots, n-1\} \) and multiplying by \( \nu_n \) we get

\[
\nu_n \frac{\partial^2 w}{\partial x_j \partial x_k} - \nu_k \frac{\partial^2 w}{\partial x_j \partial x_n} = 0,
\]

for \( 1 \leq j \leq n \). Let \( j = k \) in \((10)\). Adding we obtain

\[
\frac{\partial}{\partial \nu} \left( \frac{\partial w}{\partial x_n} \right) = \nu_n \Delta w = -\nu_n
\]

for \( 1 \leq j \leq n \). Multiplying \((10)\) by \( \nu_k \) and adding we obtain

\[
\frac{\partial^2 w}{\partial x_j \partial x_n} = \nu_n \frac{\partial}{\partial \nu} \left( \frac{\partial w}{\partial x_j} \right),
\]

for \( 1 \leq j \leq n \). \((11)\) and \((12)\) imply that \( \partial^2 w/\partial x_n^2 = -\nu_n^2 \). Then \((10)\) implies that \( \partial^2 w/\partial x_k \partial x_n = -\nu_k \nu_n \). Using \((10)\) again the lemma follows.

Equation \((11)\) in the proof of Lemma 3 implies that for all \( x \in \partial \Omega \cap \mathbb{R}^n_+ \) there exists \( r > 0 \) such that \( \partial w/\partial x_n > 0 \) on \( B(x, r) \cap \Omega \). If \( C = \{ x \in A : \partial w/\partial x_n(x) < 0 \} \neq \emptyset \), then \( \partial w/\partial x_n = 0 \) on \( \partial C \) and we get a contradiction as in i). If \( C = \emptyset \), then as in i) we show that \( w < 0 \) in \( \Omega \) and Theorem 2 implies that \( \Omega \) is a ball, a contradiction to Proposition 1. We conclude that \( b = 0 \) cannot occur.
Case 2: \( \lambda = \lambda_2 \). Suppose first that \( a = 0 \). Then \( b \neq 0 \) since \( u \) is nonconstant and \( u \) is an eigenfunction corresponding to \( \lambda_2 \). \( u \) is symmetric in \( x_n \); hence \( \partial u / \partial x_n = 0 \) on \( \Omega \cap T \). We can assume that \( b > 0 \). Since \( \partial u / \partial x_n = b \nu_n \) on \( \partial \Omega \), we have \( \partial u / \partial x_n > 0 \) on \( \partial \Omega \cap \mathbb{R}^n_n \). If \( D = \{ x \in A ; \partial u / \partial x_n < 0 \} \neq \emptyset \), we have \( \partial u / \partial x_n = 0 \) on \( \partial D \) and we obtain a contradiction as in Case 1 i). Therefore \( \partial u / \partial x_n \geq 0 \) in \( A \). Arguing as in Case 1 i) we obtain that \( u < 0 \) in \( \Omega \), and we reach a contradiction since the second eigenfunction changes sign. Therefore \( a \neq 0 \). As before \( w = (u - a) / \lambda a \) satisfies (7). If \( ab > 0 \) (resp. \( ab < 0 \)), as in Case 1 i) (resp. ii)) we obtain that \( w < 0 \) (resp. \( w > 0 \)), and we conclude that \( \Omega \) is a ball, a contradiction to Proposition 2. Therefore \( b = 0 \). Arguing as in Case 1 iii) we show that \( \Omega \) is a ball. \( \square \)

Remark 2. Theorem 1 (2) complements Berenstein’s result (see Remark 1).

4. Concluding remarks

Below we give an alternative proof of Theorem 1 (1) when \( \lambda < \lambda_1 \), which seems of independent interest.

Let \( \chi \) be the torsion function relative to \( \Omega \), that is,

\[
\Delta \chi + 1 = 0 \quad \text{in } \Omega, \quad \chi = 0 \quad \text{on } \partial \Omega.
\]

Since \( \chi = \varphi = 0 \) on \( \partial \Omega \) and \( \partial \varphi / \partial \nu \neq 0 \) on \( \partial \Omega \), the function \( \psi \) defined by

\[
\psi = \frac{\chi}{\varphi} \quad \text{in } \Omega
\]

extends to a continuous function on \( \overline{\Omega} \).

Now we define a sequence of functions \( (\chi_n)_{n \in \mathbb{N}} \) as follows: \( \chi_0 = \chi \) and, for \( n \geq 0 \),

\[
\Delta \chi_{n+1} + \chi_n = 0 \quad \text{in } \Omega, \quad \chi_n = 0 \quad \text{on } \partial \Omega.
\]

We begin with the following lemma.

Lemma 4. We have \( \lambda^n \| \chi_n \|_\infty \leq \| \psi \|_\infty \| \varphi \|_\infty \).

Proof. The following estimate implies the lemma:

\[
(13) \quad \lambda^n \chi_n(x) \leq \| \psi \|_\infty \varphi(x) \quad \forall x \in \overline{\Omega}.
\]

(13) holds for \( n = 0 \). Assume that (13) holds for \( 0, \ldots, n-1, \ n \geq 1 \). Let \( G \) denote the Green function of the operator \( -\Delta \) for the Dirichlet problem in \( \Omega \). For \( x \in \overline{\Omega} \) we write

\[
\lambda^n \chi_n(x) = \lambda^n \int_{\Omega} G(x, y) \chi_{n-1}(y)dy
\]

\[
\leq \lambda_1 \| \psi \|_\infty \int_{\Omega} G(x, y) \varphi(y)dy = \| \psi \|_\infty \varphi(x),
\]

and (13) is proved. \( \square \)

Our assumption and Lemma 4 imply that the series

\[
\sum_{n \geq 0} \lambda^n \chi_n
\]

is uniformly convergent on \( \overline{\Omega} \) and thus defines a function \( v \in C(\overline{\Omega}) \) such that \( v = 0 \) on \( \partial \Omega \). Since

\[
\Delta v + \lambda v + 1 = 0 \quad \text{in } \mathcal{D}'(\Omega),
\]

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we conclude that \( v \) is analytic in \( \Omega \) and that

\[
\Delta v + \lambda v + 1 = 0 \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial \Omega.
\]

\( a \neq 0 \) because \( u \) is nonconstant. Then \( w = (u - a)/\lambda a \) satisfies (7). Now we have

\[
\Delta (v - w) + \lambda (v - w) = 0 \quad \text{in} \quad \Omega, \quad v - w = 0 \quad \text{on} \quad \partial \Omega,
\]

which implies that \( v = w \). It follows from the maximum principle that \( \chi_n > 0 \) in \( \Omega \) for \( n \geq 0 \). Therefore \( v > 0 \) in \( \Omega \). Since \( \partial v/\partial \nu = \partial w/\partial \nu = b/\lambda a \) on \( \partial \Omega \), Theorem 2 implies that \( \Omega \) is a ball.

**Remark 3.** Notice that the series

\[
\sum_{n \geq 0} \lambda^n \chi_n
\]

cannot converge uniformly on \( \Omega \) for \( \lambda \geq \lambda_1 \). Indeed we have

\[
\int_{\Omega} \varphi \, dx = - \int_{\Omega} \varphi \Delta \chi \, dx = - \int_{\Omega} \chi \Delta \varphi \, dx = \lambda_1 \int_{\Omega} \chi \varphi \, dx
\]

\[
= - \lambda_1 \int_{\Omega} \varphi \Delta \chi_1 \, dx = - \lambda_1 \int_{\Omega} \chi_1 \Delta \varphi \, dx
\]

\[
= \lambda_1^2 \int_{\Omega} \chi_1 \varphi \, dx = \cdots = \lambda_1^{n+1} \int_{\Omega} \chi_n \varphi \, dx \leq \lambda_1^{n+1} ||\chi_n||_{\infty} \int_{\Omega} \varphi \, dx,
\]

for \( n \geq 0 \), which implies that \( \lambda_1^{n+1} ||\chi_n||_{\infty} \geq 1 \).

**Remark 4.** When \( \Omega = B(0, 1) \) we have

\[
\sum_{n \geq 0} \lambda^n ||\chi_n||_{\infty} = \sum_{n \geq 0} \lambda^n \chi_n(0) = v(0) = \frac{\lambda^{n+1}}{2^{n+1}\Gamma(\frac{n+1}{2})} \frac{\lambda^{n+1}}{2^{\frac{n+1}{2}}} - \frac{1}{\lambda},
\]

for \( 0 < \lambda < \lambda_1 \).

**References**


Laboratoire Jean Kuntzmann, Equipe EDP, 51 rue des Mathématiques, Domaine Universitaire, BP 53, 38041 Grenoble Cedex 9, France

*E-mail address*: robert.dalmasso@imag.fr

*Current address*: L’Eden, 17 Boulevard Maurice Maeterlinck, 06300 Nice, France