SHARP COUNTEREXAMPLES RELATED TO THE DE GIORGI CONJECTURE IN DIMENSIONS $4 \leq n \leq 8$

AMIR MORADIFAM

(Communicated by James E. Colliander)

Abstract. In this note, we show that in dimensions $n \geq 4$ there exists a smooth bounded potential $V$ such that $(\Delta + V)w = 0$ has a positive solution $u$ as well as a bounded sign-changing solution $v$ satisfying

$$\int_{B_R} v^2 \leq CR^3 \quad \forall R > 0,$$

for some $C > 0$ independent of $R$. This in particular implies that the Ambrosio-Cabré proof of the De Giorgi conjecture in dimension $n = 3$ cannot be extended to dimensions $4 \leq n \leq 8$. We also answer an open question of L. Moschini [L. Moschini, New Liouville theorems for linear second order degenerate elliptic equations in divergence form, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), 11-23].

1. Introduction

Berestycki-Caffarelli-Nirenberg [3] in a paper on the qualitative properties of the non-linear partial differential equation $\Delta u + F(u) = 0$ formulated the following problems in the linear theory of elliptic partial differential equations.

**Conjecture A.** Let $L = -\Delta - V$ be a Schrödinger operator on $\mathbb{R}^n$ with a smooth and bounded potential $V$. Suppose that $u \in W^{2,p}$ for some $p > n$ is a bounded and sign-changing solution for $Lu = 0$. Then $L$ has negative spectrum: i.e.,

$$\lambda_1(V) = \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla \psi|^2 - V|\psi|^2}{\int_{\mathbb{R}^n} |\psi|^2} ; \psi \in C^\infty_0(\mathbb{R}^n) \right\} < 0.$$

**Conjecture B.** Suppose $\varphi > 0$. Then any function $u \in C^2$ such that

$$(1) \quad \varphi u \in L^\infty(\mathbb{R}^n)$$

and satisfying $\nabla (\varphi^2 \nabla u) = 0$ is necessarily constant.

It is well known that Conjecture A holds in bounded domains of $\mathbb{R}^n$, while Conjecture B holds whenever $\varphi$ is bounded and away from zero (i.e., $\varphi > \delta > 0$ on $\mathbb{R}^n$). In [3] Berestycki, Caffarelli, and Nirenberg verified that Conjectures A and B hold in dimensions $n = 1, 2$. Indeed they proved the following proposition and inquired whether this proposition remains valid in higher dimensions.
Proposition 1.1 ([3]). Suppose \( \varphi \) is a \( C^2 \) positive function on \( \mathbb{R}^n \), and let \( u \) be a solution \( u \in C^2(\mathbb{R}^n) \) of the linear equation
\[
\nabla(\varphi^2 \nabla u) = 0 \quad \text{in} \quad \mathbb{R}^n
\]
such that
\[
\varphi u \in L^\infty(\mathbb{R}^n).
\]
If the dimension \( n = 1 \) or 2, then \( u \) is necessarily constant.

Ghoussoub and Gui [7] used Proposition 1.1 to prove DeGiorgi’s conjecture (see [4] for its statement) in dimension 2. They also proved that Proposition 1.1 does not hold in higher dimensions by giving explicit counterexamples in dimensions \( n \geq 7 \), and conjectured that it should also be false in dimension \( n = 3, \ldots, 6 \). Shortly after, Barlow [2] did indeed show that the result is false for all dimensions \( n \geq 3 \). Barlow’s counterexamples were however not explicit, since he used a probabilistic approach that exploits the correspondence between symmetric diffusions \( \tilde{X} = (X_t)_t, P^x = X_t \) and divergence-form operators \( L_\varphi = \frac{1}{2} \nabla(\varphi^2 \nabla) \). Since \( \varphi \)-harmonic functions \( u \) on \( \mathbb{R}^n \) (i.e., those \( u \) that satisfy (2)) are exactly those functions such that \( u(X_t)_t \) is a \( P^x \)-martingale for all \( x \in \mathbb{R}^n \), he then reduced the problem to a construction of an appropriate diffusion with a non-trivial tail \( \sigma \)-field.

On another front, Ambrosio and Cabré [1] managed to prove DeGiorgi’s conjecture in dimension 3 by extending the above proposition in the following way. Letting \( B_R = \{|x| < R\} \) denote a ball of \( \mathbb{R}^n \) of radius \( R > 0 \) and centered at the origin, they showed the following.

Proposition 1.2 ([1]). Let \( \varphi \) be a positive function in \( L^\infty_{\text{loc}}(\mathbb{R}^n) \), and suppose that \( u \in H^1_{\text{loc}}(\mathbb{R}^n) \) satisfies
\[
u(\varphi^2 \nabla u) \geq 0 \quad \text{in} \quad \mathbb{R}^n\]
in the distributional sense, while
\[
\int_{B_R} (\varphi u)^2 \, dx \leq CR^2
\]
for some constant \( C \) independent of \( R \). Then \( u \) is necessarily constant.

They also showed that DeGiorgi’s conjecture would hold in a dimension \( n \) provided one can show that the above proposition holds with (5) replaced by
\[
\int_{B_R} (\varphi u)^2 \, dx \leq CR^{n-1}.
\]

Moschini [8] improved Proposition 1.2 and showed that (5) can be replaced by
\[
\int_{B_R} (\varphi u)^2 \, dx \leq CR^2 \log(R).
\]
She also showed that Proposition 1.2 is not valid in dimensions \( n \geq 9 \) if one replaces (5) by (6) and inquired whether in this case Proposition 1.2 remains valid in dimensions \( 4 \leq n \leq 8 \) (see Remark 5.5 in [8]). In this note we give a negative answer to the above question. Indeed, we show the following proposition.
Proposition 1.3. For any \( n \geq 3 \), there exists a positive function \( \varphi \in C^\infty(\mathbb{R}^n) \) and a non-constant function \( u \) such that \( u \nabla (\varphi^2 \nabla u) \geq 0 \) in the classical sense on \( \mathbb{R}^n \) and for every \( R > 0 \) we have
\[
\int_{B_R} (\varphi u)^2 \, dx \leq CR^2 (\log(R))^2,
\]
for some \( C > 0 \) independent of \( R \).

A similar counterexample is well known in dimension \( n = 2 \), where one can simply take \( \varphi = 1 \) (see Remark 5.4 in [8] and the references therein). See also [5] for another explicit counterexample to a different blow-up rate.

Recently Del Pino, Kowalczyk, and Wei [6] in a very interesting paper proved that De Giorgi’s conjecture [4] is false in dimensions \( n \geq 9 \). So it remained an open question whether Proposition 1.2 remains valid if one replaces (5) by (6) for dimensions \( 4 \leq n \leq 8 \). In fact, the original question raised in [1] concerns the very same problem as Proposition 1.2 for the corresponding equation, namely the largest rate of blowup allowed in the right hand side of (5) when, instead of (4), equation (2) is indeed satisfied. This is answered by the following proposition, as explained in Remark 2.1 below.

Proposition 1.4. Let \( n \geq 4 \). Then there exists a smooth bounded potential \( V \) such that \( (\Delta + V)w = 0 \) has a positive solution \( u \) as well as a bounded sign changing solution \( v \) satisfying
\[
\int_{B_R} v^2 \, dx \leq CR^3 \quad \forall R > 0,
\]
for some \( C > 0 \) independent of \( R \).

We finally mention that even though as a consequence of Proposition 1.4 above, where the Ambrosio-Cabrea proof does not extend slightly forward in dimension \( n \geq 4 \), De Giorgi’s conjecture has been recently proved in dimension \( 4 \leq n \leq 8 \) by Savin (see [9]) under a weak additional assumption.

2. Counterexamples

In this section we provide our counterexamples and prove Propositions 1.3 and 1.4.

Proof of Proposition 1.3. Let \( R_0 > e^{\frac{3}{4}} \) and define \( u \in C^2(\mathbb{R}^2) \) as follows:
\[
u(x) := \begin{cases} \log R_0 - \frac{3}{4} + \frac{|x|^2}{R_0^2} - \frac{|x|^4}{4R_0^2}, & |x| < R_0, \\ \log(|x|), & |x| \geq R_0. \end{cases}
\]

It is easy to check that \( u, \Delta u \geq 0 \) in \( \mathbb{R}^2 \) and for \( R \) sufficiently large
\[
\int_{B_R} u^2 \, dx \leq CR^2 (\log(R))^2,
\]
for some \( C > 0 \) independent of \( R \) (see Remark 5.4 in [8]). Let \( X = (x, y) := (x_1, x_2, y_1, y_2, \ldots, y_{n-2}) \in \mathbb{R}^n \), where \( x \in \mathbb{R}^2 \) and \( y \in \mathbb{R}^{n-2} \). Choose \( \alpha > 0 \) sufficiently large such that
\[
\int_{\mathbb{R}^{n-2}} (1 + |y|^2)^{-2\alpha} \, dy = C_1 < \infty.
\]
Let \( \varphi(y) = (1 + |y|^2)^{-\alpha} \). Since \( \nabla u \cdot \nabla \varphi = 0 \), we have
\[
    u \nabla (\varphi^2 \nabla u) = u \varphi^2 \nabla (\nabla u) \geq 0.
\]
To complete the proof it is enough to show that \( \varphi u \) satisfies (8) for some \( C > 0 \).

For \( R > 0 \) sufficiently large we have
\[
    \int_{B_R} (\varphi u)^2 \leq \int_{\mathbb{R}^{n-2}} (1 + |y|^2)^{-2\alpha} \left( \int_{B_R^2} u^2 \, dx \, dy \right)
    \leq CR^2 (\log(R))^2 \int_{\mathbb{R}^{n-2}} (1 + |y|^2)^{-2\alpha} \, dy
    = C_1 R^2 (\log(R))^2,
\]
where \( B_R^2 \) is a ball of radius \( R \) in \( \mathbb{R}^2 \) centered at the origin. \( \square \)

**Proof of Proposition 1.4.** Let \( \psi(x) \) and \( \varphi(x) \) be a pair of bounded sign-changing and positive solutions for the equation
\[
    \Delta u + \bar{V} u = 0,
\]
for some smooth bounded potential \( \bar{V} \) in \( \mathbb{R}^3 \), respectively (the existence of such solutions is guaranteed by Barlow’s work [2]). Let \( X = (x, y) := (x_1, x_2, x_3, y_1, \ldots, y_{n-3}) \in \mathbb{R}^n \), where \( x \in \mathbb{R}^3 \) and \( y \in \mathbb{R}^{n-3} \). Choose \( \alpha > 0 \) sufficiently large such that
\[
    \int_{\mathbb{R}^{n-3}} (1 + |y|^2)^{-2\alpha} \, dy = C_1 < \infty.
\]
Define
\[
    v(X) = \psi(x)(1 + |y|^2)^{-\alpha}, \quad u(X) = \varphi(x)(1 + |y|^2)^{-\alpha}.
\]
Then we have
\[
    -V := \frac{\Delta v(X)}{v(X)} = \frac{\Delta(\psi(x))}{\psi(x)} + \frac{\Delta((1 + |y|^2)^{-\alpha})}{(1 + |y|^2)^{-\alpha}}
    = \frac{\Delta(\varphi(x))}{\varphi(x)} + \frac{\Delta((1 + |y|^2)^{-\alpha})}{(1 + |y|^2)^{-\alpha}}
    = \frac{\Delta u(X)}{u(X)}.
\]
Therefore \( V \) is bounded and \( \Delta w + V w = 0 \) has a bounded sign-changing solution \( v \) as well as a positive solution \( u \) in \( \mathbb{R}^n \). To complete the proof it is enough to show that \( v \) satisfies (9) for some \( C > 0 \) independent of \( R \). Since \( \psi \) is bounded, we have \( |\psi(x)| \leq L \) for some \( L > 0 \). Let \( R > 0 \). We have
\[
    \int_{B_R} v^2 \leq \int_{\mathbb{R}^{n-3}} (1 + |y|^2)^{-2\alpha} \left( \int_{B_R} (\psi(x))^2 \, dx \right) \, dy
    \leq \omega_3 C_1 L^2 R^3,
\]
where \( B_R^3 \) is a ball of radius \( R \) in \( \mathbb{R}^3 \) centered at the origin and \( w_3 \) is the volume of the unit ball in \( \mathbb{R}^3 \). \( \square \)

Remark 2.1. Proposition 1.4 also shows that (5) in Proposition 1.2 cannot be replaced by (6) for \( n \geq 4 \) even though we require equation (2) to be satisfied. The proof follows from Proposition 1.4 and a standard argument (Proposition 2.8 in [7]; see also [3]). Indeed, let \( u = \frac{v}{\varphi} \), where \( \varphi \) is the positive solution (resp., \( v \) is the sign-changing solution) for the Schrödinger equation \( \Delta + V w = 0 \) on \( \mathbb{R}^n \), where
$V$ is the potential considered in the previous proposition. Then, $\nabla(\varphi^2\nabla u) = 0$ and $\varphi u = v$ satisfies (9).

Remark 2.2. If one can find a bounded potential $V$ such that $(\Delta + V)w = 0$ has a positive solution $u$ as well as a bounded sign-changing solution $v$ in $\mathbb{R}^3$ satisfying

\begin{equation}
\int_{B_R} v^2 \leq C f(R),
\end{equation}

then (8) and (9) in Propositions 1.3 and 1.4 can be replaced by (12).

ACKNOWLEDGEMENTS

The author is grateful to Professor Nassif Ghoussoub for many useful comments and insightful discussions. Also, the author would like to thank the anonymous referee for constructive comments that improved the presentation of the paper.

REFERENCES


DEPARTMENT OF APPLIED PHYSICS AND APPLIED MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027

E-mail address: am3937@columbia.edu

Current address: Department of Mathematics, University of Toronto, Toronto, ON M5S 1A1, Canada