PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 1, January 2014, Pages 101–112
S 0002-9939(2013)12059-9
Article electronically published on October 2, 2013

LENGTH ASYMPTOTICS IN HIGHER TEICHMÜLLER THEORY

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(Communicated by Nimish Shah)

Abstract. In this note, we recover a recent result of Sambarino by showing that certain length functions arising in higher Teichmüller theory satisfy a prime geodesic theorem analogous to that of Huber in the classical case. We also show that there are more sophisticated distributional and limiting results.

1. Introduction

The classical Teichmüller theory for compact surfaces S can be described in terms of the representation of the fundamental group $\pi_1(S)$ in $\mathrm{PSL}(2,\mathbb{R})$. A recent development deals with representations in $\mathrm{PSL}(n,\mathbb{R})$. In this note, we will explore some associated asymptotic properties using dynamical techniques.

Let S be a compact surface of genus $g \geq 2$. The universal covering space corresponds to the unit disk $\mathbb{D}^2$, equipped with the Poincaré metric. One can then associate to S the Teichmüller space $\mathcal{T}(S)$, the space of equivalence classes of marked Riemannian metrics on S. This defines a deformation space for Riemann metrics on S and is homeomorphic to $\mathbb{R}^{6g-6}$. Alternatively, and more conveniently from our perspective, $\mathcal{T}(S)$ may also be described in terms of the faithful representations of $\pi_1(S)$ in $\mathrm{PSL}(2,\mathbb{R})$. More precisely, $\mathcal{T}(S)$ is a connected component of the representation space

$$\mathrm{Rep}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

consisting of the representations of $\pi_1(S)$ in $\mathrm{PSL}(2,\mathbb{R})$. Given a metric $m \in \mathcal{T}(S)$, there is a discrete faithful representation $\rho_m : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ such that (S,m) is isometric to $\mathbb{H}^2/\rho_m(\pi_1(S))$.

Following ideas of Hitchin [11] and Labourie [15], one can introduce a higher-dimensional Teichmüller theory. The group $\mathrm{PSL}(2,\mathbb{R})$ acts on the space of homogeneous polynomials of degree $n-1$ by $\gamma \cdot P(x,y) = P(\gamma(x,y))$. This then defines a canonical irreducible representation $\iota : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ (cf. Example 2.1). Then $m \mapsto \iota \circ \rho_m$ gives an embedding of $\mathcal{T}(S)$ into

$$\mathrm{Rep}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R}).$$

This space of representations is not connected. The component of $\mathrm{Rep}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$ containing the image of $\mathcal{T}(S)$ is called the *Hitchin component* [11].

We can call any representation in the Hitchin component a quasi-Fuchsian representation. Labourie showed that every such representation is Anosov (in the sense of the associated covering flow, [3], §8). Furthermore, he also showed the following simplicity result for the eigenvalues.

Received by the editors March 8, 2012.
2010 *Mathematics Subject Classification.* Primary 20H10, 22E40, 37C30, 37D20, 37D40.

Proposition 1.1 (Labourie [15], Theorem 4.1)**.** *Any representation ρ of $\pi_1(S)$ in the Hitchin component is discrete and faithful. Furthermore, for any $\alpha \in \pi_1(S)$, the matrix $\rho(\alpha)$ is diagonalizable and has n distinct eigenvalues $\lambda_1(\alpha), \ldots, \lambda_n(\alpha)$, which may be arranged so that*

$$|\lambda_1(\alpha)| < \cdots < |\lambda_n(\alpha)|;$$

i.e. the representation is real split.

This result allows us to introduce a natural generalization of the lengths of closed geodesics on a hyperbolic surface (S, m). We recall that there is a one-to-one correspondence between closed geodesics and non-trivial conjugacy classes in $\pi_1(S)$.

Definition 1.2. Let α denote a non-trivial conjugacy class in $\pi_1(S)$. (The eigenvalues $\lambda_i(\alpha)$ depend only on the conjugacy class of α.) We define the *length functions* $l_i(\alpha) = \log|\lambda_i(\alpha)|$, for $i = 1, \cdots, n$. We can also associate to ρ the *width*

$$l_\rho(\alpha) = l_1(\alpha) - l_n(\alpha).$$

(The widths l_ρ have an alternative formulation as the periods of a suitable cross ratio [16].)

We note that if we reverse the orientation of the geodesic, then α is replaced by α^{-1} and we have that $l_\rho(\alpha^{-1}) = l_\rho(\alpha)$.

It is natural to consider counting problems associated to the above lengths and the width. This has been the subject of recent work by Sambarino [24, 25], in the more general context of so-called strictly convex representations, which include Hitchin representations of surface groups as a particular case. For example, he obtains an asymptotic formula for conjugacy classes ordered by the logarithm of the maximal eigenvalue, which corresponds to the length function l_1, as well as other orderings (and for group elements counted by the norm of the associated matrix). While Sambarino considers a more general situation, our approach in this note, exploiting a construction of Dreyer [6], leads to a shorter proof and also to other distributional and statistical results.

The first result we consider (which appears as Proposition 7.8 in [24]) is the following.

Theorem 1.3. *For any ρ in the Hitchin component, there exists $\delta(\rho) > 0$ such that*

$$\#\{\alpha : l_\rho(\alpha) \le T\} \sim \frac{e^{\delta(\rho)T}}{\delta(\rho)T}, \quad \text{as } T \to +\infty.$$

Moreover, the value of $\delta(\rho)$ is an analytic function on the Hitchin component.

A key ingredient in our approach depends on a recent construction of Dreyer [6], which enables us to relate the length functions $l_i(\alpha)$ to Hölder continuous functions. The method of proof depends on studying a zeta function

$$\zeta(s) = \prod_\alpha \left(1 - e^{-s l_\rho(\alpha)}\right)^{-1},$$

where the product is taken over all primitive conjugacy classes in $\pi_1(S)$. This converges for $\mathrm{Re}(s) > \delta(\rho)$. In particular, we show the following.

Theorem 1.4. *The zeta function has a simple pole at $\delta(\rho)$ and a non-zero analytic extension to a neighbourhood of $\{s : \mathrm{Re}(s) = \delta(\rho)\} - \{\delta(\rho)\}$.*

We can compare this with the classical setting.

Example 1.5. In the case $n = 2$ the width reduces to the length of the closed geodesic in the free homotopy class of α, i.e.

$$l_\rho(\alpha) = l_1(\alpha) - l_2(\alpha) = 2\cosh^{-1}\mathrm{tr}(\alpha).$$

Theorem 1.3 then reduces to the classical theorem of Huber [12] counting lengths of closed geodesics, in which case $\delta(\rho) = 1$. In this setting, Theorem 1.4 follows from classical results on the Selberg zeta function [8, 27].

The proof involves some familiar dynamical ingredients, but additionally has some interesting new features. We cannot directly apply results on Anosov flows, since we have to consider Hölder reparameterizations. Moreover, we need to show that the associated flow is weak mixing.

2. Examples

In this section we discuss a number of examples.

Example 2.1 (Canonical representation)**.** If ρ is an embedded Teichmüller representation, then the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on the homogeneous polynomials $P(x, y) = \sum_{i=1}^n a_i x^{n-i} y^{i-1}$ (of degree $n - 1$) by

$$A : x^{n-i}y^{i-1} \mapsto (ax + by)^{n-i}(cx + dy)^{i-1}.$$

In particular, if $A = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ with $a > 1$, then

$$A : x^{n-i}y^{i-1} \mapsto a^{n-2i+1}x^{n-i}y^{i-1},$$

i.e. multiplication by a^{n-2i+1}. The set of lengths is then $(n - 2i + 1)\log a$, for $i = 1, \ldots, n$. Setting $a = e^{l/2}$ we have $l_i = (n - 2i + 1)l/2$.

The standard presentation of $\pi_1(S)$ of a surface of genus g has generators a_i, b_i, where $1 \le i \le g$, and a relation $\prod_{i=1}^4 [a_i, b_i] = I$. Starting from the canonical representation of a surface group $\pi_1(S) \to SL(2, \mathbb{R}) \to SL(3, \mathbb{R})$ in Example 2.1, we can make a perturbation within the Hitchin component to get a continuous family of examples. Moreover, we saw in Example 2.1 that the width was proportional to the length. However, for typical ρ within the Hitchin component this no longer happens. In particular, given a_1 and b_1 we can ask that the ratio of the widths to the lengths be different, i.e. $l_\rho(a_1)/l(a_1) \neq l_\rho(b_1)/l(b_1)$. This is illustrated in the next example.

Example 2.2 (Triangle groups)**.** Consider the $(3, 3, 4)$-triangle group

$$\langle a, b | a^3 = b^3 = (ab)^4 = 1\rangle.$$

We can choose a compact surface S such that $\Gamma = \pi_1(S)$ is a finite index subgroup. In particular, following [18] and [19], there is a continuous family of representations ρ_t, $t \in \mathbb{R}$, corresponding to

$$\rho_t(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \rho_t(b) = \begin{pmatrix} 1 & 2 - t + t^2 & 3 + t^2 \\ 0 & -t + 2t - t^2 & -1 + t - t^2 \\ 0 & 3 - 3t + t^2 & (-1 + t)^2 \end{pmatrix}.$$

We can consider the hyperbolic elements $g_1 = ab^2$ and $g_2 = a(ab)^2b$, say; then we can consider the ratio of widths $r(t) := l_{\rho_t}(g_1)/l_{\rho_t}(g_2)$. This is continuous and non-constant (since, for example, we can explicitly compute $r(0) = 1.78944\ldots$ and $r(1) = 1.51784\ldots$). In particular, we can choose t such that the widths are not integer multiples of a single value.

Remark 2.3. When $n = 3$, the representations in the Hitchin component have been related to real projective structures on S [4].

Remark 2.4. Other examples of Anosov representations are Hitchin representations into split real Lie groups, maximal representations into Lie groups of Hermitian type, quasi-Fuchsian representations and small deformations of co-compact lattices in rank one Lie groups into Lie groups of higher rank [3].

3. The Hitchin component, reparametrizations and symbolic dynamics

Our approach to the counting result in Theorem 1.3 is based on the dynamics of the geodesic flow $\psi_t : T^1S \to T^1S$ on the bundle of tangent vectors of unit length over the surface S and a certain lifted flow that we shall describe in the next paragraph. It is worth recalling the following simple result.

Lemma 3.1. *The canonical projection from T^1S to S induces a one-to-one correspondence between closed orbits of the geodesic flow and closed geodesics on S (and hence non-trivial conjugacy classes in $\pi_1(S)$). Furthermore, the period of a closed orbit is equal to the length of the corresponding closed geodesic.*

Let $\rho : \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ be a representation lying in a Hitchin component. Since ρ lies in the same component as a Fuchsian representation, it lifts to a representation, which we still denote by ρ, taking values in $\mathrm{SL}(n, \mathbb{R})$ [7]. Consider the trivial $\mathbb{R}^n$-bundle over the unit tangent bundle to the universal cover $T^1\widetilde{S} \times \mathbb{R}^n$. There is a natural action of $\pi_1(S)$ given by

$$\alpha \cdot (u, v) = (\alpha u, \rho(\alpha)v).$$

Forming the quotient space by this action, we get the twisted bundle

$$T^1S \times_\rho \mathbb{R}^n = T^1\widetilde{S} \times \mathbb{R}^n/\pi_1(S).$$

Let $p : T^1S \times_\rho \mathbb{R}^n \to T^1S$ be the natural projection map. The geodesic flow $\psi_t : T^1S \to T^1S$ lifts to a flow ϕ_t on $T^1S \times_\rho \mathbb{R}^n$.

We recall the following elegant result of Labourie [15].

Theorem 3.2. *Let ρ be a representation in the Hitchin component. Then*:

(1) *The bundle splits as a direct sum of one-dimensional ψ_t-invariant bundles*

$$T^1S \times_\rho \mathbb{R}^n = V_1 \oplus \cdots \oplus V_n.$$

(2) *Let γ be the closed orbit of the geodesic flow corresponding to the conjugacy class α. Then the holonomy induced by the action of the lifted flow ϕ_t on V_i as it traverses the lift of γ is given by multiplication by $1/\lambda_i(\alpha)$.*

(3) *Each bundle $V_i(u)$ depends Hölder continuously on $u \in T^1S$.*

We could reparametrize the geodesic flow by a Hölder continuous function associated to the splitting so that the new flow has least periods which are now equal to the widths. However, although it is not difficult to adapt the known results on C^1 Anosov flows to this setting, there is no appropriate reference in the literature (although the article of Parry [20] on synchronization is prescient). Instead, we will proceed as follows.

Let A be a $k \times k$ matrix with entries 0 or 1, which is aperiodic (i.e. there exists $m \geq 1$ such that A^m has all entries positive). We can associate to A the space of sequences

$$\Sigma = \left\{x = (x_n)_{n=-\infty}^{\infty} \in \{0, \cdots, k-1\}^{\mathbb{Z}} : A(x_n, x_{n+1}) = 1, \forall n \in \mathbb{Z}\right\}$$

with metric

$$d(x,y) = \sum_{n=-\infty}^{\infty} \frac{1 - \delta(x_n, y_n)}{2^{|n|}},$$

where $\delta_{i,j}$ is the Kronecker symbol, and the subshift of finite type $\sigma : \Sigma \to \Sigma$ given by $(\sigma x)_n = x_{n+1}$, for $n \in \mathbb{Z}$. Given a strictly positive function $r : \Sigma \to \mathbb{R}$, we can define the r-suspension space

$$\Sigma^r = \{(x,s) : x \in \Sigma,\ 0 \leq s \leq r(x)\}/(x, r(x)) \sim (\sigma x, 0)$$

and suspended flow $\sigma_t^r : \Sigma^r \to \Sigma^r$ by $\sigma_t^r(x,s) = (x, s+t)$ (respecting the identifications).

We can now state the following well-known technical lemma, which gives a symbolic model for the geodesic flow.

Lemma 3.3 (Bowen [2], Ratner [22])**.** *There exists a subshift of finite type $\sigma : \Sigma \to \Sigma$, a strictly positive Hölder continuous function and a Hölder continuous surjection $\pi : \Sigma^r \to T^1S$ such that $\phi_t \circ \pi = \pi \circ \sigma_t^r$. Furthermore, the map π induces a correspondence between closed σ-orbits and closed ψ_t-orbits γ, which is one-to-one apart from at most finitely many exceptions. If $\sigma^m x = x$ corresponds to a closed orbit γ, then*

$$l(\gamma) = r^m(x) := \sum_{j=0}^{m-1} r(\sigma^j x).$$

For our subsequent analysis, we need to construct Hölder continuous functions on Σ from which the lengths l_i, $i = 1, \ldots, n$, may be recovered in the same way that the periods of closed orbits $l(\gamma)$ are recovered from the function r. To do this, we will use an approach of Dreyer [6]. Let L be a leaf of the geodesic foliation on T^1S (i.e. an orbit for the geodesic flow). Choose $u_0 \in L$ and $X_i \in V_i(u_0)$. Define a function f_{u_0,X_i} on a neighbourhood of u_0 in L by

$$f_{u_0,X_i}(\phi_t(u_0)) = \log \|\psi_t(X_i)\|_{\phi_t(u_0)}.$$

This is smooth along L. Now define a 1-form ω_i on the same neighbourhood by

$$\omega_i = -df_{u_0,X_i}.$$

The following result may be found in [6].

Lemma 3.4. *The 1-form ω_i is independent of the choice of u_0 and X_i and hence is well-defined on T^1S. Moreover, ω_i is smooth along the geodesic foliation and is Hölder continuous transverse to it.*

From this we deduce the result we require.

Lemma 3.5. *For each $i = 1, \cdots, n$, there exist Hölder continuous functions $g_i : \Sigma \to \mathbb{R}$ such that if $\sigma^m x = x$ corresponds to a conjugacy class α, then*

$$l_i(\alpha) = g_i^m(x).$$

Proof. Define $F_i : T^1S \to \mathbb{R}$ by $F_i(x, v) = \omega_i(v)$. Then F_i is Hölder continuous, and it follows from Theorem 3.2 that

$$\int_0^{\ell(\gamma)} F_i(\dot{\gamma}(t))\, dt = \ell_i(\alpha).$$

Then the required functions $g_i : \Sigma \to \mathbb{R}$ are given by

$$g_i(x) = \int_0^{r(x)} F_i \circ \pi(x, t)\, dt. \qquad \square$$

For the purposes of counting, it would be convenient to consider positive functions. To achieve this, we have the following result.

Lemma 3.6. *Let α be any non-trivial conjugacy class in $\pi_1(S)$. Then:*

(i) $l_1(\alpha) > l_2(\alpha) > \cdots > l_n(\alpha)$, *and*
(ii) $l_1(\alpha) + l_2(\alpha) + \cdots + l_n(\alpha) = 0$.

In particular, $l_\rho(\alpha) := l_1(\alpha) - l_n(\alpha) > 0$.

Proof. The first part is a simple restatement of Labourie's result, Theorem 3.2 above. The second part follows from Theorem 3 in [6], which is essentially a consequence of the action of $\mathrm{SL}(n, \mathbb{R})$ on $\mathbb{R}^n$ being volume preserving. $\square$

We recall the following.

Lemma 3.7 (Positive Livsic Theorem [26])**.** *Suppose that $g : \Sigma \to \mathbb{R}$ is a positive Hölder continuous function such that $g^n(x) > 0$ whenever $\sigma^n x = x$. Then there exists a Hölder continuous function $u : \Sigma \to \mathbb{R}$ such that $\overline{g} := g + u\sigma - u > 0$.*

We then have the following.

Lemma 3.8. *The function $r_\rho := g_1 - g_n$ is cohomologous to a strictly positive function.*

Proof. This follows immediately from Lemma 3.7. $\square$

4. Asymptotic formulae

We know that the original geodesic flow $\psi_t : T^1S \to T^1S$ is weak mixing. This can be from the geometric fact that the set of periods of closed orbits does not lie in a discrete subgroup of $\mathbb{R}$; i.e. there does not exist $c > 0$ such that $\{l(\alpha) : \alpha \in \pi_1(S)\} \subset c\mathbb{Z}^+$. We have the following analogous result for the width spectrum.

Lemma 4.1. *There does not exist $c > 0$ such that $\{l_\rho(\alpha) : \alpha \in \pi_1(S)\} \subset c\mathbb{Z}^+$.*

Proof. Since the weights are positive, we can continuously reparametrize the original geodesic flow so that the resulting Anosov flow $\chi_t : T^1S \to T^1S$ has periodic orbits with periods $l_\rho(\alpha)$. If we assume for a contradiction that there exists $c > 0$ such that $\{l_\rho(\alpha) : \alpha \in \pi_1(S)\} \subset c\mathbb{Z}^+$, then the flow is not weak mixing. However, this contradicts a theorem of Arnol'd (cf. [1], §23) that no geodesic flow can be reparametrized to make it not weak mixing. $\square$

Proof of Theorem 1.3. We have already observed that r_ρ is strictly positive (after adding a coboundary, if necessary). In addition, we deduce from the above that r_ρ is not cohomologous to a constant. Then the general main asymptotic theorem follows from the general theory of suspension flows [21]:

$$\#\{\alpha : l_\rho(\alpha) \leq T\} \sim e^{\delta(\rho)T}/\delta(\rho)T,$$

for some $\delta(\rho) > 0$.

The growth rate $\delta(\rho) > 0$ satisfies $P(-\delta(\rho)r_\rho) = 0$, where $P : C(\Sigma, \mathbb{R}) \to \mathbb{R}$ is the pressure functional given by

$$P(g) = \lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} e^{g^n(x)} \right), \text{ where } g \in C(\Sigma, \mathbb{R}).$$

Thus we see that $\rho \mapsto \delta(\rho)$ is analytic given the construction of r_ρ (cf. [21]). □

There is also a natural result for the asymptotics of conjugacy classes corresponding to null homologous closed geodesics.

Theorem 4.2. *There exists $C > 0$ such that (denoting by $[\alpha] \in H_1(S, \mathbb{Z})$ the corresponding element in homology)*

$$\#\{\alpha : l_\rho(\alpha) \leq T, [\alpha] = 0\} \sim Ce^{\delta(\rho)T}/T^{g+1}.$$

Again this is a consequence of the symbolic dynamics and the results on distribution of closed orbits in homology for Anosov flows [13, 28]. (A little care is needed as $H_1(T^1S, \mathbb{Z})$ and $H_1(S, \mathbb{Z})$ only agree up to torsion.) The fact that the growth rate is $\delta(\rho)$ follows from the symmetry $l_\rho(\alpha^{-1}) = -l_\rho(\alpha)$.

Remark 4.3. Clearly we have from Labourie's theorem that

$$l_1(\alpha) + \cdots + l_n(\alpha) = 0 \text{ and } l_1(\alpha) > \cdots > l_n(\alpha).$$

Therefore each of the various weights of the closed geodesics given by

$$l_1(\alpha), l_1(\alpha) + l_2(\alpha), \cdots, \sum_{j=1}^{n-1} l_j(\alpha)$$

are all positive and the asymptotic formula from the theorem generalizes.

Because of the dynamical nature of the proofs, we can also change the metric on S to be of variable negative curvature, and the results will immediately generalize.

5. Equidistribution and statistical results

We know by the classical equidistribution results that the closed orbits γ of the geodesic flow over a compact hyperbolic surface ordered by their periods $l(\gamma)$ are equidistributed with respect to the Haar measure. More precisely, given a nontrivial conjugacy class α in $\pi_1(S)$, with associated closed orbit γ and a continuous function $F : T^1S \to \mathbb{R}$, we can define the weight $l^F(\alpha) = \int_0^{l(\alpha)} F(\psi_t x_\gamma)\, dt$, where x_γ is a point on γ. Then

$$\lim_{T\to+\infty} \frac{\sum_{l(\alpha)\leq T} \frac{l^F(\alpha)}{l(\alpha)}}{\sum_{l(\alpha)\leq T} 1} = \int F\, d\mu,$$

where μ is the normalized volume measure on T^1S, which is also the measure of maximal entropy for the geodesic flow $\psi_t : T^1S \to T^1S$.

We can also consider closed orbits γ ordered by the associated weights $l_\rho(\alpha)$ and with weights obtained from the reparameterized Anosov flow $\chi_t : T^1S \to T^1S$ for which the least periods are $l_\rho(\alpha)$:

$$l_\rho^F(\alpha) = \int_0^{l_\rho(\alpha)} F(\chi_t x_\gamma)\, dt.$$

Theorem 5.1 (Equidistribution)**.** *There exists a fully supported probability measure μ_ρ on T^1S such that if $F : T^1S \to \mathbb{R}$ is a continuous function, then*

$$\lim_{T\to+\infty} \frac{\sum_{l_\rho(\alpha)\le T} \frac{l_\rho^F(\alpha)}{l_\rho(\alpha)}}{\sum_{l_\rho(\alpha)\le T} 1} = \int F\, d\mu_\rho.$$

Proof. To see this, we consider the symbolic model for the reparametrization of an Anosov flow for which the least periods are $l_\rho(\alpha)$. At the symbolic level we replace the roof function r by r_ρ and write

$$f(x) = \int_0^{r_\rho(x)} F(x,t)\, dt,$$

where we abuse notation by regarding F as a function on the r_ρ-suspension space over Σ. We can then consider

$$\eta(s,t) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{-s r_\rho^n(x) + t f^n(x)} = \frac{1}{1 - e^{P(-s r_\rho + t f)}} + A(s,t),$$

where $A(s,t)$ is analytic in a neighbourhood of $s = \delta(\rho)$ and $t = 0$. Thus

$$\frac{\partial}{\partial t}\eta(s,t)\bigg|_{t=0} = \frac{\int f\, d\nu}{\int r_\rho\, d\nu} \frac{1}{1-\delta(\rho)} + A_1(s,t) = \int_1^\infty t^{-s}\, d\pi_f(t),$$

where $A(s,t)$ is analytic in a neighbourhood of $s = \delta(\rho)$ and $t = 0$, $\pi_f(T) = \sum_{r_\rho^n(x))\le T} f^n(x)$ and ν is the equilibrium state for $-\delta(\rho) r_\rho$. The result then follows using standard techniques (see [21]) with μ_ρ equal to the push-forward of the measure $(\nu \times dt)/\int r_\rho\, d\nu$. □

We also have the following large deviations type property, which improves on the basic equidistribution result.

Theorem 5.2 (Large deviations)**.** *Given $\epsilon > 0$,*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \left(\frac{\#\left\{\alpha : \left|\frac{l_\rho^F(\alpha)}{l(\alpha)} - \int F\, d\mu_\rho\right| > \epsilon \text{ and } l_\rho(\alpha) \le T\right\}}{\#\{\gamma : l_\rho(\alpha) \le T\}} \right) < 0.$$

Proof. This is based on the symbolic dynamics using the roof function r_ρ (cf. [14], [17]). □

There is also a central limit theorem under a suitable hypothesis.

Theorem 5.3 (Central Limit Theorem)**.** *Suppose that F is not cohomologous to a constant function, i.e. that the weights $l_\rho^F(\alpha)$ are not contained in a discrete subgroup of $\mathbb{R}$. Then there exists $\sigma > 0$ such that*

$$\lim_{T\to+\infty} \frac{\#\left\{\alpha : \frac{l_\rho^F(\alpha)}{l_\rho(\alpha)} - \int F d\mu_\rho < \frac{y}{\sqrt{l_\rho(\alpha)}} \text{ and } l_\rho(\alpha) \le T\right\}}{\#\{\alpha : l_\rho(\alpha) \le T\}} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{y} e^{-u^2/2\sigma^2}\, du.$$

Proof. This follows from classical results for the suspension flows using Hölder functions which are used to model the reparameterized flows (cf. [23] and [5] for the standard CLT and [17] for a periodic orbit version). □

Remark 5.4. There are also more general invariance principles associated to the reparameterized flow.

Theorem 5.5 (Local Central Limit Theorems)**.** *For $a, b \in \mathbb{R}$, suppose that the numbers $al_\rho(\alpha) - bl_\rho^F(\alpha)$ all lie in a discrete subgroup of $\mathbb{R}$ only if $a = b = 0$. Then there exists $C > 0$ such that*

$$\frac{\#\left\{\alpha : p < \frac{l_\rho^F(\alpha)}{l_\rho(\alpha)} - \int F d\mu_\rho < q \text{ and } l_\rho(\alpha) \leq T\right\}}{\#\{\alpha : l_\rho(\alpha) \leq T\}} \sim \frac{C(q-p)}{\sqrt{T}}, \text{ as } T \to +\infty.$$

Proof. This again follows from classical results on Local Central Limit Theorems for the suspension flows using Hölder functions which are used to model the reparameterized flows (cf. [17]). □

6. Limiting lengths

In this final section we recover an interesting result of Dreyer [6] using symbolic dynamics in place of currents, and then use this approach to deduce various natural generalizations.

Theorem 6.1 (Dreyer [6])**.** *Let A, B be representatives of elements of $\pi_1(S)$. For each $i = 1, \cdots, n$, we have that the limit*

$$c_i(A, B) = \lim_{k \to +\infty} \frac{l_i(A^k B)}{l_i(A^k)}$$

exists.

Proof. Assume for simplicity that the coding is chosen so that we can associate to the generators A, B the symbols $0, 1$ in the alphabet for Σ and that $A(0,0) = A(1,1) = A(0,1) = A(1,0) = 1$. We can consider the Hölder continuous functions $g_i : \Sigma \to \mathbb{R}$, for $i = 1, \cdots, n$, on the shift space.

For the two symbols 0 and 1 and for each k, associate a periodic point of period $k+1$, which we denote by $\overline{0^k 1}$ on Σ, by concatenating the word $0^k 1$ infinitely often. If we add the function g_i around the $k+1$ points in this σ-orbit, then we get

$$\sum_{i=0}^{k} g_i\left(\overline{0^{k-i}10^i}\right),$$

where $\overline{0^{k-i}10^i}$ again represents the period point given by concatenating $0^{k-i}10^i$ (i.e. the first $\overline{0^k 1}$-orbit shifted i-times). Given $\epsilon > 0$, for m sufficiently large $\|g_i\|\theta^m/(1-\theta) < \epsilon$, where $\|g_i\|$ is the Hölder norm of g_i. If $\overline{0}$ represents the fixed point (concatenating 0 infinitely often), then the result is equivalent to showing that the following sequence (in k) has a limit:

$$\left(\sum_{i=0}^{k}[g_i(\overline{0^{k-i}10^i}) - g_i(\overline{0})]\right)_{k=1}^{\infty}.$$

We can break the kth term into the sum of the two parts,

$$\sum_{j=0}^{m-1}[g_i(\overline{0^j 10^{k-j}}) - g_i(\overline{0})] \text{ and } \sum_{j=m}^{k}[g_i(\overline{0^j 10}) - g_i(\overline{0})],$$

for $k > m$. By the choice of m the last term is smaller than ϵ (uniformly in m). Letting $m \to +\infty$ (and thus $k \to +\infty$) the first term converges to

$$\sum_{j=0}^{\infty}[g_i(0^j 10^\infty) - g_i(\overline{0})],$$

where $0^j 10^\infty$ finishes with infinitely many 0s. However, since $k \to +\infty$, we see that the second term tends to zero and the original sequence actually converges to the limit $c_i := \sum_{j=0}^{\infty}[g_i(0^j 10^\infty) - g_i(\overline{0})]$. This suffices to complete the proof. □

This is a special case of the following more general result.

Theorem 6.2. *Let $A_1, \cdots, A_m$ be representatives of elements of $\pi_1(S)$. Let $w_k = A_{x_1}A_{x_2}\cdots A_{x_k}$, for $k \geq 1$, be a sequence with word length tending to infinity, and let $B \in \pi_1(S)$. For each $i = 1, \cdots, n$, we have that the limit*

$$c_i(\{w_k\}, B) = \lim_{k\to+\infty} \frac{l_i(w_k B)}{l_i(w_k)}$$

exists.

Proof. The proof is very similar to the proof of Theorem 6.1. Assume for simplicity that the coding is chosen so that we can associate to the generator B the symbol 1 in the alphabet for Σ and we can write the words $w_k = A_{x_1}A_{x_2}\cdots A_{x_k}$ as a product of generators corresponding to a finite word $x_1x_2\cdots x_k$ in Σ. Assume that $A(1,1) = A(x_k,1) = A(1,x_1) = 1$, for $k \geq 1$. We can consider the Hölder continuous functions $g_i : \Sigma \to \mathbb{R}$, for $i = 1, \cdots, n$, on the shift space.

For each k associate a periodic point of period $k+1$, which we denote by $\overline{x_1x_2\cdots x_k1}$ on Σ, by concatenating the word $x_1x_2\cdots x_k1$ infinitely often. If we add the function g_i around the $k+1$ points in this σ-orbit, then we get

$$\sum_{i=0}^{k} g_i\left(\overline{x_{i+1}x_{i+2}\cdots x_k1x_1x_2\cdots x_i}\right),$$

where $\overline{x_{i+1}x_{i+2}\cdots x_k1x_1x_2\cdots x_i}$ again represents the periodic point given by concatenating $x_{i+1}x_{i+2}\cdots x_k1x_1x_2\cdots x_i$ (and is just the first orbit $\overline{x_1x_2\cdots x_k1}$ shifted i-times). Given $\epsilon > 0$, for m sufficiently large we have that $\|g_i\|\theta^m/(1-\theta) < \epsilon$, where $\|g_i\|$ is the Hölder norm of g_i.

Let $w = (x_n)_{n=0}^{\infty} \in \Sigma$ represent the infinite sequence. Then the result is equivalent to showing that the following sequence (in k) has a limit:

$$\left(\sum_{i=0}^{k}[g_i\left(\overline{x_{i+1}x_{i+2}\cdots x_k1x_1x_2\cdots x_i}\right) - g_i(\sigma^i w)]\right)_{k=1}^{\infty}.$$

We can break the kth term into the sum of the two parts,

$$\sum_{i=0}^{m-1}[g_i\left(\overline{x_{i+1}x_{i+2}\cdots x_m1x_1x_2\cdots x_i}\right) - g_i(\sigma^i w)]$$

and

$$\sum_{i=m}^{k}[g_i\left(\overline{x_{i+1}x_{i+2}\cdots x_m 1 x_1 x_2\cdots x_i}\right) - g_i(\sigma^i w)],$$

for $k > m$. By the choice of m the last term is smaller than ϵ (uniformly in m). Letting $m \to +\infty$ (and this $k \to +\infty$) the first term converges to the finite expression

$$\sum_{i=0}^{\infty}[g_i\left(\overline{x_{i+1}x_{i+2}\cdots x_k 1 x_1 x_2\cdots x_i}\right) - g_i(\sigma^i w)].$$

However, since $k \to +\infty$, the second term tends to zero and the original sequence converges to

$$c_i(\{w_k\}, B) := \sum_{i=0}^{\infty}[g_i\left(\sigma^i x_1 x_2\cdots x_k 1 w\right) - g_i(\sigma^i w)].$$

This completes the proof. □

7. Some questions

We summarize below some natural outstanding questions associated with the work in this paper.

Question 7.1. Does $\delta(\rho)$ have another geometric interpretation in terms of some geometric growth rate?

Question 7.2. For which representations ρ is $\delta(\rho)$ maximized?

Question 7.3. Does the zeta function $\zeta(s)$ have a meromorphic extension to the entire complex plane?

Question 7.4. Is there a natural characterization of the limits $c_i(\cdot,\cdot)$, $i = 1, \cdots, n$?

Acknowledgments

The authors are grateful to François Labourie for informing them of Sambarino's work and for other helpful comments.

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