ON THE ANALYTIC INTEGRABILITY
OF THE 5–DIMENSIONAL LORENZ SYSTEM
FOR THE GRAVITY–WAVE ACTIVITY

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Abstract. For the 5-dimensional Lorenz system
\begin{align*}
\frac{dU}{dT} &= -VW + bVZ, \\
\frac{dV}{dT} &= UW - bUZ, \\
\frac{dW}{dT} &= -UV, \\
\frac{dX}{dT} &= -Z, \\
\frac{dZ}{dT} &= bUV + X
\end{align*}
(with \(b \in \mathbb{R}\) a parameter), describing coupled Rosby waves and gravity waves, we prove that it has at most three functionally independent global analytic first integrals and exactly three functionally independent global analytic first integrals when \(b = 0\). In this last case the system is completely integrable with an additional functionally independent first integral which is not globally analytic.

1. Introduction

E.N. Lorenz constructed in [4] the following 5–dimensional differential system in \(\mathbb{R}^5\):
\begin{align*}
\frac{dU}{dT} &= -VW + bVZ, \\
\frac{dV}{dT} &= UW - bUZ, \\
\frac{dW}{dT} &= -UV, \\
\frac{dX}{dT} &= -Z, \\
\frac{dZ}{dT} &= bUV + X
\end{align*}
(1.1)
(where \(b \in \mathbb{R}\) is a parameter), describing coupled Rosby waves and gravity waves. He was mainly interested in its slow manifolds. Here our interest will be in studying its integrability and in particular its analytic integrability; i.e. what is the maximal number of functionally independent analytic first integrals that the system (1.1) can exhibit? This question has been considered for many other relevant differential systems and other classes of first integrals not necessarily analytic; see for instance \([2,5,6]\) and the references therein.
Let \( \Omega \) be an open subset of \( \mathbb{R}^5 \) invariant by the flow of the differential system (1.1); i.e., if a solution of system (1.1) has a point in \( \Omega \), then all the points of this solution are contained in \( U \). A first integral of the differential system (1.1) in \( \Omega \) is a continuous function \( H = H(U,V,W,X,Z) : \Omega \to \mathbb{R} \) non-constant on any open subset of \( \Omega \) and satisfying that it is constant on every solution of system (1.1) contained in \( \Omega \). In other words, a continuously differentiable function \( H \) is a first integral of system (1.1) in \( \Omega \) if and only if
\[
(-VW + bVZ) \frac{\partial H}{\partial U} + (UW - bUZ) \frac{\partial H}{\partial V} - UV \frac{\partial H}{\partial W} - Z \frac{\partial H}{\partial X} + (bUV + X) \frac{\partial H}{\partial z} \equiv 0, \quad \text{in } \Omega.
\]
The maximal open set \( \Omega \) for which \( H : \Omega \to \mathbb{R} \) is a first integral of system (1.1) is called the domain of definition of the first integral \( H \).

Of course, when the first integral \( H \) is an analytic function, we say that \( H \) is an analytic first integral. If the first integral \( H \) is analytic in \( \mathbb{R}^5 \), we say that \( H \) is a global analytic first integral.

Let \( H_1 : \Omega_1 \to \mathbb{R} \) and \( H_2 : \Omega_2 \to \mathbb{R} \) be two first integrals of the 5-dimensional Lorenz system (1.1). They are functionally independent in \( \Omega_1 \cap \Omega_2 \) if their gradients are linearly independent over a full Lebesgue measure subset of \( \Omega_1 \cap \Omega_2 \).

It is well known (see [4]) and easy to check that the 5-dimensional Lorenz system (1.1) has the two functionally independent global analytic first integrals
\[
U^2 + V^2
\]
and
\[
V^2 + W^2 + X^2 + Z^2,
\]
having domains of definition equal to \( \mathbb{R}^5 \).

Our main result on the global analytic first integrals of system (1.1) is the following one.

**Theorem 1.1.** For the 5-dimensional Lorenz differential systems (1.1) the following statements hold:

(a) If \( b = 0 \), system (1.1) has three functionally independent global analytic first integrals, namely \( U^2 + V^2 \), \( V^2 + W^2 \) and \( X^2 + Z^2 \). Furthermore, any other global analytic first integral is an analytic function in the variables \( U^2 + V^2 \), \( V^2 + W^2 \) and \( X^2 + Z^2 \).

(b) If \( b \neq 0 \), the number of functionally independent global analytic first integrals of system (1.1) is either 2 or 3. Two of these are \( U^2 + V^2 \) and \( V^2 + W^2 + X^2 + Z^2 \).

Theorem 1.1 is proved in section 2.

We say that system (1.1) is completely integrable in an invariant open set \( U \) of \( \mathbb{R}^5 \) if there exist 4 first integrals functionally independent in \( U \).

**Proposition 1.2.** The 5-dimensional Lorenz differential system with \( b = 0 \) is completely integrable.

Proposition 1.2 is proved in section 3.
2. Proof of Theorem 1.1

In the proof of Theorem 1.1 we will use a characterization of the maximum number of local analytic first integrals that an analytic differential system can have in a neighborhood of a singularity.

We denote by \( \mathbb{Z}_+ \) the set of non-negative integers. The following result, due to Zhang [7], will be used in the proof of Theorem 1.1.

**Theorem 2.1.** For an analytic differential system defined in a neighborhood of the origin of \( \mathbb{R}^n \) for which the origin is a singularity, let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of its linear part at the origin. Set

\[
G = \left\{ (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n : \sum_{i=1}^{n} k_i \lambda_i = 0, \sum_{i=1}^{n} k_i > 0 \right\}.
\]

Assume that the differential system has \( r < n \) functionally independent analytic first integrals \( \Phi_1(x), \ldots, \Phi_r(x) \) in a neighborhood of the origin. If the \( \mathbb{R} \)-linear space generated by \( G \) has dimension \( r \), then any non-trivial analytic first integral of the system in a neighborhood of the origin is an analytic function of \( \Phi_1(x), \ldots, \Phi_r(x) \).

**Proof of Theorem 1.1** For the singularities \((0,0,W_0,0,0)\) with \( W_0 \in \mathbb{R} \), system (1.1) has the eigenvalues \( \lambda_1 = 0, \lambda_2,3 = \pm W_0i, \lambda_3,4 = \pm i \), where \( i = \sqrt{-1} \). Consider the set

\[
G_{W_0} = \left\{ (k_1, k_2, k_3, k_4, k_5) \in \mathbb{Z}_+^5 : \sum_{j=1}^{k_5} k_j \lambda_j = 0, \sum_{j=1}^{k_5} k_j \geq 1 \right\},
\]

with \( \lambda_1 = 0, \lambda_2 = W_0i, \lambda_3 = -W_0i, \lambda_4 = i, \lambda_5 = -i \). Clearly, if \( W_0 \) is irrational, the maximal number of linearly independent elements of \( G_{W_0} \) is three. So, by Theorem 2.1 system (1.1) has at most three functionally independent analytic first integrals in some neighborhood of \((0,0,W_0,0,0)\) with \( W_0 \) irrational, and consequently system (1.1) has at most three functionally independent global analytic first integrals.

If \( b = 0 \) it is easy to check that \( U^2 + V^2, V^2 + W^2 \) and \( X^2 + Z^2 \) are three functionally independent global analytic first integrals. Therefore, no other additional independent global analytic first integral can exist, and any other global analytic first integral must be a function in the variables \( U^2 + V^2, V^2 + W^2 \) and \( X^2 + Z^2 \). This completes the proof of statement (a) of Theorem 1.1.

In fact, when \( b = 0 \) we can provide a precise proof of the claim that any other global analytic first integral must be a function in the variables \( U^2 + V^2, V^2 + W^2 \) and \( X^2 + Z^2 \). Indeed, let \( H \) be an analytic first integral of system (1.1) with \( b = 0 \). Then by definition we have

\[
-VW \frac{\partial H}{\partial U} + UW \frac{\partial H}{\partial V} - UV \frac{\partial H}{\partial W} - Z \frac{\partial H}{\partial X} + X \frac{\partial H}{\partial Z} \equiv 0.
\]

We expand \( H \) in the Taylor series,

\[
H = \sum_{j=m}^{\infty} H_j(U, V, W, X, Z),
\]

where \( m \geq 1 \) is a positive integer, and \( H_j, j = m, m + 1, \ldots, \) are homogenous polynomials of degree \( j \).
Comparing the homogenous polynomials in (2.1) of the same degree, we get
\begin{align*}
-\partial H_m \partial X + X \partial H_m \partial Z &= 0, \quad (2.2) \\
-\partial H_{j+1} \partial X + X \partial H_{j+1} \partial Z &= VW \partial H_j \partial U - UW \partial H_j \partial V + UV \partial H_j \partial W, \quad (2.3)
\end{align*}
for \( j = m, m + 1, \ldots \).

The characteristic equation associated with the linear partial differential equation (2.2) has the first integral
\[ X^2 + Z^2, \]
so by the method of characteristic curves for solving linear partial differential equations we get that the general solution of (2.2) is
\[ H_m(U, V, W, X, Z) = G_m(X^2 + Z^2, U, V, W), \]
where \( G_m \) must be a polynomial function in its variables because \( H_m \) is a homogenous polynomial in the variables \( U, V, W, X, Z \).

For \( j = m \), since \( X^2 + Z^2 \) is a first integral of the characteristic equation associated with \( -\partial H_{m+1} \partial X + X \partial H_{m+1} \partial Z = 0 \), we make the change of variables
\[ A = X^2 + Z^2, \quad Z = Z, \]
with its inverse \( X = \sqrt{A - Z^2} \) and \( Z = Z \). Then equation (2.3) with \( j = m \) becomes the ordinary differential equation
\[ \sqrt{A - Z^2} \frac{d\tilde{H}_{m+1}}{dZ} = VW \frac{\partial G_m}{\partial U} - UW \frac{\partial G_m}{\partial V} + UV \frac{\partial G_m}{\partial W}, \]
taking \( U, V, W \) as constants, where \( \tilde{H}_{m+1} \) is \( H_{m+1} \) written in \( U, V, W, A, Z \) instead of \( U, V, W, X, Z \). This last equation can be written as
\[ \frac{d\tilde{H}_{m+1}}{dZ} = \left( VW \frac{\partial G_m}{\partial U} - UW \frac{\partial G_m}{\partial V} + UV \frac{\partial G_m}{\partial W} \right) \frac{1}{\sqrt{A - Z^2}}. \]
Integrating this ordinary differential equation with respect to \( Z \), we get
\[ \tilde{H}_{m+1}(U, V, W, A, Z) = \left( VW \frac{\partial G_m}{\partial U} - UW \frac{\partial G_m}{\partial V} + UV \frac{\partial G_m}{\partial W} \right) \arcsin \frac{Z}{\sqrt{A}} + \tilde{G}_{m+1}(U, V, W, A), \]
where \( \tilde{G}_{m+1} \) is an integrating constant w.r.t. \( Z \). Since \( H_{m+1}(U, V, W, X, Z) \) is a homogenous polynomial, we must have
\[ VW \frac{\partial G_m}{\partial U} - UW \frac{\partial G_m}{\partial V} + UV \frac{\partial G_m}{\partial W} \equiv 0. \]
The characteristic equations associated with this last partial differential equation have the two functionally independent first integrals
\[ B = U^2 + V^2, \quad C = V^2 + W^2. \]
Hence the general solution of the linear partial differential equation (2.6) is \( g(B, C) \), with \( g \) any continuous differentiable function. This forces that
\[ H_m(U, V, W, X, Z) = G_m(U, V, W, A) = R_m(A, B, C), \]
with \( R_m \) a homogenous polynomial in \( A, B, C \). So \( m \) must be even. From (2.5) we have
\[ \tilde{H}_{m+1}(U, V, W, A, Z) = \tilde{G}_{m+1}(U, V, W, A). \]
This shows that
\[ H_{m+1}(U, V, W, X, Z) = G_{m+1}(X^2 + Z^2, U, V, W), \]
where \( G_{m+1} \) is \( \tilde{G}_{m+1} \) written in \( U, V, W, X, Z \) instead of \( U, V, W, A, Z \).

Now equation (2.3) with \( j = m + 1 \) is
\[ -Z \frac{\partial H_{m+2}}{\partial X} + X \frac{\partial H_{m+2}}{\partial Z} = VW \frac{\partial G_{m+1}}{\partial U} - UW \frac{\partial G_{m+1}}{\partial V} + UV \frac{\partial G_{m+1}}{\partial W}. \]
Working in a similar way to get \( H_{m+1} \), we apply the method of characteristic curves to the last linear partial differential. In order to obtain a polynomial solution we must have
\[ VW \frac{\partial G_{m+1}}{\partial U} - UW \frac{\partial G_{m+1}}{\partial V} + UV \frac{\partial G_{m+1}}{\partial W} \equiv 0. \]
This implies that
\[ G_{m+1}(X^2 + Z^2, U, V, W) = R_{m+1}(X^2 + Z^2, U^2 + V^2, V^2 + W^2), \]
with \( R_{m+1} \) a homogenous polynomial in \( A, B, C \). Since \( H_m \) is a homogenous polynomial of even degree, we must have
\[ G_{m+1} \equiv 0, \]
because \( G_{m+1} \) cannot be of odd degree by its expression. Furthermore we have
\[ H_{m+2}(U, V, W, X, Z) = G_{m+2}(U, V, W, A), \]
where \( H_{m+2} \) is a homogenous polynomial of degree \( m + 2 \).

By induction we can prove from (2.3) that
\[ G_{m+2k}(U, V, W, X, Z) = R_{m+2k}(A, B, C), \quad k = 1, 2, \ldots, \]
where \( R_{m+2k} \) are homogeneous functions in \( A, B, C \). This proves the claim.

Finally, statement (b) of Theorem 1.1 follows immediately from the fact that \( U^2 + V^2 \) and \( V^2 + W^2 + X^2 + Z^2 \) are two independent global analytic first integrals of system (1.1) when \( b \neq 0 \), and the fact that system (1.1) has at most three functionally independent global analytic first integrals. \( \square \)

3. PROOF OF PROPOSITION 1.2

The following result is due to Jacobi. For a proof in a more general setting, see Theorem 2.7 of [3].

**Theorem 3.1.** Consider an analytic differential system in \( \mathbb{R}^n \) of the form
\[ \frac{dx}{dt} = \dot{x} = P(x), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \]
with \( P(x) = (P_1(x), \ldots, P_n(x)) \). Assume that
\[ \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} = 0 \quad (i.e. \ it \ has \ zero \ divergence) \]
and that it admits \( n - 2 \) first integrals, \( I_i(x) = c_i \) with \( i = 1, \ldots, n - 2 \) functionally independent. These integrals define, up to a relabeling of the variables, an invertible transformation mapping from \( (x_1, \ldots, x_n) \) to \( (c_1, \ldots, c_{n-2}, x_{n-1}, x_n) \) given by
\[ y_i = I_i(x), \quad i = 1, \ldots, n - 2, \quad y_{n-1} = x_{n-1}, \quad y_n = x_n. \]
Let $\Delta$ be the Jacobian of the transformation

$$
\Delta = \det \begin{pmatrix}
\frac{\partial x_1}{\partial I_1} & \frac{\partial x_1}{\partial I_2} & \cdots & \frac{\partial x_1}{\partial I_{n-2}} \\
\frac{\partial x_2}{\partial I_1} & \frac{\partial x_2}{\partial I_2} & \cdots & \frac{\partial x_2}{\partial I_{n-2}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial I_1} & \frac{\partial x_n}{\partial I_2} & \cdots & \frac{\partial x_n}{\partial I_{n-2}}
\end{pmatrix}.
$$

Then system (3.1) admits an extra first integral given by

$$
I_{n-1} = \int \frac{1}{\Delta} (\dot{P}_n \ dx_{n-1} - \dot{P}_{n-1} \ dx_n),
$$

where the tilde denotes the quantities expressed in the variables $(c_1, \ldots, c_{n-2}, x_{n-1}, x_n)$. Moreover, this first integral is functionally independent with the previous $n-2$ first integrals; that is, the system is completely integrable.

**Proof of Proposition 1.2.** It is immediate to verify that the Lorenz system (1.1) in $\mathbb{R}^5$ has zero divergence. If $b = 0$ by statement (a) of Theorem 1.1, the Lorenz system has $3 = 5-2$ first integrals functionally independent. So, the Lorenz system satisfies the assumptions of Theorem 3.1. Therefore, if $b = 0$, it is completely integrable in the open and dense set of $\mathbb{R}^5$ where the additional first integral $I_4$ is defined. □

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