PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 2, February 2014, Pages 531–537
S 0002-9939(2013)11773-9
Article electronically published on October 22, 2013

ON THE ANALYTIC INTEGRABILITY OF THE 5–DIMENSIONAL LORENZ SYSTEM FOR THE GRAVITY–WAVE ACTIVITY

JAUME LLIBRE, RADU SAGHIN, AND XIANG ZHANG

(Communicated by Yingfei Yi)

Abstract. For the 5-dimensional Lorenz system

$$\begin{aligned} dU/dT &= -VW + bVZ, \\ dV/dT &= UW - bUZ, \\ dW/dT &= -UV, \\ dX/dT &= -Z, \\ dZ/dT &= bUV + X \end{aligned}$$

(with $b \in \mathbb{R}$ a parameter), describing coupled Rosby and gravity waves, we prove that it has at most three functionally independent global analytic first integrals and exactly three functionally independent global analytic first integrals when $b = 0$. In this last case the system is completely integrable with an additional functionally independent first integral which is not globally analytic.

1. Introduction

E.N. Lorenz constructed in [4] the following 5–dimensional differential system in $\mathbb{R}^5$:

$$\begin{aligned} \frac{dU}{dT} &= -VW + bVZ, \\ \frac{dV}{dT} &= UW - bUZ, \\ \frac{dW}{dT} &= -UV, \\ \frac{dX}{dT} &= -Z, \\ \frac{dZ}{dT} &= bUV + X \end{aligned} \tag{1.1}$$

(where $b \in \mathbb{R}$ is a parameter), describing coupled Rosby waves and gravity waves. He was mainly interested in its slow manifolds. Here our interest will be in studying its integrability and in particular its analytic integrability; i.e. *what is the maximal number of functionally independent analytic first integrals that the system* (1.1) *can exhibit*? This question has been considered for many other relevant differential systems and other classes of first integrals not necessarily analytic; see for instance [2, 5, 6] and the references therein.

Received by the editors September 4, 2011 and, in revised form, March 19, 2012.
2010 *Mathematics Subject Classification.* Primary 34C05, 34A34, 34C14.

Let Ω be an open subset of $\mathbb{R}^5$ invariant by the flow of the differential system (1.1); i.e. if a solution of system (1.1) has a point in Ω, then all the points of this solution are contained in U. A *first integral* of the differential system (1.1) in Ω is a continuous function $H = H(U,V,W,X,Z) : \Omega \to \mathbb{R}$ non–constant on any open subset of Ω and satisfying that it is constant on every solution of system (1.1) contained in Ω. In other words, a continuously differentiable function H is a first integral of system (1.1) in Ω if and only if

$$\begin{aligned}(-VW + bVZ)\frac{\partial H}{\partial U} &+ (UW - bUZ)\frac{\partial H}{\partial V} \\ &- UV\frac{\partial H}{\partial W} - Z\frac{\partial H}{\partial X} + (bUV + X)\frac{\partial H}{\partial z} \equiv 0, \quad \text{in } \Omega.\end{aligned}$$

The maximal open set Ω for which $H : \Omega \to \mathbb{R}$ is a first integral of system (1.1) is called the *domain of definition* of the first integral H.

Of course, when the first integral H is an analytic function, we say that H is an *analytic first integral*. If the first integral H is analytic in $\mathbb{R}^5$, we say that H is a *global analytic first integral*.

Let $H_1 : \Omega_1 \to \mathbb{R}$ and $H_2 : \Omega_2 \to \mathbb{R}$ be two first integrals of the 5–dimensional Lorenz system (1.1). They are *functionally independent* in $\Omega_1 \cap \Omega_2$ if their gradients are linearly independent over a full Lebesgue measure subset of $\Omega_1 \cap \Omega_2$.

It is well known (see [4]) and easy to check that the 5–dimensional Lorenz system (1.1) has the two functionally independent global analytic first integrals

$$U^2 + V^2$$

and

$$V^2 + W^2 + X^2 + Z^2,$$

having domains of definition equal to $\mathbb{R}^5$.

Our main result on the global analytic first integrals of system (1.1) is the following one.

Theorem 1.1. *For the* 5*–dimensional Lorenz differential systems* (1.1) *the following statements hold:*

- (a) *If* $b = 0$*, system* (1.1) *has three functionally independent global analytic first integrals, namely* $U^2 + V^2$, $V^2 + W^2$ *and* $X^2 + Z^2$. *Furthermore, any other global analytic first integral is an analytic function in the variables* $U^2 + V^2$, $V^2 + W^2$ *and* $X^2 + Z^2$.
- (b) *If* $b \neq 0$*, the number of functionally independent global analytic first integrals of system* (1.1) *is either* 2 *or* 3. *Two of these are* $U^2 + V^2$ *and* $V^2 + W^2 + X^2 + Z^2$.

Theorem 1.1 is proved in section 2.

We say that system (1.1) is *completely integrable* in an invariant open set U of $\mathbb{R}^5$ if there exist 4 first integrals functionally independent in U.

Proposition 1.2. *The* 5*–dimensional Lorenz differential system with* $b = 0$ *is completely integrable.*

Proposition 1.2 is proved in section 3.

2. Proof of Theorem 1.1

In the proof of Theorem 1.1 we will use a characterization of the maximum number of local analytic first integrals that an analytic differential system can have in a neighborhood of a singularity.

We denote by $\mathbb{Z}_+$ the set of non–negative integers. The following result, due to Zhang [7], will be used in the proof of Theorem 1.1.

Theorem 2.1. *For an analytic differential system defined in a neighborhood of the origin of $\mathbb{R}^n$ for which the origin is a singularity, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of its linear part at the origin. Set*

$$\mathcal{G} = \left\{ (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n : \sum_{i=1}^{n} k_i \lambda_i = 0, \sum_{i=1}^{n} k_i > 0 \right\}.$$

Assume that the differential system has $r < n$ functionally independent analytic first integrals $\Phi_1(x), \ldots, \Phi_r(x)$ in a neighborhood of the origin. If the $\mathbb{R}$–linear space generated by $\mathcal{G}$ has dimension r, then any non-trivial analytic first integral of the system in a neighborhood of the origin is an analytic function of $\Phi_1(x), \ldots, \Phi_r(x)$.

Proof of Theorem 1.1. For the singularities $(0, 0, W_0, 0, 0)$ with $W_0 \in \mathbb{R}$, system (1.1) has the eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm W_0 i$, $\lambda_{3,4} = \pm i$, where $i = \sqrt{-1}$. Consider the set

$$\mathcal{G}_{W_0} = \left\{ (k_1, k_2, k_3, k_4, k_5) \in \mathbb{Z}_+^5 : \sum_{j=1}^{5} k_j \lambda_j = 0, \sum_{j=1}^{5} k_j \geq 1 \right\},$$

with $\lambda_1 = 0$, $\lambda_2 = W_0 i$, $\lambda_3 = -W_0 i$, $\lambda_4 = i$, $\lambda_5 = -i$. Clearly, if W_0 is irrational, the maximal number of linearly independent elements of $\mathcal{G}_{W_0}$ is three. So, by Theorem 2.1, system (1.1) has at most three functionally independent analytic first integrals in some neighborhood of $(0, 0, W_0, 0, 0)$ with W_0 irrational, and consequently system (1.1) has at most three functionally independent global analytic first integrals.

If $b = 0$ it is easy to check that $U^2 + V^2$, $V^2 + W^2$ and $X^2 + Z^2$ are three functionally independent global analytic first integrals. Therefore, no other additional independent global analytic first integral can exist, and any other global analytic first integral must be a function in the variables $U^2 + V^2$, $V^2 + W^2$ and $X^2 + Z^2$. This completes the proof of statement (a) of Theorem 1.1.

In fact, when $b = 0$ we can provide a precise proof of the claim that *any other global analytic first integral must be a function in the variables $U^2 + V^2$, $V^2 + W^2$* and $X^2 + Z^2$. Indeed, let H be an analytic first integral of system (1.1) with $b = 0$. Then by definition we have

$$-VW\frac{\partial H}{\partial U} + UW\frac{\partial H}{\partial V} - UV\frac{\partial H}{\partial W} - Z\frac{\partial H}{\partial X} + X\frac{\partial H}{\partial Z} \equiv 0. \tag{2.1}$$

We expand H in the Taylor series,

$$H = \sum_{j=m}^{\infty} H_j(U, V, W, X, Z),$$

where $m \geq 1$ is a positive integer, and H_j, $j = m, m + 1, \ldots$, are homogenous polynomials of degree j.

Comparing the homogenous polynomials in (2.1) of the same degree, we get

$$-Z\frac{\partial H_m}{\partial X}+X\frac{\partial H_m}{\partial Z}=0, \tag{2.2}$$

$$-Z\frac{\partial H_{j+1}}{\partial X}+X\frac{\partial H_{j+1}}{\partial Z}=VW\frac{\partial H_j}{\partial U}-UW\frac{\partial H_j}{\partial V}+UV\frac{\partial H_j}{\partial W}, \tag{2.3}$$

for $j=m,m+1,\dots$.

The characteristic equation associated with the linear partial differential equation (2.2) has the first integral X^2+Z^2, so by the method of characteristic curves for solving linear partial differential equations we get that the general solution of (2.2) is

$$H_m(U,V,W,X,Z)=G_m(X^2+Z^2,U,V,W),$$

where G_m must be a polynomial function in its variables because H_m is a homogenous polynomial in the variables U,V,W,X,Z.

For $j=m$, since X^2+Z^2 is a first integral of the characteristic equation associated with $-Z\partial H_{m+1}/\partial X+X\partial H_{m+1}/\partial Z=0$, we make the change of variables

$$A=X^2+Z^2,\quad Z=Z,$$

with its inverse $X=\sqrt{A-Z^2}$ and $Z=Z$. Then equation (2.3) with $j=m$ becomes the ordinary differential equation

$$\sqrt{A-Z^2}\,\frac{d\widetilde{H}_{m+1}}{dZ}=VW\frac{\partial G_m}{\partial U}-UW\frac{\partial G_m}{\partial V}+UV\frac{\partial G_m}{\partial W},$$

taking U,V,W as constants, where $\widetilde{H}_{m+1}$ is H_{m+1} written in U,V,W,A,Z instead of U,V,W,X,Z. This last equation can be written as

$$\frac{d\widetilde{H}_{m+1}}{dZ}=\left(VW\frac{\partial G_m}{\partial U}-UW\frac{\partial G_m}{\partial V}+UV\frac{\partial G_m}{\partial W}\right)\frac{1}{\sqrt{A-Z^2}}. \tag{2.4}$$

Integrating this ordinary differential equation with respect to Z, we get

$$\begin{aligned}\widetilde{H}_{m+1}&(U,V,W,A,Z)\\ &=\left(VW\frac{\partial G_m}{\partial U}-UW\frac{\partial G_m}{\partial V}+UV\frac{\partial G_m}{\partial W}\right)\arcsin\frac{Z}{\sqrt{A}}\\ &\quad+\widetilde{G}_{m+1}(U,V,W,A),\end{aligned} \tag{2.5}$$

where $\widetilde{G}_{m+1}$ is an integrating constant w.r.t. Z. Since $H_{m+1}(U,\ V,W,X,Z)$ is a homogenous polynomial, we must have

$$VW\frac{\partial G_m}{\partial U}-UW\frac{\partial G_m}{\partial V}+UV\frac{\partial G_m}{\partial W}\equiv 0. \tag{2.6}$$

The characteristic equations associated with this last partial differential equation have the two functionally independent first integrals

$$B=U^2+V^2,\quad C=V^2+W^2.$$

Hence the general solution of the linear partial differential equation (2.6) is $g(B,C)$, with g any continuous differentiable function. This forces that

$$H_m(U,V,W,X,Z)=G_m(U,V,W,A)=R_m(A,B,C),$$

with R_m a homogenous polynomial in A,B,C. So m must be even. From (2.5) we have

$$\widetilde{H}_{m+1}(U,V,W,A,Z)=\widetilde{G}_{m+1}(U,V,W,A).$$

This shows that

$$H_{m+1}(U,V,W,X,Z) = G_{m+1}(X^2+Z^2,U,V,W),$$

where G_{m+1} is $\widetilde{G}_{m+1}$ written in U,V,W,X,Z instead of U,V,W,A,Z.

Now equation (2.3) with $j=m+1$ is

$$-Z\frac{\partial H_{m+2}}{\partial X} + X\frac{\partial H_{m+2}}{\partial Z} = VW\frac{\partial G_{m+1}}{\partial U} - UW\frac{\partial G_{m+1}}{\partial V} + UV\frac{\partial G_{m+1}}{\partial W}.$$

Working in a similar way to get H_{m+1}, we apply the method of characteristic curves to the last linear partial differential. In order to obtain a polynomial solution we must have

$$VW\frac{\partial G_{m+1}}{\partial U} - UW\frac{\partial G_{m+1}}{\partial V} + UV\frac{\partial G_{m+1}}{\partial W} \equiv 0.$$

This implies that

$$G_{m+1}(X^2+Z^2,U,V,W) = R_{m+1}(X^2+Z^2,U^2+V^2,V^2+W^2),$$

with R_{m+1} a homogenous polynomial in A,B,C. Since H_m is a homogenous polynomial of even degree, we must have

$$G_{m+1} \equiv 0,$$

because G_{m+1} cannot be of odd degree by its expression. Furthermore we have

$$H_{m+2}(U,V,W,X,Z) = G_{m+2}(U,V,W,A),$$

where H_{m+2} is a homogenous polynomial of degree $m+2$.

By induction we can prove from (2.3) that

$$\begin{array}{l} G_{m+2k}(U,V,W,X,Z) = R_{m+2k}(A,B,C), \\ G_{m+2k-1}(U,V,W,X,Z) = 0, \end{array} \qquad k=1,2,\ldots,$$

where R_{m+2k} are homogeneous functions in A,B,C. This proves the claim.

Finally, statement (b) of Theorem 1.1 follows immediately from the fact that U^2+V^2 and $V^2+W^2+X^2+Z^2$ are two independent global analytic first integrals of system (1.1) when $b \neq 0$, and the fact that system (1.1) has at most three functionally independent global analytic first integrals. □

3. Proof of Proposition 1.2

The following result is due to Jacobi. For a proof in a more general setting, see Theorem 2.7 of [3].

Theorem 3.1. *Consider an analytic differential system in* $\mathbb{R}^n$ *of the form*

$$\frac{dx}{dt} = \dot{x} = P(x), \quad x=(x_1,\ldots,x_n)\in\mathbb{R}^n, \tag{3.1}$$

with $P(x) = (P_1(x),\ldots,P_n(x))$. *Assume that*

$$\sum_{i=1}^{n}\frac{\partial P_i}{\partial x_i} = 0 \quad (\textit{i.e. it has zero divergence})$$

and that it admits $n-2$ *first integrals,* $I_i(x)=c_i$ *with* $i=1,\ldots,n-2$ *functionally independent. These integrals define, up to a relabeling of the variables, an invertible transformation mapping from* $(x_1,\ldots,x_n)$ *to* $(c_1,\ldots,c_{n-2},x_{n-1},x_n)$ *given by*

$$y_i = I_i(x), \quad i=1,\ldots,n-2, \quad y_{n-1}=x_{n-1}, \quad y_n = x_n.$$

Let Δ *be the Jacobian of the transformation*

$$\Delta = \det \begin{pmatrix} \partial_{x_1} I_1 & \partial_{x_2} I_1 & \cdots & \partial_{x_{n-2}} I_1 \\ \partial_{x_1} I_2 & \partial_{x_2} I_2 & \cdots & \partial_{x_{n-2}} I_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} I_{n-2} & \partial_{x_2} I_{n-2} & \cdots & \partial_{x_{n-2}} I_{n-2} \end{pmatrix}.$$

Then system (3.1) *admits an extra first integral given by*

$$I_{n-1} = \int \frac{1}{\tilde{\Delta}} (\tilde{P}_n \, dx_{n-1} - \tilde{P}_{n-1} \, dx_n),$$

where the tilde denotes the quantities expressed in the variables $(c_1, \ldots, c_{n-2}, x_{n-1}, x_n)$. *Moreover, this first integral is functionally independent with the previous* $n-2$ *first integrals; that is, the system is completely integrable.*

Proof of Proposition 1.2. It is immediate to verify that the Lorenz system (1.1) in $\mathbb{R}^5$ has zero divergence. If $b = 0$ by statement (a) of Theorem 1.1, the Lorenz system has $3 = 5-2$ first integrals functionally independent. So, the Lorenz system satisfies the assumptions of Theorem 3.1. Therefore, if $b = 0$, it is completely integrable in the open and dense set of $\mathbb{R}^5$ where the additional first integral I_4 is defined. □

Acknowledgments

The first author has been partially supported by MCINN/FEDER grant No. MTM 2008-03437, by AGAUR grant No. 2009SGR410, and by ICREA Academia.

The second author has been partially supported by Marie Curie grant IEF-234559, and would like to thank the Centre de Recerca Matemàtica, Barcelona, for its hospitality.

Finally, the third author is partially supported by NNSF of China grant No. 11271252, by RFDP of Higher Education of China grant No. 20110073110054, and by FP7-PEOPLE-2012-IRSES-316338.

References

[1] Vladimir I. Arnold, *Ordinary differential equations*, Universitext, Springer-Verlag, Berlin, 2006. Translated from the Russian by Roger Cooke; Second printing of the 1992 edition. MR2242407 (2007b:34001)

[2] Isaac A. García, Maite Grau, and Jaume Llibre, *First integrals and Darboux polynomials of natural polynomial Hamiltonian systems*, Phys. Lett. A **374** (2010), no. 47, 4746–4748, DOI 10.1016/j.physleta.2010.09.060. MR2728364 (2011m:34001)

[3] Alain Goriely, *Integrability and nonintegrability of dynamical systems*, Advanced Series in Nonlinear Dynamics, vol. 19, World Scientific Publishing Co. Inc., River Edge, NJ, 2001. MR1857742 (2002k:37001)

[4] Edward N. Lorenz, *On the existence of a slow manifold*, J. Atmospheric Sci. **43** (1986), no. 15, 1547–1557, DOI 10.1175/1520-0469(1986)043⟨1547:OTEOAS⟩2.0.CO;2. MR855190 (87h:76037)

[5] Andrzej J. Maciejewski and Maria Przybylska, *Differential Galois obstructions for non-commutative integrability*, Phys. Lett. A **372** (2008), no. 33, 5431–5435, DOI 10.1016/j.physleta.2008.06.050. MR2439693 (2009j:37086)

[6] S. I. Popov, W. Respondek, and J.-M. Strelcyn, *On rational integrability of Euler equations on Lie algebra* so$(4, \mathbb{C})$*, revisited*, Phys. Lett. A **373** (2009), no. 29, 2445–2453, DOI 10.1016/j.physleta.2009.04.075. MR2543752 (2010g:37092)

[7] Xiang Zhang, *Local first integrals for systems of differential equations*, J. Phys. A **36** (2003), no. 49, 12243–12253, DOI 10.1088/0305-4470/36/49/007. MR2025828 (2004i:34094)

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, Chile

E-mail address: rsaghin@gmail.com

MOE-LSC, Shanghai Jiao Tong University, Shanghai, 200240, People's Republic of China

E-mail address: xzhang@sjtu.edu.cn