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LOWER BOUNDS FOR THE CONSTANTS IN THE BOHNENBLUST–HILLE INEQUALITY: THE CASE OF REAL SCALARS

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Abstract. The Bohnenblust–Hille inequality was obtained in 1931 and (in the case of real scalars) asserts that for every positive integer m there is a constant C_m so that

$$\left(\sum_{i_1,\dots,i_m=1}^{N}\left|T(e_{i_1},\dots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq C_m\left\|T\right\|$$

for all positive integers N and every m-linear mapping $T:\ell_\infty^N\times\cdots\times\ell_\infty^N\to\mathbb{R}$. Since then, several authors have obtained upper estimates for the values of C_m. However, the novelty presented in this short note is that we provide lower (and non-trivial) bounds for C_m.

1. Introduction

The Bohnenblust–Hille inequality [2] for real scalars asserts that for every positive integer m there is a constant C_m such that

$$\left(\sum_{i_1,\dots,i_m=1}^{N}\left|T(e_{i_1},\dots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq C_m\left\|T\right\| \tag{1.1}$$

for all positive integers N and every m-linear mapping $T:\ell_\infty^N\times\cdots\times\ell_\infty^N\to\mathbb{R}$. This inequality has important applications in various fields of mathematics and physics. Of course, there is no need to mention how many applications this theory has in analysis (see [5]), although it also has several links to quantum information theory.

Remark 1.1. As we mentioned above, the Bohnenblust–Hille inequality also has its applications in quantum information theory. For instance, the exact growth of C_m is related to a conjecture of Aaronson and Ambainis [1] about classical simulations of quantum query algorithms (as stated in a lecture delivered by A. Montanaro in

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January 2012 at the ICMAT, Madrid, Spain). Moreover, since the Bohnenblust–Hille inequality gives control of the operator norm (usually intractable) by an ℓ_p-norm, it is of potential use in the study of multipartite Bell inequalities and XOR-games (see [8]).

When $m = 2$ it is interesting to note that the Bohnenblust–Hille inequality is precisely the well–known Littlewood 4/3 inequality [10].

Since the 1930's many authors have obtained estimates for upper bounds of C_m in the case of real and complex scalars (see, e.g., [2, 3, 5, 6, 9, 11, 12]). The constants of the polynomial version of the Bohnenblust–Hille inequality (for the complex case) were recently investigated in [4]. Until now, the more accurate upper bounds for C_m (in the real case) were given in [12]:

m	upper bounds for C_m
2	$\sqrt{2} \approx 1.414$
3	≈ 1.782
4	2
5	≈ 2.298
6	≈ 2.520
7	≈ 2.692
8	≈ 2.828
9	≈ 3.055
10	≈ 3.249

Also, it has very recently been proved ([7]) that the sequence of constants $(C_m)_m$ has the best possible asymptotic behavior, that is,

$$\lim_{m\to\infty} \frac{C_m}{C_{m-1}} = 1.$$

However, and to the best of our knowledge, there is absolutely no work presenting estimates for (non-trivial) lower bounds for the constants C_m. In this short note we obtain lower bounds for C_m which we believe are (especially) interesting for the cases $m = 2, 3, 4, 5$.

In the following, e_k denotes the k-th canonical vector in $\mathbb{R}^N$.

2. The case $m = 2$

Let $T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$ be given by

$$T_2(x, y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2.$$

Note that $\|T_2\| = 2$. In fact,

$$\begin{aligned}|T_2(x, y)| &= |x_1(y_1 + y_2) + x_2(y_1 - y_2)| \\ &\le \|x\|(|y_1 + y_2| + |y_1 - y_2|) \\ &= 2\|x\| \max\{|y_1|, |y_2|\} \\ &= 2\|x\|\|y\|,\end{aligned}$$

and since $T_2(e_1 + e_2, e_1 + e_2) = 2$ it follows that $\|T_2\| = 2$.

Now the inequality

$$\left(\sum_{i_1,i_2=1}^{2} \left| T_2(e_{i_1}, e_{i_2}) \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq C_2 \, \| T_2 \|$$

can be re-written as

$$4^{3/4} \leq 2C_2,$$

which gives

$$C_2 \geq 2^{1/2}.$$

Since it is well-known that $C_2 \leq 2^{1/2}$, we conclude that $C_2 = 2^{1/2}$.

3. The case $m = 3$

Now, let $T_3 : \ell_\infty^4 \times \ell_\infty^4 \times \ell_\infty^4 \to \mathbb{R}$ be given by

$$\begin{aligned} T_3(x,y,z) = & (z_1 + z_2)\left(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2\right) \\ & + (z_1 - z_2)\left(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4\right). \end{aligned}$$

We have

$$\begin{aligned} |T_3(x,y,z)| &= |(z_1 + z_2)\left(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2\right) \\ &\quad + (z_1 - z_2)\left(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4\right)| \\ &\leq |z_1 + z_2| \left(|x_1||y_1 + y_2| + |x_2||y_1 - y_2|\right) \\ &\quad + |z_1 - z_2| \left(|x_3||y_3 + y_4| + |x_4||y_3 - y_4|\right) \\ &\leq \|x\| \Big\{ |z_1 + z_2| \left(|y_1 + y_2| + |y_1 - y_2|\right) + |z_1 - z_2| \left(|y_3 + y_4| + |y_3 - y_4|\right) \Big\} \\ &= 2\|x\| \Big\{ |z_1 + z_2| \max\{|y_1|, |y_2|\} + |z_1 - z_2|| \max\{|y_3|, |y_4|\} \Big\} \\ &\leq 2\|x\|\|y\| \left(|z_1 + z_2| + |z_1 - z_2|\right) \\ &= 4\|x\|\|y\| \max\{|z_1|, |z_2|\} \\ &\leq 4\|x\|\|y\|\|z\|. \end{aligned}$$

Since $T_3(e_1 + e_2 + e_3, e_1 + e_2 + e_3, e_1 + e_2 + e_3) = 4$, then $\|T_3\| = 4$. Also,

$$\left(\sum_{i_1,i_2,i_3=1}^{4} \left| T_3(e_{i_1}, e_{i_2}, e_{i_3}) \right|^{\frac{6}{4}} \right)^{\frac{4}{6}} \leq C_3 \, \| T_3 \|$$

becomes

$$16^{2/3} \leq 4C_3,$$

which gives

$$C_3 \geq 2^{2/3} \approx 1.587.$$

4. The case $m = 4$

In this case, let us consider $T_4 : \ell_\infty^8 \times \ell_\infty^8 \times \ell_\infty^8 \times \ell_\infty^8 \to \mathbb{R}$ given by

$$\begin{aligned} &T_4(x, y, z, w) \\ &= (w_1 + w_2) \left(\begin{array}{c} (z_1 + z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) \\ +(z_1 - z_2)(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4) \end{array} \right) \\ &+ (w_1 - w_2) \left(\begin{array}{c} (z_3 + z_4)(x_5y_5 + x_5y_6 + x_6y_5 - x_6y_6) \\ +(z_3 - z_4)(x_7y_7 + x_7y_8 + x_8y_7 - x_8y_8) \end{array} \right). \end{aligned}$$

As in Sections 2 and 3 we see that $\|T_4\| = 8$, and from (1.1) we obtain

$$64^{5/8} \le 8C_4.$$

Hence

$$C_4 \ge 2^{3/4} \approx 1.681.$$

5. The general case

From the previous results it is not difficult to prove that in general

$$C_m \ge 2^{\frac{m-1}{m}}$$

for every $m \ge 2$. Indeed, let us define the m-linear forms $T_m : \ell_\infty^{2^{m-1}} \times \overset{m}{\cdots} \times \ell_\infty^{2^{m-1}} \to \mathbb{R}$ by induction as

$$\begin{aligned} T_2(x_1, x_2) &= x_1^1x_2^1 + x_1^1x_2^2 + x_1^2x_2^1 - x_1^2x_2^2, \\ T_m(x_1, \ldots, x_m) &= (x_m^1 + x_m^2)T_{m-1}(x_1, \ldots, x_{m-1}) \\ &\quad + (x_m^1 - x_m^2)T_{m-1}(B^{2^{m-2}}(x_1), B^{2^{m-2}}(x_2), \\ &\qquad B^{2^{m-3}}(x_3), \ldots, B^2(x_{m-1})), \end{aligned}$$

where $x_k = (x_k^n)_n \in \ell_\infty^{2^{m-1}}$ for $1 \le k \le m$, $1 \le n \le 2^{m-1}$ and B is the backward shift operator in $\ell_\infty^{2^{m-1}}$. It has been proved in Section 2 that $\|T_2\| = 2$. If we assume that $\|T_{m-1}\| = 2^{m-2}$, then

$$\begin{aligned} |T_m(x_1, \ldots, x_m) &\le |x_m^1 + x_m^2||T_{m-1}(x_1, \ldots, x_{m-1})| \\ &\quad + |x_m^1 - x_m^2||T_{m-1}(B^{2^{m-2}}(x_1), B^{2^{m-2}}(x_2), \\ &\qquad B^{2^{m-3}}(x_3), \ldots, B^2(x_{m-1}))| \\ &\le 2^{m-2}[|x_m^1 + x_m^2|\|x_1\| \cdots \|x_{m-1}\| \\ &\quad + |x_m^1 - x_m^2|\|B^{2^{m-2}}(x_1)\|\|B^{2^{m-2}}(x_2)\|\|B^{2^{m-1}}(x_3)\| \\ &\quad \cdots \|B^2(x_{m-1})\|] \\ &\le 2^{m-2}[|x_m^1 + x_m^2| + |x_m^1 - x_m^2|]\|x_1\| \cdots \|x_{m-1}\| \\ &= 2^{m-1}\|x_1\| \cdots \|x_{m-1}\| \max\{|x_m^1|, |x_m^2|\} \\ &\le 2^{m-1}\|x_1\| \cdots \|x_m\|. \end{aligned}$$

This induction argument shows that $\|T_m\| \le 2^{m-1}$ for all $m \in \mathbb{N}$. Using a similar induction argument it is easy to prove that $T_m(x_1, \ldots, x_m) = 2^{m-1}$ for $x_1, \ldots, x_m$ such that $\|x_1\| = \ldots = \|x_m\| = 1$ and $x_j^k = 1$ with $1 \le j, k \le m$, which proves that $\|T_m\| = 2^{m-1}$ for all $m \in \mathbb{N}$.

On the other hand, from (1.1) we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{2^{m-1}} \left| T_m(e_{i_1},...,e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|T_m\| = 2^{m-1} C_m.$$

To finish, we shall prove that $|T_m(e_{i_1},...,e_{i_m})|$ is either 0 or 1 and that $|T_m(e_{i_1},...,e_{i_m})| = 1$ for exactly 4^{m-1} choices of the vectors $e_{i_1},\ldots,e_{i_m}$. Working again by induction, the reader can easily check that the latter is true for $m = 2$ (see Section 2). If we assume that the result is true for $m-1$ and $e_{i_1},\ldots,e_{i_{m-1}}$ is one of the 4^{m-2} choices of the unit vectors such that $|T_{m-1}(e_{i_1},\ldots,e_{i_{m-1}})| = 1$, then, using the definition of T_m,

$$|T_m(e_{i_1},\ldots,e_{i_{m-1}},e_k)| = |T_{m-1}(e_{i_1},\ldots,e_{i_{m-1}})| = 1,$$
$$|T_m(e_{i_1+2^{m-2}},e_{i_2+2^{m-2}},e_{i_3+2^{m-3}},\ldots,e_{i_{m-1}+2},e_k)| = |T_{m-1}(e_{i_1},\ldots,e_{i_{m-1}})| = 1,$$

for $k = 1, 2$. Hence we have found $4 \cdot 4^{m-2} = 4^{m-1}$ choices of a unit vector where $|T_m|$ takes the value 1. A simple inspection of the problem shows that $|T_m|$ vanishes at any other choice of canonical vector.

6. Final remarks

Notice that our estimate $2^{\frac{m-1}{m}}$ seems inaccurate as $m \to \infty$, since it is a common feeling that the optimal values for the constants C_m should tend to infinity as $m \to \infty$. However, and as a matter of fact, we must say that this common feeling seems to be supported by the estimates of the upper bounds for C_m obtained throughout the last decades, but there does not seem to be any particular result supporting this "fact". In any case (and at least for $m = 2, 3, 4, 5$), our estimates are clearly interesting. Summarizing:

$$\begin{array}{rcl} & C_2 & = \sqrt{2}, \\ 1.587 \leq & C_3 & \leq 1.782, \\ 1.681 \leq & C_4 & \leq 2, \\ 1.741 \leq & C_5 & \leq 2.298. \end{array}$$

We can also conclude that $C_3 > C_2$, which seems not to have been known until now.

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