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ON THE CONTINUITY OF THE LUXEMBURG NORM OF THE GRADIENT IN $L^{p(\cdot)}$ WITH RESPECT TO $p(\cdot)$

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ABSTRACT. The asymptotic behavior of a sequence of functionals involving the Luxemburg norm of the gradient in variable exponent Lebesgue spaces is studied in the framework of Γ-convergence. As a consequence, we prove the convergence of minima for closely related functionals to a corresponding quantity associated to the Γ-limit.

1. Introduction and main results

Motivated by applications to elasticity [29], electrorheological fluids [12], [26], [27], polycrystal plasticity [3], and image processing [7], the theory of variable exponent Lebesgue and Sobolev spaces has received a great deal of attention during the last decades. We refer to the recent monograph [11] for an excellent survey of the field. In recent years, a number of related studies have been devoted to understanding the asymptotic behavior of power-law functionals with variable exponents and of solutions to the associated partial differential equations (see, e.g., [2], [4], [17], [19], [20], [21], [25]).

In this paper we study, via De Giorgi's Γ-convergence, the asymptotic behavior of a sequence of functionals defined on variable exponent Lebesgue spaces and indicate some consequences. Our analysis is motivated in part by a recent paper of Parini [24] where the question of continuity of the variational eigenvalues of the p-Laplacian with respect to p is answered in the affirmative. The result in [24] is a direct consequence of a Γ-convergence theorem for a sequence of functionals in L^1 involving the L^q norms of the gradient of Sobolev mappings as $q \to p$ for some $p \in (1, \infty)$, coupled with a result of Champion and De Pascale [5].

In Theorem 1.1 below, we generalize Parini's Γ-convergence result to the setting of variable exponents Lebesgue spaces. Under more stringent restrictions on the exponents, we then use our Γ-convergence theorem to establish the convergence of minima for a sequence of "Rayleigh quotient" type functionals adapted to our setting to the corresponding quantity associated to the Γ-limit.

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Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with sufficiently smooth boundary. Throughout this paper we denote by $\mathcal{P}(\Omega)$ the class of variable exponents $p : \Omega \to (1, \infty)$ that are continuous and satisfy $p^- := \inf_{x\in\Omega} p(x) > 1$ and $p^+ := \sup_{x\in\Omega} p(x) < +\infty$. Let $p \in \mathcal{P}(\Omega)$, and let $\{p_n\} \subset \mathcal{P}(\Omega)$ be a sequence such that

$$p_n \text{ converges uniformly to } p \text{ in } \Omega. \tag{1.1}$$

In what follows $|v|_{q(\cdot)}$ will denote the Luxemburg norm of v in the variable exponent Lebesgue space $L^{q(\cdot)}(\Omega;\mathbb{R}^m), m \in \mathbb{N}$. We refer to Section 2 of the paper for the precise definitions, as well as for more details on variable exponent Lebesgue and Sobolev spaces. For $n \in \mathbb{N}$, consider the functional $I_n : L^1(\Omega) \to [0, +\infty]$ defined by

$$I_n(u) = \begin{cases} |\nabla u|_{p_n(\cdot)} & \text{if } u \in W_0^{1,p_n(\cdot)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

In Section 3 of the paper we prove the following Γ-convergence result for the sequence $\{I_n\}$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain, and let $\{p_n\} \subset \mathcal{P}(\Omega)$ and $p \in \mathcal{P}(\Omega)$ be such that* (1.1) *holds. Define $I : L^1(\Omega) \to [0, +\infty]$ by*

$$I(u) = \begin{cases} |\nabla u|_{p(\cdot)} & \textit{if } u \in W_0^{1,p(\cdot)}(\Omega), \\ +\infty & \textit{otherwise.} \end{cases}$$

Then $\Gamma(L^{p(\cdot)}(\Omega)) - \lim_{n\to\infty} I_n = I$.

We say that $p \in \mathcal{P}^{\log}(\Omega)$ if $p \in \mathcal{P}(\Omega)$ and, in addition, p satisfies the global log-Hölder continuity condition: there exists $c_1, c_2 > 0$, and $p_\infty \in \mathbb{R}$ such that

$$|p(x) - p(y)| \leq \frac{c_1}{\log(e + 1/|x-y|)} \text{ for all } x, y \in \Omega$$

and

$$|p(x) - p_\infty| \leq \frac{c_2}{\log(e + |x|)} \text{ for all } x \in \Omega.$$

It is well-known (see, e.g., [11, Theorem 8.2.4]) that if $p_n \in \mathcal{P}^{\log}(\Omega)$, then there exists a positive constant $c_n > 0$ such that

$$c_n = \inf\left\{ \frac{|\nabla u|_{p_n(\cdot)}}{|u|_{p_n(\cdot)}} \ : \ u \in W_0^{1,p_n(\cdot)}(\Omega) \setminus \{0\} \right\}.$$

The following theorem, proved in Section 4 of the paper, identifies the limit of c_n as $n \to \infty$ under the uniform convergence hypothesis (1.1) on the sequence of exponents $\{p_n\}$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain, and let $\{p_n\} \subset \mathcal{P}^{\log}(\Omega)$ be such that* (1.1) *holds for some $p \in \mathcal{P}^{\log}(\Omega)$. Let*

$$c := \inf\left\{ \frac{|\nabla u|_{p(\cdot)}}{|u|_{p(\cdot)}} \ : \ u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\} \right\} > 0.$$

Then $c_n \to c$ as $n \to \infty$.

2. Variable exponent Lebesgue and Sobolev spaces and Γ-convergence

We begin this section with a brief review of the basic properties of the variable exponent Lebesgue-Sobolev spaces. For more details we refer to the books by Diening, Harjulehto, Hästö and M. Ružička [11], Musielak [23] and the papers by Edmunds *et al.* [13–15], Kovacik and Rákosník [18], Mihăilescu and Rădulescu [22], and Samko and Vakulov [28].

Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $|\Omega|$ the N-dimensional Lebesgue measure of Ω. For any Lipschitz continuous function $p : \overline{\Omega} \to (1, \infty)$ let

$$p^- := \inf_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \sup_{x\in\Omega} p(x).$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

$L^{p(\cdot)}(\Omega)$ is a Banach space when endowed with the *Luxemburg norm*, defined by

$$|u|_{p(\cdot)} := \inf \left\{ \mu > 0 \ : \ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is a special case of an Orlicz-Musielak space. For constant functions p, $L^{p(\cdot)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, endowed with the standard norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{1/p}.$$

We recall that if $1 < p^- \le p^+ < +\infty$, the space $L^{p(\cdot)}(\Omega)$ is separable and reflexive. If $0 < |\Omega| < \infty$, and if p_1, p_2 are such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous and its norm does not exceed $|\Omega|+1$.

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality holds:

$$\left| \int_\Omega uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)}. \tag{2.1}$$

The *modular* of the space $L^{p(\cdot)}(\Omega)$ is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx.$$

Lebesgue and Sobolev spaces with $p^+ = +\infty$ have been investigated in [13] and [18]. In this case one defines $\Omega_\infty := \{x \in \Omega;\ p(x) = +\infty\}$, and the modular is given by

$$\rho_{p(\cdot)}(u) := \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}\, dx + \operatorname{ess\,sup}_{x\in\Omega_\infty} |u(x)|.$$

In the particular case $\Omega_\infty = \Omega$ we recover the usual Lebesgue space $L^\infty(\Omega)$. If $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold:

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|^{p^-}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p^+}_{p(\cdot)}, \tag{2.2}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^-}, \tag{2.3}$$

$$|u|_{p(\cdot)} = 1 \Leftrightarrow \rho_{p(\cdot)}(u) = 1. \tag{2.4}$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : \ |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

On $W^{1,p(\cdot)}(\Omega)$ we can consider one of the following equivalent norms:

$$\|u\|_{p(\cdot)} := |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}$$

or

$$\|u\| := \inf \left\{ \mu > 0; \ \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) \, dx \leq 1 \right\},$$

where, in the definition of $\|u\|_{p(\cdot)}$, $|\nabla u|_{p(\cdot)}$ stands for the Luxemburg norm of $|\nabla u|$. It is well-known (see, e.g., [11]) that under very mild assumptions on the variable exponent p the space $W^{1,p(\cdot)}(\Omega)$ is also a separable and reflexive Banach space. Finally, the Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{p(\cdot)}$.

Next, we recall the definition of Γ-convergence (introduced in [9], [10]) in metric spaces. The reader is referred to [8] for a comprehensive introduction to the subject.

Definition 2.1. Let X be a metric space. A sequence $\{F_n\}$ of functionals $F_n : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to $\Gamma(X)$-converge to $F : X \to \overline{\mathbb{R}}$, and we write $\Gamma(X) - \lim_{n\to\infty} F_n = F_\infty$ if the following hold:

(i) for every $u \in X$ and $\{u_n\} \subset X$ such that $u_n \to u$ in X, we have

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n);$$

(ii) for every $u \in X$ there exists a sequence $\{u_n\} \subset X$ (called a recovery sequence) such that $u_n \to u$ in X and

$$F(u) \geq \limsup_{n\to\infty} F_n(u_n).$$

The following result is well-known and can be found, for example, in [16, Lemma 6.1.1].

Proposition 2.1. *Let X be a topological space that satisfies the first axiom of countability, and assume that $\{u_n\}$ is a sequence such that $u_n \to u$ in X as $n \to \infty$,*

$$\limsup_{n\to\infty} F(u_n) \leq F(u),$$

and such that for every $m \in \mathbb{N}$ there exists a sequence $\{u_{m,n}\}_n$, $u_{m,n} \to u_m$ as $n \to \infty$, with

$$\limsup_{n\to\infty} F_n(u_{m,n}) \leq F(u_m).$$

Then there exists a recovering sequence for u in the sense of (ii) of Definition 2.1.

3. Proof of Theorem 1.1

We start by recalling a result proved in [11, Theorem 3.5.7].

Proposition 3.1. *For $n \in \mathbb{N}$, let p, $p_n : \overline{\Omega} \to (1,\infty)$ be continuous functions such that*

$$p_n(x) \leq p(x), \quad \forall\, n \geq 1 \text{ and } \forall\, x \in \overline{\Omega}, \tag{3.1}$$

and

$$p_n(x) \to p(x) \text{ as } n \to \infty, \ \ \text{for a.e. } x \in \Omega. \tag{3.2}$$

Then, for each $u \in L^{p(\cdot)}(\Omega)$, we have

$$\lim_{n\to\infty} |u|_{p_n(\cdot)} = |u|_{p(\cdot)}.$$

Proposition 3.2. *For $n \in \mathbb{N}$, let p, $p_n : \overline{\Omega} \to (1,\infty)$ be continuous functions satisfying* (1.1)*. Then, for each $u \in C_0^\infty(\Omega)$, we have $\lim\limits_{n\to\infty} |u|_{p_n(\cdot)} = |u|_{p(\cdot)}$.*

Proof. By (1.1), we have that for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $|p_n(x) - p(x)| < \epsilon$, $\forall\, x \in \Omega,\ n \geq N_\epsilon$. In particular,

$$p_n(x) < p(x) + \epsilon, \quad \forall\, x \in \Omega,\ n \geq N_\epsilon. \tag{3.3}$$

Since $u \in C_0^\infty(\Omega)$, this gives that $u \in L^{p(\cdot)+\epsilon}(\Omega) \subset L^{p_n(\cdot)}(\Omega)$ for all $n \geq N_\epsilon$. Furthermore, (3.3) also implies that

$$\left|\frac{u(x)}{|u|_{p(\cdot)}}\right|^{p_n(x)} \leq \left|\frac{u(x)}{|u|_{p(\cdot)}}\right|^{p(x)+\epsilon} + 1 \in L^1(\Omega),\ \forall\, x \in \Omega,\ n \geq N_\epsilon.$$

On the other hand, for a.e. $x \in \Omega$, (1.1) yields

$$\left|\frac{u(x)}{|u|_{p(\cdot)}}\right|^{p_n(x)} \to \left|\frac{u(x)}{|u|_{p(\cdot)}}\right|^{p(x)} \quad \text{as } n \to \infty.$$

Set $v(x) := \frac{u(x)}{|u|_{p(\cdot)}}$. By Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_\Omega |v(x)|^{p_n(x)}\, dx = \lim_{n\to\infty} \int_\Omega \left|\frac{u(x)}{|u|_{p(\cdot)}}\right|^{p_n(x)} dx = \int_\Omega \left|\frac{u(x)}{|u|_{p(\cdot)}}\right|^{p(x)} dx = 1. \tag{3.4}$$

To conclude the proof it suffices to show that $\lim\limits_{n\to\infty} |v|_{p_n(\cdot)} = 1$. To this end, note that (2.2) and (2.3) imply that

$$\min\{\rho_{p_n(\cdot)}(v)^{1/p_n^-}, \rho_{p_n(\cdot)}(v)^{1/p_n^+}\} \leq |v|_{p_n(\cdot)} \leq \max\{\rho_{p_n(\cdot)}(v)^{1/p_n^-}, \rho_{p_n(\cdot)}(v)^{1/p_n^+}\},$$

for all $n \in \mathbb{N}$. Since

$$\frac{1}{p^+ + \epsilon} < \frac{1}{p_n^+} \leq \frac{1}{p_n^-} \leq 1, \forall\, n \geq N_\epsilon,$$

and in view of (3.4), we have

$$\lim_{n\to\infty} \left(\int_\Omega |v(x)|^{p_n(x)}\, dx\right)^{1/p_n^\pm} = 1.$$

$\square$

We are now ready to prove Theorem 1.1. Let $u_n \to u$ in $L^{p(\cdot)}(\Omega)$. If we have $\liminf_{n\to\infty} I_n(u_n) = +\infty$, there is nothing to prove. Thus, we may assume, without loss of generality, that $u_n \in W_0^{1,p_n(\cdot)}(\Omega)$ and, after eventually extracting a subsequence,

$$\liminf_{n\to\infty} I_n(u_n) = \lim_{n\to\infty} I_n(u_n) =: L < +\infty. \tag{3.5}$$

Let $\epsilon > 0$ be such that $1 < p^- - \epsilon$. From (1.1) we deduce that there exists $N_\epsilon \in \mathbb{N}$ such that

$$p(x) - \epsilon < p_n(x), \quad \forall\, x \in \Omega, \ \ \forall\, n \geq N_\epsilon. \tag{3.6}$$

Thus, for all $n \geq N_\epsilon$, we have that $u_n \in W_0^{1,p_n(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)-\epsilon}(\Omega)$ and

$$|\nabla u_n|_{p(\cdot)-\epsilon} \leq (1 + |\Omega|)|\nabla u_n|_{p_n(\cdot)} \leq D, \tag{3.7}$$

where $D > 0$ is a constant which does not depend on ϵ. Consequently, $\{u_n\}$ is a bounded sequence in the reflexive Sobolev space $W_0^{1,p(\cdot)-\epsilon}(\Omega)$, and thus, after possibly extracting a subsequence (not relabelled), $u_n \rightharpoonup u$ weakly in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$, with $u \in W_0^{1,p(\cdot)-\epsilon}(\Omega)$ for all $\epsilon > 0$ sufficiently small. Taking into account (2.2), (2.3), and (3.7), we have

$$\begin{aligned}
&\int_\Omega |\nabla u(x)|^{p(x)-\epsilon}\, dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n(x)|^{p(x)-\epsilon}\, dx \\
&\leq \liminf_{n\to\infty} \max\left\{|\nabla u_n|_{p(\cdot)-\epsilon}^{p^+-\epsilon}, |\nabla u_n|_{p(\cdot)-\epsilon}^{p^--\epsilon}\right\} \\
&\leq \liminf_{n\to\infty} \max\{[(1+|\Omega|)|\nabla u_n|_{p_n(\cdot)}]^{p^+-\epsilon}, [(1+|\Omega|)|\nabla u_n|_{p_n(\cdot)}]^{p^--\epsilon}\} \\
&\leq \max\{D^{p^+-\epsilon}, D^{p^--\epsilon}\} < +\infty.
\end{aligned}$$

Thus,

$$\sup_{\epsilon>0} \int_\Omega |\nabla u(x)|^{p(x)-\epsilon}\, dx < +\infty,$$

and hence, by Fatou's lemma, $|\nabla u(x)|^{p(x)} = \liminf_{\epsilon\to 0} |\nabla u(x)|^{p(x)-\epsilon} \in L^1(\Omega)$. It follows that $u \in W_0^{1,p(\cdot)}(\Omega)$.

Next, taking into account (3.6) and applying Young's inequality, we have

$$\begin{aligned}
\int_\Omega |\nabla u_n(x)|^{p(x)-\epsilon}\, dx &\leq \int_\Omega \frac{p(x)-\epsilon}{p_n(x)} |\nabla u_n(x)|^{p_n(x)}\, dx + \int_\Omega \frac{p_n(x)-p(x)+\epsilon}{p_n(x)}\, dx \\
&\leq \int_\Omega |\nabla u_n(x)|^{p_n(x)}\, dx + \epsilon \int_\Omega \frac{1}{p_n(x)}\, dx + \frac{1}{p_n^-}\|p_n - p\|_\infty.
\end{aligned} \tag{3.8}$$

Letting $n \to \infty$, we find, owing to (1.1), the weak convergence of u_n to u in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$, and by (3.8) that

$$\int_\Omega |\nabla u(x)|^{p(x)-\epsilon}\, dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n(x)|^{p_n(x)}\, dx + \epsilon \frac{|\Omega|}{p^-}.$$

Since this inequality holds for each $\epsilon > 0$, letting $\epsilon \searrow 0$ we deduce that

$$\int_\Omega |\nabla u(x)|^{p(x)}\, dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n(x)|^{p_n(x)}\, dx, \tag{3.9}$$

where we have used again Fatou's lemma.

Since $L = \lim\limits_{n\to\infty} I_n(u_n)$ we have, for each $\lambda > L$, that $|\nabla u_n|_{p_n(\cdot)} \le \lambda$ for all $n \in \mathbb{N}$ large enough. Thus,

$$\int_\Omega \left| \frac{\nabla u_n(x)}{\lambda} \right|^{p_n(x)} dx \le 1,$$

for all $n \in \mathbb{N}$ sufficiently large. We may repeat the arguments used to obtain (3.9) with u and u_n replaced by u/λ and u_n/λ, respectively, to deduce that

$$\int_\Omega \left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} dx \le 1,$$

which gives $|\nabla u|_{p(\cdot)} \le \lambda$, $\forall\, \lambda > L$. Letting $\lambda \searrow L$ we find

$$|\nabla u|_{p(\cdot)} \le L = \liminf_{n\to\infty} |\nabla u_n|_{p_n(\cdot)}.$$

It remains to prove the existence of a recovery sequence for the Γ-limit. To this aim, let $u \in L^1(\Omega)$ be arbitrary, and note that if $u \notin W_0^{1,p(\cdot)}(\Omega)$ there is nothing to prove, since $I(u) = +\infty$ in this case. Next, assume that $u \in W_0^{1,p(\cdot)}(\Omega)$, and let $\{u_n\} \subset C_0^\infty(\Omega)$ be such that $u_n \to u$ as $n \to \infty$ in $W_0^{1,p(\cdot)}(\Omega)$. In particular, we have $\lim\limits_{n\to\infty} I(u_n) = I(u)$. On the other hand, by Proposition 3.2 we have, for each $m \in \mathbb{N}$, that $\lim\limits_{n\to\infty} |u_m|_{p_n(\cdot)} = |u_m|_{p(\cdot)}$. Equivalently, $\lim\limits_{n\to\infty} I_n(u_m) = I(u_m)$. In view of Proposition 2.1, we deduce that

$$I(u) \ge \limsup_{n\to\infty} I_n(u_n).$$

This concludes the proof of Theorem 1.1.

Remark 3.1. If we assume that in addition to (1.1), condition (3.1) is satisfied, then, by Proposition 3.1, the constant sequence $\{u\}$ can serve as a recovery sequence.

4. Proof of Theorem 1.2

The proof is based on Propositions 4.1 and 4.2 below. We will first establish an auxiliary result whose proof is similar to the first part of the proof of Theorem 1.1.

Lemma 4.1. *Assume that* (1.1) *holds. If* $u \in W_0^{1,p(\cdot)}(\Omega)$ *and* $\{u_n\} \subset C_0^\infty(\Omega)$ *are such that* $u_n \to u$ *in* $W_0^{1,p(\cdot)}(\Omega)$, *then*

$$\liminf_{n\to\infty} |u_n|_{p_n(\cdot)} \ge |u|_{p(\cdot)}.$$

Proof. Let $l := \liminf_{n\to\infty} |u_n|_{p_n(\cdot)} < +\infty$, and extract a subsequence (not relabelled) such that $\liminf\limits_{n\to\infty} |u_n|_{p_n(\cdot)} = \lim\limits_{n\to\infty} |u_n|_{p_n(\cdot)}$. Let $\epsilon > 0$ be small enough such that $1 < p^- - \epsilon$. By (1.1), there exists $N_\epsilon \in \mathbb{N}$ such that $p(x) - \epsilon < p_n(x)$ for all $x \in \Omega$ and $n \ge N_\epsilon$. Next, applying Young's inequality we have

$$\begin{aligned}\int_\Omega |u_n(x)|^{p(x)-\epsilon}\, dx &\le \int_\Omega \frac{p(x)-\epsilon}{p_n(x)} |u_n(x)|^{p_n(x)}\, dx + \int_\Omega \frac{p_n(x) - p(x) + \epsilon}{p_n(x)}\, dx \\ &\le \int_\Omega |u_n(x)|^{p_n(x)}\, dx + \epsilon \int_\Omega \frac{1}{p_n(x)}\, dx + \frac{1}{p_n^-}\|p_n - p\|_\infty.\end{aligned}$$

By (1.1), the convergence of u_n to u in $L^{p(\cdot)}(\Omega)$ (and thus in $L^{p(\cdot)-\epsilon}(\Omega)$), and the above inequality, we find, letting $n \to \infty$, that

$$\int_\Omega |u(x)|^{p(x)-\epsilon}\, dx \le \liminf_{n\to\infty} \int_\Omega |u_n(x)|^{p_n(x)}\, dx + \epsilon \frac{|\Omega|}{p^-}.$$

Passing to the limit $\epsilon \searrow 0$, we deduce, by Proposition 3.1, that

$$\int_\Omega |u(x)|^{p(x)}\,dx \le \liminf_{n\to\infty} \int_\Omega |u_n(x)|^{p_n(x)}\,dx. \tag{4.1}$$

Since $l = \lim_{n\to\infty} |u_n|_{p_n(\cdot)}$, for each $\lambda > l$ we have $|u_n|_{p_n(\cdot)} \le \lambda$, and thus

$$\int_\Omega \left|\frac{u_n(x)}{\lambda}\right|^{p_n(x)} dx \le \left|\frac{u_n}{\lambda}\right|^{p_n^-}_{p_n(\cdot)} \le 1,$$

for all $n \in \mathbb{N}$ sufficiently large. Repeating the previous arguments with u and u_n replaced by u/λ and u_n/λ, respectively, we obtain that (4.1) still holds with these choices, and thus for each $\lambda > l$ we have

$$\int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1,$$

which gives $|u|_{p(\cdot)} \le \lambda$, $\forall\, \lambda > l$. Letting $\lambda \searrow l$ we obtain $|u|_{p(\cdot)} \le l = \lim_{n\to\infty} |u_n|_{p_n(\cdot)}$. □

Proposition 4.1.

$$c \ge \limsup_{n\to\infty} c_n.$$

Proof. The Direct Method of the Calculus of Variations (see, e.g., [6]) ensures that there exists $u \in W_0^{1,p(\cdot)}(\Omega)$ such that $|u|_{p(\cdot)} = 1$ and $|\nabla u|_{p(\cdot)} = c$. Let $\{u_n\} \subset C_0^\infty(\Omega)$ be such that $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$. We can apply Proposition 2.1 with $X = L^{p(\cdot)}(\Omega)$, $F_n(u) = |u|_{p_n(\cdot)}$ and $F(u) = |u|_{p(\cdot)}$. Indeed, we have $\limsup_{n\to\infty} |u_n|_{p(\cdot)} = |u|_{p(\cdot)}$, and since by Proposition 3.2 for each $m \in \mathbb{N}$ the constant sequence $\{u_m\}_n$ satisfies

$$\limsup_{n\to\infty} |u_m|_{p_n(\cdot)} \le |u_m|_{p(\cdot)},$$

we deduce that the hypotheses of Proposition 2.1 are satisfied. Thus,

$$|u|_{p(\cdot)} \ge \limsup_{n\to\infty} |u_n|_{p_n(\cdot)}. \tag{4.2}$$

Taking into account Lemma 4.1, we obtain that $\lim_{n\to\infty} |u_n|_{p_n(\cdot)} = |u|_{p(\cdot)}$. Next, since $\{u_n\}$ is a recovery sequence for the Γ-limit (corresponding to u), the arguments in the second part of the proof of Theorem 1.1 give

$$c = |\nabla u|_{p(\cdot)} = \lim_{n\to\infty} |\nabla u_n|_{p_n(\cdot)}. \tag{4.3}$$

The definition of c_n ensures that

$$|\nabla u_n|_{p_n(\cdot)} \ge c_n |u_n|_{p_n(\cdot)},\ \forall\, n \in \mathbb{N}. \tag{4.4}$$

On the other hand, using the fact that $\lim_{n\to\infty} |u_n|_{p_n(\cdot)} = |u|_{p(\cdot)}$, and since $|u|_{p(\cdot)} = 1$, we deduce that for all $\epsilon > 0$ sufficiently small there exists $N_\epsilon \in \mathbb{N}$ such that $|u_n|_{p_n(\cdot)} > 1 - \epsilon$, $\forall\, n \ge N_\epsilon$. Consequently, $c_n |u_n|_{p_n(\cdot)} > (1-\epsilon) c_n$, $\forall\, n \ge N_\epsilon$, and thus, taking into account (4.3) and (4.4), we get

$$c \ge (1-\epsilon) \limsup_{n\to\infty} c_n.$$

Letting $\epsilon \searrow 0$ above gives $c \ge \limsup_{n\to\infty} c_n$. □

Proposition 4.2.

$$c \leq \liminf_{n \to \infty} c_n.$$

Proof. For each $n \in \mathbb{N}$, let $v_n \in W_0^{1,p_n(\cdot)}$ be such that $|v_n|_{p_n(\cdot)} = 1$ and $|\nabla v_n|_{p_n(\cdot)} = c_n$. Fix $\epsilon > 0$ small enough such that $1 < p^- - \epsilon$ and $(p^-)^2 > 2N\epsilon + \epsilon^2$. In particular, this implies that we have $p(x) + \epsilon < \frac{N(p(x)-\epsilon)}{N-p(x)+\epsilon} = (p(x)-\epsilon)^\star$ for all $x \in \Omega$, which ensures that the embedding $W_0^{1,p(\cdot)-\epsilon}(\Omega) \subset L^{p(\cdot)+\epsilon}(\Omega)$ is compact. By (1.1), there exists $N_\epsilon \in \mathbb{N}$ such that

$$p(x) - \epsilon < p_n(x) < p(x) + \epsilon, \ \forall \, x \in \Omega, \ n \geq N_\epsilon. \tag{4.5}$$

Thus, for all $n \geq N_\epsilon$ we have $v_n \in W_0^{1,p_n(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)-\epsilon}(\Omega)$ and

$$|\nabla v_n|_{p(\cdot)-\epsilon} \leq (1 + |\Omega|)|\nabla v_n|_{p_n(\cdot)} \leq M, \tag{4.6}$$

where $M > 0$ is a constant which is independent of ϵ. Consequently, the sequence $\{v_n\}$ is bounded in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$, and thus we may extract a subsequence (not relabelled) of $\{v_n\}$ such that $v_n \rightharpoonup v$ weakly in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$, with $v \in W_0^{1,p(\cdot)-\epsilon}(\Omega)$. Due to the compactness of the embedding of $W_0^{1,p(\cdot)-\epsilon}(\Omega)$ into $L^{p(\cdot)+\epsilon}(\Omega)$, we have that $v_n \to v$ strongly in $L^{p(\cdot)+\epsilon}(\Omega)$ for all $\epsilon > 0$ satisfying $(p^-)^2 > 2N\epsilon + \epsilon^2$. The weak convergence of v_n to v in $W_0^{1,p(\cdot)-\epsilon}(\Omega)$, together with (2.2), (2.3), and (4.6), gives

$$\begin{aligned}
&\int_\Omega |\nabla v(x)|^{p(x)-\epsilon} \, dx \leq \liminf_{n \to \infty} \int_\Omega |\nabla v_n(x)|^{p(x)-\epsilon} \, dx \\
&\leq \liminf_{n \to \infty} \max\{|\nabla v_n|_{p(\cdot)-\epsilon}^{p^+-\epsilon}, |\nabla v_n|_{p(\cdot)-\epsilon}^{p^--\epsilon}\} \\
&\leq \liminf_{n \to \infty} \max\{[(1+|\Omega|)|\nabla v_n|_{p_n(\cdot)}]^{p^+-\epsilon}, [(1+|\Omega|)|\nabla v_n|_{p_n(\cdot)}]^{p^--\epsilon}\} \\
&\leq \max\{M^{p^+-\epsilon}, M^{p^--\epsilon}\} < +\infty,
\end{aligned}$$

for $\epsilon > 0$ sufficiently small. Thus, $\sup_{\epsilon > 0} \int_\Omega |\nabla v(x)|^{p(x)-\epsilon} \, dx < +\infty$, and in view of Fatou's lemma we deduce that $|\nabla v(x)|^{p(x)} = \liminf_{\epsilon \to 0} |\nabla v(x)|^{p(x)-\epsilon} \in L^1(\Omega)$. Hence, $v \in W_0^{1,p(\cdot)}(\Omega)$. Since $v_n \to v$ strongly in $L^{p(\cdot)}(\Omega)$, Theorem 1.1 implies that $I(v) \leq \liminf_{n \to \infty} I_n(v_n)$, which gives

$$|\nabla v|_{p(\cdot)} \leq \liminf_{n \to \infty} |\nabla v_n|_{p_n(\cdot)} = \liminf_{n \to \infty} c_n.$$

To finish the proof it suffices to show that $|v|_{p(\cdot)} = 1$ and to take into account that in this case we have $c \leq |\nabla v|_{p(\cdot)}$.

To show that indeed $|v|_{p(\cdot)} = 1$, we first note that since v_n converges strongly to v in $L^{p(\cdot)+\epsilon}(\Omega)$ and taking into account (1.1), we have, after eventually extracting a subsequence, that $|v_n(x)|^{p_n(x)} \to |v(x)|^{p(x)}$ a.e. $x \in \Omega$ and that there exists $g \in L^1(\Omega)$ such that $|v_n(x)|^{p(x)+\epsilon} \leq g(x)$ for a.e. $x \in \Omega$. This, together with (4.5), implies that

$$|v_n(x)|^{p_n(x)} \leq 1 + |v_n(x)|^{p(x)+\epsilon} \leq 1 + g(x) \in L^1(\Omega) \text{ for all } n \in \mathbb{N} \text{ and a.e. } x \in \Omega.$$

Thus, we may apply Lebesgue's Dominated Convergence Theorem to deduce that

$$\lim_{n\to\infty}\int_\Omega |v_n(x)|^{p_n(x)}dx = \int_\Omega |v(x)|^{p(x)}dx.$$

Since $|v_n|_{p_n(\cdot)} = 1$, we have, in view of (2.4), that $\int_\Omega |\nabla v_n(x)|^{p_n(x)}dx = 1$ for all $n \in \mathbb{N}$. We deduce that $\int_\Omega |\nabla v(x)|^{p(x)}\,dx = 1$ or, equivalently, $|v|_{p(\cdot)} = 1$. $\square$

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