

KOSZUL HOMOLOGY OF CODIMENSION 3 GORENSTEIN IDEALS

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ABSTRACT. In this note, we calculate the Koszul homology of the codimension 3 Gorenstein ideals. We find filtrations for the Koszul homology in terms of modules with pure free resolutions and completely describe these resolutions. We also consider the Huneke–Ulrich deviation 2 ideals.

INTRODUCTION

For the codimension 3 Pfaffian ideal of $2n \times 2n$ Pfaffians of a $(2n + 1) \times (2n + 1)$ generic skew-symmetric matrix, we give an explicit description of the Koszul homology modules. By a result of Buchsbaum–Eisenbud [BE], the general case of codimension 3 Gorenstein ideals reduces to this case. They are filtered by equivariant modules M_i with self-dual pure free resolutions of length 3 supported in the ideal of Pfaffians. The free resolutions of the modules M_i give natural generalizations of the Buchsbaum–Eisenbud complexes for codimension 3 Gorenstein ideals and are interesting in their own right. It was known that the Koszul homology modules of codimension 3 Pfaffian ideals are Cohen–Macaulay [Hun1, Example 2.2], but no explicit description was given. The only other example we could find in the literature of explicit calculations of Koszul homology is the paper of Avramov–Herzog [AH], which handles the case of codimension 2 perfect ideals. The Koszul homology modules of codimension 3 Pfaffian ideals also gives examples of modules with pure filtrations that do not follow from the results in [EES]. Finally, we calculate the Koszul homology modules for the Huneke–Ulrich deviation 2 ideals which were studied by Kustin [Kus].

1. KOSZUL HOMOLOGY

Throughout R is a Cohen–Macaulay (graded) local ring. After this section, we will be working over polynomial rings with \mathbf{Z} -coefficients, which we pretend is a graded local ring by saying that its maximal ideal is the one generated by the variables. Let $I \subset R$ be a (graded) ideal of grade g , and let $\mu(I)$ denote the smallest size of a generating set of I . The Koszul homology of I depends on a set of generators, but any two choices of *minimal* generating sets yield isomorphic Koszul homology. In the case of a minimal generating set, we denote the Koszul homology by $H_\bullet(I; R)$. We will only be interested in Koszul homology for minimal generating sets of I . We say that I is **strongly Cohen–Macaulay** if the Koszul homology of I is Cohen–Macaulay.

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If R is Gorenstein and R/I is Cohen–Macaulay, then the top nonvanishing Koszul homology $H_{\mu(I)-g}(I; R)$ is the canonical module $\omega_{R/I}$ of R/I (see [Hun2, Remark 1.2]). Furthermore, the exterior multiplication on the Koszul complex induces maps

$$(1.1) \quad H_i(I; R) \rightarrow \text{Hom}_R(H_{\mu(I)-g-i}(I; R), H_{\mu(I)-g}(I; R)),$$

and these maps are isomorphisms in the case that I is strongly Cohen–Macaulay. This is also true if we only assume that the Koszul homology modules are reflexive [Hun2, Proposition 2.7].

2. CODIMENSION 3 PFAFFIAN IDEALS

In this section we work over the integers \mathbf{Z} and set $A = \text{Sym}(\bigwedge^2 E)$, where E is a free \mathbf{Z} -module of rank $2n + 1$. We consider the ideal $I = \text{Pf}_{2n}(\varphi)$ of $2n \times 2n$ Pfaffians of the generic skew-symmetric matrix

$$\varphi = (\varphi_{i,j})_{1 \leq i,j \leq n},$$

where $\varphi_{i,j}$ are the variables satisfying $\varphi_{i,j} = -\varphi_{j,i}$. The free resolution for this ideal and its main properties can be found in [BE] (the quotient A/I is also the module M_0 defined in the next section). Thus if $\{e_1, \dots, e_{2n+1}\}$ is a basis in E , we can think of $\varphi_{i,j} = e_i \wedge e_j \in \bigwedge^2 E$. Denote the $2n \times 2n$ Pfaffians of φ by

$$Y_i = (-1)^{i+1} \text{Pf } \varphi(i),$$

where $\varphi(i)$ is the skew-symmetric matrix we get from φ by omitting the i -th row and i -th column.

Consider the Koszul complex $\mathbf{K}_\bullet = \mathbf{K}(Y_1, \dots, Y_{2n+1}; A)$. In this case,

$$\begin{aligned} \mathbf{K}_i &= \bigwedge^i \left(\bigwedge^{2n} E \otimes A(-in) \right) = \bigwedge^i E^* \otimes (\det E)^i \otimes A(-in) \\ &= \bigwedge^{2n+1-i} E \otimes (\det E)^{i-1} \otimes A(-in). \end{aligned}$$

2.1. Modules M_i . Before we start we describe a family of A -modules supported in the ideal I . For $i = 0, \dots, n - 1$, we get equivariant inclusions

$$\begin{aligned} d_1: \bigwedge^{2n-i} E &\subset \bigwedge^i E \otimes \bigwedge^{2n-2i} E \subset \bigwedge^i E \otimes \text{Sym}^{n-i}(\bigwedge^2 E), \\ d_2: \det E \otimes \bigwedge^{i+1} E &\subset \bigwedge^{2n-i} E \otimes \bigwedge^{2i+2} E \subset \bigwedge^{2n-i} E \otimes \text{Sym}^{i+1}(\bigwedge^2 E), \\ d_3: \det E \otimes \bigwedge^{2n+1-i} E &\subset \det E \otimes \bigwedge^{i+1} E \otimes \bigwedge^{2n-2i} E \subset \det E \otimes \bigwedge^{i+1} E \otimes \text{Sym}^{n-i}(\bigwedge^2 E), \end{aligned}$$

where in each case the first inclusion can be defined in terms of comultiplication, and the second is given by Pfaffians. We make these maps more explicit. Let e_1, \dots, e_{2n+1} be an ordered basis for E compatible with φ . For an ordered sequence $I = (i_1, \dots, i_n)$ consisting of elements from $[1, 2n]$ we denote by e_I the decomposable tensor $e_{i_1} \wedge \dots \wedge e_{i_n}$. The embedding $\bigwedge^{2d} E \subseteq \text{Sym}^d(\bigwedge^2 E)$ sends the tensor e_I ($\#I = 2d$) to the Pfaffian of the $2d \times 2d$ skew-symmetric submatrix of φ corresponding to the rows and columns indexed by I . We will denote this

Pfaffian by $\text{Pf}(I)$. With these conventions, the maps d_1, d_2, d_3 are given by the formulas

$$\begin{aligned} d_1(e_I) &= \sum_{I' \subset I} \text{sgn}(I', I'') e_{I'} \otimes \text{Pf}(I''), \\ d_2(e_{[1, 2n+1]} \otimes e_J) &= \sum_{I' \subset [1, 2n+1]} \text{sgn}(I', I'') \text{sgn}(I'', J) e_{I'} \otimes \text{Pf}(I'' \cup J), \\ d_3(e_I) &= \sum_{I' \subset I} \text{sgn}(I', I'') e_{I'} \otimes \text{Pf}(I''), \end{aligned}$$

where I'' is the complement of I' in I , all subsets are listed in increasing order, and $\text{sgn}(I', I'')$ is the sign of the permutation that reorders (I', I'') in its natural order. The symbol $\text{Pf}(I'' \cup J)$ is by convention 0 if $I'' \cap J \neq \emptyset$.

Proposition 2.1. *For $i = 0, \dots, n - 1$, we define the complex C^i ,*

$$0 \rightarrow \begin{array}{c} (\det E) \otimes \bigwedge^{2n+1-i} E \\ \otimes A(-2n+i-1) \end{array} \xrightarrow{d_3} \begin{array}{c} (\det E) \otimes \bigwedge^{i+1} E \\ \otimes A(-n-1) \end{array} \xrightarrow{d_2} \begin{array}{c} \bigwedge^{2n-i} E \\ \otimes A(-n+i) \end{array} \xrightarrow{d_1} \begin{array}{c} \bigwedge^i E \\ \otimes A \end{array},$$

using the inclusions defined above. This complex is acyclic, and the cokernel M_i is supported in the variety defined by the Pfaffians of size $2n$.

Proof. To check that the above is a complex, it is enough to extend scalars to \mathbf{Q} . In this case, we can use representation theory (namely, Pieri’s formula [Wey, Corollary 2.3.5] and the decomposition of $\text{Sym}(\bigwedge^2)$ into Schur functors [Wey, Proposition 2.3.8]) to see that these maps define a complex.

To prove acyclicity, we use the Buchsbaum–Eisenbud exactness criterion. The formulation of the result that we use, which is a consequence of [Eis, Theorem 20.9], is: Given a finite free resolution \mathbf{F}_\bullet of length n , \mathbf{F}_\bullet is acyclic if and only if the localization $(\mathbf{F}_\bullet)_P$ is acyclic for all primes P with $\text{depth } A_P < n$. Localizing at a prime P with depth at most 2, some variable becomes a unit, so using row and column operations, we can reduce φ to the matrix

$$\hat{\varphi} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \varphi' \end{bmatrix},$$

where φ' is a generic $(2n - 1) \times (2n - 1)$ skew-symmetric matrix. Let C^{i0}, \dots, C^{in-2} be the complexes in Proposition 2.1 defined for the matrix φ' . Then

$$(C^i)_P \cong C^{i0} \oplus 2C^{i-1} \oplus C^{i-2},$$

with the convention that $C^{in-1} = 0$ and $C^{ij} = 0$ for $j < 0$. By induction on the size of φ , we see that each C^i is acyclic. □

2.2. Results and proofs.

Theorem 2.2. *Set $\mathbf{K}_\bullet = \mathbf{K}(Y_1, \dots, Y_{2n+1}; A)_\bullet$.*

(a) *For $0 \leq j \leq n - 1$ we have a filtration $\dots \subset \mathcal{F}_1 H_j \subset \mathcal{F}_0 H_j = H_j(\mathbf{K}_\bullet)$ such that*

$$\mathcal{F}_i H_j / \mathcal{F}_{i+1} H_j \cong M_{j-2i} \otimes (\det E)^j.$$

(b) *For $0 \leq j \leq n - 2$ we have a filtration $0 = \mathcal{F}_0 H_{2n-2-j} \subset \mathcal{F}_1 H_{2n-2-j} \subset \dots$ such that*

$$\mathcal{F}_{i+1} H_{2n-2-j} / \mathcal{F}_i H_{2n-2-j} \cong M_{j-2i} \otimes (\det E)^{2n-2-j}.$$

Proof. We assume that $n > 0$ since the case $n = 0$ is trivial. We will construct a sequence of complexes $\mathbf{F}(r)_\bullet$ for $r = 0, \dots, n - 1$ such that

- (1) $\mathbf{F}(0)_\bullet = \mathbf{K}_\bullet$,
- (2) $\mathbf{F}(r)_\bullet$ is concentrated in degrees $[r, 2n + 1]$,
- (3) the cokernel of $\mathbf{F}(r)_\bullet$ has a filtration as specified by the theorem. Letting $\mathbf{G}(r)_\bullet$ be its minimal free resolution, we have that $\mathbf{F}(r + 1)_\bullet$ is the minimal subcomplex of the mapping cone $\mathbf{F}(r)_\bullet \rightarrow \mathbf{G}(r)_\bullet$.

The existence of this sequence implies the first part of the theorem. For the second part, we appeal to (1.1), which says that $H_{2n-2-i}(\mathbf{K}_\bullet)$ is the A -dual of $H_i(\mathbf{K}_\bullet)$ (note that the M_i are self-dual by the form of their free resolutions). We construct this sequence by induction on r .

For $r = 0$, there is nothing to check, so assume that $r > 0$ and that $\mathbf{F}(r - 1)_\bullet$ has the listed properties. Then $\mathbf{F}(r - 1)$ is the minimal subcomplex of some extension of

$$\mathbf{K}_\bullet \oplus \bigoplus_{i=0}^{r-2} ((\det E)^i \otimes \bigoplus_k C^{i-2k}[-i + 1])$$

which is concentrated in degrees $[r - 1, 2n + 1]$. Since each complex C has length 3, we see that $\mathbf{F}(r - 1)_i = \mathbf{K}_i$ for all $i \geq r + 1$. Recall that $r \leq n - 1$. Then we see that from the structure of the representations in the resolutions of the M_i that after cancellations, we get (there are no cancellations in homological degree $r + 1$)

$$\mathbf{F}(r - 1)_{r-1} = \bigoplus_k \bigwedge^{r-1-2k} E \otimes (\det E)^{r-1} \otimes A,$$

$$\mathbf{F}(r - 1)_r = \left(\bigoplus_k \bigwedge^{2n-r+1+2k} E \otimes (\det E)^{r-1} \otimes A \right) \oplus \begin{cases} 0 & \text{if } r - 1 \text{ is even} \\ (\det E)^r \otimes A & \text{if } r - 1 \text{ is odd,} \end{cases}$$

$$\mathbf{F}(r - 1)_{r+1} = \mathbf{K}_{r+1} = \bigwedge^{2n-r} E \otimes (\det E)^r \otimes A$$

(we ignore the grading since it is determined by the degree of the functor on E). By our induction hypothesis, up to a change of basis, we can write the presentation matrix for $\mathbf{F}(r - 1)$ in “upper-triangular form”; i.e., the map from $\bigwedge^{2n-r+1+2k'} E$ to $\bigwedge^{r-1-2k} E$ is nonzero if and only if $k' \geq k$. Also, when $r - 1$ is odd, the extra term $(\det E)^r \otimes A$ is a redundant relation. Now consider the mapping cone

$$\begin{array}{ccccccc} \mathbf{F}(r - 1)_{r-1} & \longleftarrow & \mathbf{F}(r - 1)_r & \longleftarrow & \mathbf{F}(r - 1)_{r+1} & \longleftarrow & \dots \\ & \swarrow & & \swarrow & & \swarrow & \\ \mathbf{G}(r - 1)_0 & \longleftarrow & \mathbf{G}(r - 1)_1 & \longleftarrow & \mathbf{G}(r - 1)_2 & \longleftarrow & \mathbf{G}(r - 1)_3 \longleftarrow 0. \end{array}$$

The maps $\mathbf{F}(r - 1)_{r-1} \rightarrow \mathbf{G}(r - 1)_0$ and $\mathbf{F}(r - 1)_r \rightarrow \mathbf{G}(r - 1)_1$ are isomorphisms, except when $r - 1$ is odd, in which case the term $(\det E)^r \otimes A$ is in the kernel of the second map. When $r = n - 1$, there is an additional cancellation involving the terms $\bigwedge^n E \otimes (\det E)^n \otimes A$ in $\mathbf{F}(n - 2)_n$ and $\mathbf{G}(n - 2)_2$.

Finally, we can rearrange the resulting presentation matrix into upper-triangular form as follows. Note that all of the maps in the presentation matrix are saturated maps; i.e., their cokernels are free \mathbf{Z} -modules. This can be shown by induction

on r . Let N_r be the cokernel of the presentation matrix. Consider the submodule of N_r generated by $\bigoplus_{k>0} \bigwedge^{r-2k} E \otimes (\det E)^r \otimes A$. The quotient is generated by $\bigwedge^r E \otimes (\det E)^r \otimes A$. By induction, the cokernel of $\mathbf{G}(r-1)_\bullet$ has M_{r-1} as a factor, so this implies that in the diagonal maps, the map $\bigwedge^{2n-r} E \otimes (\det E)^r \otimes A \rightarrow \bigwedge^r E \otimes (\det E)^r \otimes A$ is nonzero. Since all of the maps from the relation module to this term are saturated, they all factor through the relations given by $\bigwedge^{2n-r} E \otimes (\det E)^r \otimes A$ (this follows from the uniqueness of such maps up to sign by Pieri's rule [Wey, Corollary 2.3.5]). Hence $M_r \otimes (\det E)^r$ is a quotient, and continuing in this way, one can show that N_r has the desired filtration. This finishes the induction and the proof. \square

Remark 2.3. Since the M_i have pure resolutions, the above result shows that the Koszul homology of the codimension 3 Pfaffians have a pure filtration in the sense of [EES]. \square

3. HUNEKE–ULRICH IDEALS

We continue to work over the integers \mathbf{Z} .

In this section, we study the Huneke–Ulrich ideals, which are defined as follows. Let Φ be a generic skew-symmetric matrix of size $2n$ and let \mathbf{v} be a generic column vector of size $2n$. The Huneke–Ulrich ideal J is generated by the Pfaffian of Φ along with the entries of $\Phi\mathbf{v}$. It is well known that the ideal J is Gorenstein of codimension $2n-1$ with $2n+1$ minimal generators; i.e., it has deviation 2. Since H_2 is the canonical module, the only interesting Koszul homology group to calculate is H_1 .

The notation is as follows. Let F be a free \mathbf{Z} -module of rank $2n$. We work over the polynomial ring

$$A = \text{Sym}\left(\bigwedge^2 F\right) \otimes \text{Sym}(F^*) = \mathbf{Z}[x_{i,j}, y_i]_{1 \leq i < j \leq 2n},$$

where the variables $x_{i,j}$ are the entries of the generic skew-symmetric matrix Φ and y_i are coordinates of the generic vector \mathbf{v} . Both A and J are naturally bigraded.

The minimal free resolution \mathbf{F}_\bullet of Huneke–Ulrich ideals was calculated by Kustin [Kus]. When $n = 2$, the ideal J is a codimension 3 Gorenstein ideal, and so is covered by the previous section via specialization. We will use $V(-d, -e)$ to denote $V \otimes A(-d, -e)$. For $n \geq 4$, the first three terms of the minimal free resolution are given by

$$\begin{aligned} \mathbf{F}_1 &= F(-1, -1) \oplus (\det F)(-n, 0), \\ \mathbf{F}_2 &= \bigwedge^2 F(-2, -2) \oplus \bigwedge^{2n-1} F(-n, -1) \oplus A(-1, -2), \\ \mathbf{F}_3 &= \bigwedge^3 F(-3, -3) \oplus \bigwedge^{2n-2} F(-n, -2) \oplus F(-2, -3) \oplus (\det F)(-n-1, -2). \end{aligned}$$

When $n = 3$, the same is true except that we omit the term $\bigwedge^3 F(-3, -3)$ from \mathbf{F}_3 .

Now we consider the Koszul complex \mathbf{K}_\bullet on the minimal generating set of J . Since J has deviation 2, there are only 2 nonzero Koszul homology modules. We already know that H_2 is the canonical module of A/J . More precisely, we have $H_2 = (\det F) \otimes A/J(-n-1, -2)$. Let us describe the cycle giving H_2 precisely. Denote

Inside the affine space $X = \text{Spec } A = \bigwedge^2 F^* \oplus F$ the subvariety defined by J is

$$Y = \{(\varphi, v) \in X \mid \text{rank } \varphi \leq 2n - 2, \varphi(v) = 0\}.$$

Let us consider the Grassmannian $\mathbf{Gr}(2, F)$ with the tautological sequence

$$0 \rightarrow \mathcal{R} \rightarrow F \times \mathbf{Gr}(2, F) \rightarrow \mathcal{Q} \rightarrow 0,$$

where $\mathcal{R} = \{(f, W) \mid f \in W\}$. Consider the incidence variety

$$Z = \{(\varphi, v, W) \in X \times \mathbf{Gr}(2, F) \mid v \in W \subset \ker(\varphi)\}.$$

Then $\mathcal{O}_Z = \text{Sym}(\eta)$, where $\eta = \bigwedge^2 \mathcal{Q} \oplus \mathcal{R}^*$. The first projection $q: Z \rightarrow X$ satisfies $q(Z) = Y$.

Theorem 3.2. *The nonzero homology of \mathbf{K}_\bullet is*

$$H_0(\mathbf{K}_\bullet) = H^0(\mathbf{Gr}(2, F); \text{Sym}(\eta)) = A/J,$$

$$H_1(\mathbf{K}_\bullet) = H^0(\mathbf{Gr}(2, F); \mathcal{R} \otimes \text{Sym}(\eta))(-1, -1),$$

$$H_2(\mathbf{K}_\bullet) = H^0(\mathbf{Gr}(2, F); \bigwedge^2 \mathcal{R} \otimes \text{Sym}(\eta))(-2, -2) = \det F \otimes A/J(-n - 1, -2).$$

Proof. First we work over \mathbf{Q} . Using the results in [Wey, Chapter 5], one can check that the presentation matrix for $H^0(\mathbf{Gr}(2, F); \mathcal{R} \otimes \text{Sym}(\eta))$ contains the same representations as the presentation matrix for $H_1(\mathbf{K}_\bullet)$. By equivariance, such maps are unique up to sign, so we conclude that they agree. From [Kus], we know that the coordinate ring of Y and hence its canonical module are torsion-free over \mathbf{Z} . In particular, the descriptions of H_0 and H_2 are independent of characteristic. By a Hilbert function argument, one sees that H_1 is also a torsion-free \mathbf{Z} -module, so our description extends to \mathbf{Z} -coefficients. \square

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