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A QUATERNIONIC CONSTRUCTION OF E_7

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Abstract. We give an explicit construction of the simply-connected compact real form of the Lie group of type E_7, as a group of 28×28 matrices over quaternions, acting on a 28-dimensional left quaternion vector space. This leads to a description of the simply-connected split real form, acting on a 56-dimensional real vector space, and thence to the finite quasi-simple groups of type E_7. The sign problems usually associated with constructing exceptional Lie groups are almost entirely absent from this approach.

1. Introduction

The simple Lie groups over $\mathbb{C}$ are of eight types: three classical (orthogonal, unitary and symplectic) and five exceptional (G_2, F_4, E_6, E_7 and E_8). Over $\mathbb{R}$, each type is divided into a number of 'real forms': for example, the real forms of the orthogonal groups are parametrized by the signature of the underlying quadratic form (up to sign). In every case, there is exactly one compact real form and one split real form, and there may be others in between. In the case of the orthogonal groups, the compact real form has a positive-definite quadratic form, while the split real form has quadratic form with the numbers of positive and negative terms being as nearly equal as possible.

Thus the compact real form of the orthogonal group $\mathrm{O}(n)$ acts on a real n-space preserving the quadratic form $\sum_{r=1}^{n} x_r^2$. Similarly the compact real form of the unitary group $\mathrm{SU}(n)$ acts on a complex n-space preserving the Hermitian form $\sum_{r=1}^{n} x_r\overline{x_r}$. Also, the compact real form of the symplectic group $\mathrm{Sp}(n)$ (sometimes called $\mathrm{Sp}(2n)$) acts on a quaternionic n-space preserving the quaternionic norm $\sum_{r=1}^{n} x_r\overline{x_r}$.

Each real form may further divide into different isomorphism types: there is always an adjoint group (acting on the Lie algebra) and a simply-connected group (which may or may not be the same), and sometimes others in between. In the case of the orthogonal groups, the adjoint group is the projective group $\mathrm{PO}(n)$, given by the action by conjugation on a suitable space of $n \times n$ matrices, and the simply-connected group is the spin group, that is, a double cover of the orthogonal group, acting on the Clifford algebra.

The exceptional groups are generally constructed in the adjoint action of the split real form, as this permits the most uniform approach. However, this has several drawbacks. First, the adjoint representation is not the smallest (except in the case

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of E_8); second, the centre of the group acts trivially; and third, the compact real form often has nicer properties. The following table summarizes the situation:

Type	G_2	F_4	E_6	E_7	E_8
Dimension of Lie algebra	14	52	78	133	248
Dimension of smallest representation	7	26	27	56	248
Order of centre	1	1	3	2	1

There are a number of constructions of the minimal representations in the literature, sometimes over $\mathbb{C}$ or $\mathbb{R}$, sometimes over finite fields, with or without a restriction on the characteristic. For example, in 1901 L. E. Dickson constructed groups of type E_6 over arbitrary fields by defining an invariant cubic form with 45 terms in 27 variables [5, 6]. He also constructed G_2 as the automorphism group of the Cayley numbers (octonions), first in characteristic not 2, and then in general [4]. The interpretation of F_4 as the automorphism group of the 27-dimensional exceptional Jordan algebra came later (see for example [7] or [8] for an exposition). This algebra consists of 3×3 Hermitian matrices over octonions, with product $\frac{1}{2}(AB + BA)$, and leads to an interpretation of Dickson's cubic form as the determinant of such matrices (in the case when the split form of the octonions is used).

The 56-dimensional representation of E_7 in characteristic not 2 was constructed by Brown [2], based on foundations laid by Freudenthal, and further studied by Aschbacher [1] and by Cooperstein [3], who were principally interested in the finite case. Thus they did not make use of the fact that the representation is symplectic, that is, writable over quaternions in half the number of dimensions. In this paper we significantly simplify the treatment of E_7 given in [1–3] by exploiting the quaternionic structure to the full. We first construct the compact real form, and only later convert to the split real form in order to reduce modulo p. This approach brings out the rather striking fact that this makes (the simply-connected compact real forms of) F_4, E_6, and E_7 respectively 26-dimensional real, 27-dimensional complex, and 28-dimensional quaternionic. In particular, the exceptional Jordan algebra is not the end of the line, but is only part of a much richer 28-dimensional structure.

In Section 2 we define a certain group G to be the group generated by certain explicit 28×28 matrices over quaternions. In Section 3 we prove that G is the simply-connected compact real form of E_7. In Section 4 we discuss the invariant quadrilinear form, and in Section 5 we describe some subgroups. We consider the split real form and the finite groups of type E_7 in Section 6, and conclude with Section 7, in which we make some remarks on the underlying 7-dimensional structure over the tensor product of two quaternion algebras.

2. The action of the root groups

First we describe the labelling of the 28 quaternionic coordinates in terms of the 28 pairs of opposite minimal vectors in the dual E_7 lattice E_7^*. We label the 7 coordinates of $\mathbb{R}^7$ by the elements $0, 1, \ldots, 6$ of the field $\mathbb{F}_7$, and may then take the 126 roots of the E_7 lattice to be the images under sign-changes and cyclic permutations of the coordinates of the following vectors:

- 14 images of $(2, 0, 0, 0, 0, 0, 0)$;
- 112 images of $(1, 0, 0, 1, 0, 1, 1)$.

Then the 56 minimal vectors of E_7^* (multiplied by 2 for convenience) are the images under sign-changes and rotations of $(0,1,1,0,1,0,0)$.

The Coxeter group of type E_7 is generated by the 63 reflections in the roots. For example, reflection in $\pm(2,0,0,0,0,0,0)$ negates coordinate 0, while reflection in $\pm(1,0,0,1,0,1,1)$ fixes coordinates $1,2,4$ and acts as the matrix

$$-\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

on coordinates $0,3,5,6$. The monomial subgroup of the Coxeter group is of shape $2^7{:}\mathrm{PSL}_3(2)$ and is generated by the sign-changes together with the elements which permute the coordinates by $t \mapsto t+1$, and $t \mapsto 2t$ and $(1,2)(3,6)$.

Now label the 28 pairs of minimal vectors in E_7^* by

$$\begin{aligned} h_0 &= \{\pm(0,1,1,0,1,0,0)\}, \\ i_0 &= \{\pm(0,1,-1,0,-1,0,0)\}, \\ j_0 &= \{\pm(0,-1,1,0,-1,0,0)\}, \\ k_0 &= \{\pm(0,-1,-1,0,1,0,0)\}, \end{aligned}$$

where adding 1 to the subscript (modulo 7) corresponds to rotating the coordinates backwards. (Later on, we will refine this notation by using these labels x_t for the vectors with the plus sign and jx_t for the vectors with the minus sign.) We use the same labels for the 28 coordinate vectors in the left vector space $\mathbb{H}^{28}$ and use corresponding capital letters for the coordinates of a typical vector; thus

$$(H_0, I_0, J_0, K_0, H_1, \dots, J_6, K_6) = H_0 h_0 + I_0 i_0 + \cdots + K_6 k_6 \in \mathbb{H}^{28}.$$

We are now ready to describe the action of 63 copies of $\mathrm{SU}(2)$ on this space, one for each of the 63 pairs of opposite roots. First we take the roots $\pm(2,0,0,0,0,0,0)$. This copy of $\mathrm{SU}(2)$ fixes all the coordinates with a subscript $0,3,5,6$. Now for any element $q \in \mathrm{SU}(2) \subset \mathbb{H}$, that is, $q = z + wj$ with $z, w \in \mathbb{C} = \mathbb{R}[i]$ and $q\overline{q} = z\overline{z} + w\overline{w} = 1$, we define an action of q as right-multiplication by

$$\begin{pmatrix} z & wj \\ wj & z \end{pmatrix}$$

on each of the quaternionic 2-vectors (H_1, I_1), (H_2, J_2), and (H_4, K_4), and as

$$\begin{pmatrix} \overline{z} & \overline{w}j \\ \overline{w}j & \overline{z} \end{pmatrix}$$

on each of (J_1, K_1), (K_2, I_2) and (I_4, J_4). To prove that this indeed defines an action, it suffices to check that the matrix product

$$\begin{pmatrix} u & vj \\ vj & u \end{pmatrix}\begin{pmatrix} z & wj \\ wj & z \end{pmatrix} = \begin{pmatrix} uz - v\overline{w} & (uw + v\overline{z})j \\ (uw + v\overline{z})j & uz - v\overline{w} \end{pmatrix}$$

corresponds to the quaternion product

$$(u + vj).(z + wj) = (uz - v\overline{w}) + (uw + v\overline{z})j,$$

since the other action is obtained from this by conjugation by j.

We may apply a cyclic permutation of the 7 subscripts $t \in \mathbb{F}_7$ (i.e. map $t \mapsto t+1$) to get a total of seven such fundamental $\mathrm{SU}(2)$s. It is easy to check that these $\mathrm{SU}(2)$s commute with each other, since the matrices

$$\begin{pmatrix} u & vj & & \\ vj & u & & \\ & & \overline{u} & \overline{v}j \\ & & \overline{v}j & \overline{u} \end{pmatrix}, \begin{pmatrix} z & & wj & \\ & \overline{z} & & \overline{w}j \\ wj & & z & \\ & \overline{w}j & & \overline{z} \end{pmatrix}$$

commute. Notice that there is also a symmetry of order 3 permuting these seven copies of SU(2), acting as $t \mapsto 2t$ on the suffices, and as (I, J, K) on the letters. This symmetry will be called 'triality', for reasons which will become clear later.

Next consider the element R_0 corresponding to $q = j$ in the SU(2) just constructed. It maps

$$\begin{array}{rcl} (H_1, I_1, J_1, K_1) & \mapsto & (I_1 j, H_1 j, K_1 j, J_1 j), \\ (H_2, I_2, J_2, K_2) & \mapsto & (J_2 j, K_2 j, H_2 j, I_2 j), \\ (H_4, I_4, J_4, K_4) & \mapsto & (K_4 j, J_4 j, I_4 j, H_4 j). \end{array}$$

The corresponding action in the Coxeter group is to negate coordinate 0 in the vectors $(1,1,0,1,0,0,0)$, $(1,0,1,0,0,0,1)$ and $(1,0,0,0,1,1,0)$. Hence by using these sign-changes and rotations of the seven coordinates we have transitivity on the remaining 56 pairs of roots. It is enough therefore to specify the action of one more copy of SU(2), for example the one corresponding to the root pair $\pm(1,0,0,1,0,1,1)$, which acts as right-multiplication by

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

on each of the quaternionic 2-vectors (H_1, K_3), (H_2, I_6), and (H_4, J_5), and as

$$\begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}$$

on each of (H_3, J_1), (H_6, K_2) and (H_5, I_4). Notice that this copy of SU(2) is centralized by the triality element.

Now if $R_1, \ldots, R_6$ denote the images of R_0 by repeatedly subtracting 1 from the subscripts of H_t, I_t, J_t, K_t, modulo 7, then we can easily check that this copy of SU(2) is centralized by R_1, R_2, and R_4, and normalized by $R_0 R_6 R_5 R_3$. Hence there are exactly 56 images under conjugation by the R_t and the rotation.

(The calculations are as follows. First, R_1 maps (H_1, K_3) to $(J_1 j, H_3 j)$ and (H_3, J_1) to $(K_3 j, H_1 j)$ while centralizing all the other 2-spaces on which the SU(2) acts. Since

$$\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix} = \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix},$$

we see that R_1 centralizes this SU(2). The symmetry $(1,2,4)(3,6,5)(I,J,K)$ shows the same is true for R_2 and R_4. Similarly, $R_0 R_6 R_5 R_3$ maps the pairs

$$(H_1, K_3), (H_2, I_6), (H_4, J_5)$$

to the negatives of

$$(J_1, H_3), (K_2, H_6), (I_4, H_5)$$

and vice versa, and we calculate

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{z} & -\overline{w} \\ w & z \end{pmatrix}.$$

Hence $R_0 R_6 R_5 R_3$ normalizes the SU(2) by mapping $z + wj \mapsto \overline{z} + \overline{w}j$.)

We have thus defined explicitly the actions of 63 root groups SU(2) on $\mathbb{H}^{28}$. Let G be the group generated by these 63 copies of SU(2). It remains to prove that G is a Lie group of type E_7, rather than the whole of the symplectic group Sp(28), or something else entirely. This will be proved in the next section.

3. The torus and the Weyl group

Each of the 63 root SU(2) groups has a torus (given by $q = z \in \mathbb{C}$) and a Weyl group (whose non-identity element is given by $q = j$ modulo the torus), and we can put all these together to get the torus and Weyl group for G. The two types of tori are exemplified by

- multiplying by z on $H_1, H_2, H_4, I_1, J_2, K_4$ and by $\overline{z}$ on $J_1, K_2, I_4, K_1, I_2, J_4$;
- multiplying by z on $H_1, H_2, H_4, H_3, H_6, H_5$ and by $\overline{z}$ on $K_3, I_6, J_5, J_1, K_2, I_4$.

It is straightforward to check that these 63 tori together generate a 7-dimensional torus. Indeed, the seven tori of the first type are independent (since they lie in seven commuting SU(2) subgroups), and each of the other elements listed is the square root of a suitable product of four of the first type. But $\mathbb{C}$ is algebraically closed, so all these elements are contained in the 7-dimensional torus.

The two types of reflection act as follows on the quaternionic coordinates, where the symbol $[X, Y]$ denotes the map $X \mapsto Y \mapsto -X \mapsto -Y \mapsto X$:

- $[H_1, I_1 j][H_2, J_2 j][H_4, K_4 j][J_1, K_1 j][K_2, I_2 j][I_4, J_4 j]$;
- $[K_3, H_1][I_6, H_2][J_5, H_4][H_3, J_1][H_6, K_2][H_5, I_4]$.

(In fact, these maps are given by $q = j$ in the first case and $q = -j$ in the second.) Now the permutation action, on the quaternionic coordinates, of the Weyl group in each SU(2) is clearly the same as the action on the 28 pairs of minimal vectors in E_7^* of the corresponding reflection in the Coxeter group. Hence we see that the Weyl group is indeed isomorphic to the Coxeter group of type E_7, that is, to $2 \times \mathrm{Sp}_6(2)$. The central involution of the Weyl group is represented by the element which multiplies every coordinate by j.

It turns out that the part of the Weyl group which preserves the decomposition of the 28-dimensional quaternionic space as a sum of seven 4-spaces is a subgroup of shape $2^7\mathrm{PSL}_3(2) = 2.2^3.2^3.\mathrm{PSL}_3(2)$. This group is generated (modulo the torus) by the following elements, and since it is a maximal subgroup of the Weyl group, it is the stabilizer of the decomposition, as claimed:

(1) right-multiplication by j on all coordinates;
(2) negating coordinates labelled 0, and permuting coordinates as $(H, I)(J, K)$ when the label is 4 or 6, as $(H, J)(K, I)$ when the label is 1 or 5, and as $(H, K)(I, J)$ when the label is 2 or 3;
(3) acting as $j(H, I)(J, K)$ on coordinates labelled 1, as $j(H, J)(K, I)$ when the label is 2, and as $j(H, K)(I, J)$ when the label is 4; thus $H_1 \mapsto I_1 j \mapsto -H_1$ and so on;
(4) cyclically permuting the labels by $t \mapsto t + 1$ (where $t \in \mathbb{F}_7$);
(5) permuting the labels by $t \mapsto 2t$ together with (I, J, K);
(6) $(H_0, -H_0)(I_0, -J_0)(K_0, -K_0)(I_5, K_5)$
$(H_1, H_2)(I_1, J_2)(J_1, K_2)(K_1, I_2)$
$(H_3, H_6)(I_3, K_6)(J_3, J_6)(K_3, I_6)$.

As some of these elements (particularly (3) and (6)) are slightly awkward to apply in practice, for example in getting the correct power of j in every coordinate, we shall often restrict ourselves to a smaller symmetry group $2 \times 2^3{:}7{:}3$, generated by the elements (1), (2), (4), and (5).

Our construction shows that this Weyl group is generated by any one reflection of each type together with the element (4) of order 7. We proved in the previous section that the first reflection, that is, the element (3), preserves the set of 63 root SU(2)s.

Next we show that the second reflection also preserves the set of 63 root groups. We already showed that this reflection commutes with R_1, R_2, R_4 and $R_0R_6R_5R_3$, as well as the triality element (5). Now it is easy to see that this centralizing group of order 48 has orbits of sizes

$$(1+3+3)+(1+1+3+3+12+12+12+12)$$

on the $7+56$ root groups. Hence it is enough to check the action of our reflection on one group in each orbit, which is now an easy calculation. For example, this reflection maps the coordinates

$$(H_2, I_2; H_3, J_3; H_5, K_5), (J_2, K_2; K_3, I_3; I_5, J_5)$$

respectively to

$$(-I_6, I_2; J_1, J_3; I_4, K_5), (J_2, -H_6; H_1, I_3; I_5, H_4),$$

which goes under a suitable rotation of element (3) to

$$(-I_6, H_2j; J_1, H_3j; I_4, H_5j), (K_2j, -H_6; H_1, K_3j; J_5j, H_4).$$

The action of the root group on the various 2-spaces can be computed by conjugation of matrices: for example the action on (H_1, K_3) is given by

$$\begin{pmatrix}1 & 0\\ 0 & j\end{pmatrix}\begin{pmatrix}\overline{z} & \overline{w}j\\ \overline{w}j & \overline{z}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -j\end{pmatrix}=\begin{pmatrix}\overline{z} & \overline{w}\\ -w & z\end{pmatrix}.$$

Similar conjugations on the other five 2-spaces, and rewriting the quaternion $z+wj$ as $\overline{z'}+\overline{w'}j$, give us the second copy of SU(2) defined above. (The rest of this calculation is left as an exercise for the reader. The reflection swaps two of the orbits of size 1, two of the orbits of size 3, and two of the orbits of size 12, while centralizing the rest, so there are seven cases to check, of which we have sketched one.)

Since the given reflection does not preserve the orbits of 7 and 56 pairs of roots, the Weyl group acts transitively on these 63 pairs. It follows from this and the omitted calculations that G is generated by a single SU(2) together with the Weyl group. In particular, it is of type E_7. Clearly it must be the simply-connected compact real form.

4. The quadrilinear form

The cited references [1–3] all define E_7 (in the relevant context: usually over an arbitrary field of characteristic not 2) as the stabilizer of a pair of forms, one bilinear, the other quadrilinear in the 56 complex coordinates. Although we do not need this as part of our definition, it is useful for further investigations to have the (totally symmetric) quadrilinear form explicitly. We have already taken care of the bilinear form by ensuring that our representation commutes with left-multiplication by j, so is symplectic. In fact, the quadrilinear form is invariant not only under the compact real form, but under the whole of the complexification $E_7(\mathbb{C})$.

To describe this complexification, we take as our basis for the complex 56-space the original quaternionic basis $\{h_t, i_t, j_t, k_t\}$ together with the multiples by j, that is, $\{jh_t, ji_t, jj_t, jk_t\}$. The corresponding complex coordinates of a vector will be written X'_t, X''_t, where $X_t = X'_t + X''_t j$. The symbol j now loses its quaternionic meaning and just acts as a formal symbol permuting coordinates (up to sign), though still with the understanding that $j^2 = -1$. With this interpretation, the Weyl group permutes the complex coordinates, up to sign, so it suffices to deal with one root group. We re-compute the action of $\begin{pmatrix} z & wj \\ wj & z \end{pmatrix}$ on (H_1, I_1), etc., to be $\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$ on both (H'_1, I''_1) and (I'_1, H''_1). Similarly, the action of $\begin{pmatrix} \overline{z} & \overline{w}j \\ \overline{w}j & \overline{z} \end{pmatrix}$ gives the transpose-inverse matrix $\begin{pmatrix} \overline{z} & \overline{w} \\ -w & z \end{pmatrix}$ acting on (J'_1, K''_1), etc. Next we extend from SU(2) to SL(2, $\mathbb{C}$), by allowing all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1 in this place (and, of course, the transpose-inverse $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ in the appropriate places). Note that conjugation by j can now be interpreted as the transpose-inverse map.

It turns out that we may then define the quadrilinear form by defining it on three quadruples of complex basis vectors:

$$\begin{array}{rcl} [h_0, h_0, jh_0, jh_0] & = & -1, \\ [h_0, jh_0, i_0, ji_0] & = & 1/2, \\ [h_0, i_0, j_0, k_0] & = & 1, \end{array}$$

and images under the action of the Weyl group, and to be zero on all other quadruples. Equivalently, the quartic form is defined as the sum of monomials of the form

$$-\frac{1}{4}(H'_0 H''_0)^2, \frac{1}{2} H'_0 H''_0 I'_0 I''_0, H'_0 I'_0 J'_0 K'_0.$$

(The factor of 1/4 comes from the fact that the expression $[h_0, h_0, jh_0, jh_0]$ is invariant under 4 permutations of the terms h_0, h_0, jh_0, jh_0.)

Since the Weyl group can be taken to permute our 56 coordinates, up to sign, it permutes exactly 28 monomials which are images of the first of those listed, since any sign-changes cancel out. Similarly, the second gives rise to 378 terms, one for each pair of the 28 coordinates. In order for our quartic form to be well-defined, we must show that $H'_0 H''_0 I'_0 I''_0$ is not negated by any element in the part of the Weyl group which fixes the pair $\{h_0, i_0\}$ of quaternionic coordinates. This is a straightforward calculation. Indeed, the relevant subgroup of the Weyl group is $2 \times 2^5 S_5$. This subgroup has a centre of order 4, in which the central involution of the whole Weyl group acts as right-multiplication by j, and the central reflection swaps h_0 with i_0. The outer half of S_5 may be taken to act trivially.

The third type of monomial is more interesting. The 630 monomials of this type fall into three orbits, of sizes $14 + 168 + 448$, under the action of the subgroup $2^7{:}7{:}3$ generated by the elements (1)–(5) described above. These are represented respectively by $H'_0 I'_0 J'_0 K'_0$, $H'_0 I'_0 I'_1 K'_1$ and $H'_0 I'_3 J'_6 K'_5$. For convenience we also give the orbits under $2 \times 2^3{:}7{:}3$:

- 14 images of $H'_0 I'_0 J'_0 K'_0$;
- 84 images each of $H'_0 I'_0 I'_1 K'_1$ and $H'_0 I'_0 H''_1 J''_1$;

- 112 images of $H'_0 I'_3 J'_6 K'_5$;
- 336 images of $H'_0 I'_3 H''_6 J''_5$.

(Notice that although the coefficients of all the displayed monomials are +1, nevertheless our symmetry group has minus signs, which introduces many minus signs into the quartic form. There are 105 terms with all single dashes, 105 terms with all double dashes, and 420 terms with two of each.)

In this third case, the subgroup of the Weyl group which fixes the relevant set $\{h_0, i_0, j_0, k_o\}$ of quaternion coordinates has shape $2 \times [2^5]S_3S_4$. Its action on the 4-dimensional quaternionic space $\langle h_0, i_0, j_0, k_0\rangle$ is $2 \times S_4$, generated by right-multiplication by j together with all coordinate permutations. This completes the proof that the quartic form (and the corresponding quadrilinear form) is well-defined and (therefore, by definition) invariant under the action of the Weyl group.

Now to prove that this form is invariant under the complexification of G, we must show it is invariant under an arbitrary element of our fundamental $\mathrm{SL}(2,\mathbb{C})$, corresponding to a pair of opposite roots, say $\pm(2,0,0,0,0,0)$. First we compute the orbits of the root stabilizer in the Weyl group, on the quadruples used in the definition of the form. There are just two orbits on the 28 images of the first quadruple, of lengths 16 and 12, corresponding to the quaternionic 1-spaces spanned by h_0 and h_1 respectively. The first case is obviously fixed, while the second gives

$$\begin{array}{rcl} [h_1,h_1,jh_1,jh_1] & \mapsto & [ah_1+bji_1, ah_1+bji_1, djh_1+ci_1, djh_1+ci_1] \\ & = & (ad)^2[h_1,h_1,jh_1,jh_1]+(bc)^2[ji_1,ji_1,i_1,i_1] \\ & & \quad +abcd([h_1,ji_1,jh_1,i_1]+[h_1,ji_1,i_1,jh_1] \\ & & \qquad +[ji_1,h_1,jh_1,i_1]+[ji_1,h_1,i_1,jh_1]) \\ & = & -(ad-bc)^2 = -1. \end{array}$$

On the 378 images of the second quadruple, there are four orbits, of lengths 6, 60, 120, and 192, represented respectively by

$$[h_1,jh_1,i_1,ji_1],[h_1,jh_1,j_1,jj_1],[h_0,jh_0,i_0,ji_0],[h_0,jh_0,h_1,jh_1].$$

Of these, the third is obviously fixed by the $\mathrm{SL}(2,\mathbb{C})$, and the other three are straightforward calculations. For example, the last case is

$$\begin{array}{rcl} [h_0,jh_0,h_1,jh_1] & \mapsto & [h_0,jh_0,ah_1+bji_1,djh_1+ci_1] \\ & = & ad[h_0,jh_0,h_1,jh_1]+bc[h_0,jh_0,ji_1,i_1] \\ & = & (ad-bc)[h_0,jh_0,h_1,jh_1] \end{array}$$

by applying a suitable Weyl group element such as $H_1 \mapsto jI_1 \mapsto -H_1, I_1 \mapsto jH_1 \mapsto -I_1$ to the second term. Similarly,

$$\begin{array}{rcl} [h_1,jh_1,i_1,ji_1] & \mapsto & [ah_1+bji_1, djh_1+ci_1, ai_1+bjh_1, dji_1+ch_1] \\ & = & (ad)^2[h_1,jh_1,i_1,ji_1]+(bc)^2[ji_1,i_1,jh_1,h_1] \\ & & \quad +abcd([h_1,i_1,jh_1,ji_1]+[ji_1,jh_1,i_1,h_1] \\ & & \qquad +[h_1,jh_1,jh_1,h_1]+[ji_1,i_1,i_1,ji_1]) \\ & = & \dfrac{1}{2}(ad-bc)^2 = \dfrac{1}{2}, \end{array}$$

$$\begin{array}{rcl} [h_1,jh_1,j_1,jj_1] & \mapsto & [ah_1+bji_1, djh_1+ci_1, dj_1-cjk_1, ajj_1-bk_1] \\ & = & (ad)^2[h_1,jh_1,j_1,jj_1]+(bc)^2[ji_1,i_1,jk_1,k_1] \\ & & \quad +abcd([h_1,jh_1,jk_1,k_1]-[h_1,i_1,j_1,k_1] \\ & & \qquad +[ji_1,i_1,j_1,jj_1]-[ji_1,jh_1,jk_1,jj_1]) \\ & = & (ad-bc)^2/2. \end{array}$$

On the $2 \times 315 = 630$ images of the third quadruple, there are just three orbits, of lengths 30, 120, and 480. These are represented respectively by

$$[h_1, i_1, j_1, k_1], [h_0, i_0, j_0, k_0], [h_0, i_0, i_1, k_1].$$

Again, the second is obviously fixed, and the other two are straightforward calculations:

$$\begin{array}{rcl}
[h_1, i_1, j_1, k_1] & \mapsto & [ah_1 + bji_1, ai_1 + bjh_1, dj_1 - cjk_1, dk_1 - cjj_1] \\
& = & (ad)^2[h_1, i_1, j_1, k_1] + (bc)^2[ji_1, jh_1, jk_1, jj_1] \\
& & -\, abcd([h_1, jh_1, j_1, jj_1] + [h_1, jh_1, jk_1, k_1] \\
& & \quad + [ji_1, i_1, j_1, jj_1] + [ji_1, i_1, jk_1, k_1]) \\
& = & (ad)^2 - 2abcd + (bc)^2 = 1,
\end{array}$$

$$\begin{array}{rcl}
[h_0, i_0, i_1, k_1] & \mapsto & [h_0, i_0, ai_1 + bjh_1, dk_1 - cjj_1] \\
& = & ad[h_0, i_0, i_1, k_1] - bc[h_0, i_0, jh_1, jj_1] \\
& = & (ad - bc)[h_0, i_0, i_1, k_1] \\
& = & [h_0, i_0, i_1, k_1].
\end{array}$$

In fact, in order to prove that the fundamental $\mathrm{SL}(2, \mathbb{C})$ preserves the quadrilinear form, we must also check that it preserves all the zero values. Now the only way one of these zero values could not be preserved is if one of the non-zero values we have just studied gets added to it. This means that the only zero values we need to consider are those which were actually used in the above calculations. These are as follows, up to symmetry under the root stabilizer in the Weyl group:

$$\begin{array}{c}
[h_1, h_1, jh_1, i_1], [h_0, jh_0, h_1, i_1], [h_1, h_1, i_1, i_1], \\
[h_1, i_1, jj_1, jk_1], [h_1, i_1, j_1, jj_1], [h_0, i_0, i_1, jj_1].
\end{array}$$

Thus we have a little more calculation to do of the same kind we've already done; this is left as an exercise for the reader.

Having completed these calculations, we have proved from first principles that the group G preserves the quadrilinear form defined in this section.

5. Some subgroups

If we take a subroot system of E_7, then the subgroup generated by the corresponding root $\mathrm{SU}(2)$ subgroups is often of interest. For example, the subsystem spanned by all the roots of the shape $\pm(2, 0^6)$ gives rise to a central product of seven copies of $\mathrm{SU}(2)$, in which the centre is reduced from 2^7 to 2^4. This subgroup stabilizes the seven 4-dimensional spaces $\langle h_t, i_t, j_t, k_t \rangle$, for $t \in \mathbb{F}_7$, and the Weyl group induces a transitive permutation action of $\mathrm{PSL}_3(2)$ on these seven subspaces, as well as (a different action) on the seven $\mathrm{SU}(2)$ factors of the group.

This subsystem extends to a system of type $A_1A_1A_1D_4$ by adjoining all the roots $(\pm1, 0, 0, \pm1, 0, \pm1, \pm1)$ with 0s on coordinates $1, 2, 4$. This subsystem group fixes the quaternionic 4-space $\langle h_0, i_0, j_0, k_0 \rangle$ and is normalized by a triality element which acts as

$$(i, j, k)(1, 2, 4)(3, 6, 5).$$

We may extend further to A_1D_6, consisting of all the roots equal or perpendicular to $\pm(0, 2, 0, 0, 0, 0, 0)$. For convenience, let us take instead the copy defined by $\pm(2, 0, 0, 0, 0, 0, 0)$. The corresponding subsystem group splits the space into the part with suffices $1, 2, 4$, on which the A_1 and D_6 both act naturally, that is, as

SU(2) $\otimes$ O(12); and the part with suffices $0,3,5,6$, on which the A_1 acts trivially and the D_6 acts in its spin representation.

Another maximal rank subsystem which is of interest is A_7, which corresponds to the maximal rank subgroup SL(8) used by Cooperstein [3] in his construction of E_7. We may take the roots of this A_7 subsystem to consist of the images of $(1,0,0,1,0,1,1)$ under rotations and evenly many sign-changes. Indeed, the Coxeter group of type A_7 is the symmetric group S_8, which is related to the traditional labelling of the 28 objects permuted by the Coxeter group of type E_7 by the unordered pairs from 8 points. We label the 8 points by the projective line $\mathbb{F}_7 \cup \{\infty\}$ and may then label

$$h_0 = \{\infty, 0\}, i_0 = \{1,3\}, j_0 = \{2,6\}, k_0 = \{4,5\}$$

and let the map $t \mapsto t+1$ on subscripts also act as $t \mapsto t+1$ on the projective line. The subgroup S_8 of the Coxeter group acts naturally by permuting the set $\{\infty,0,1,2,3,4,5,6\}$. It is now straightforward to identify our quadrilinear form with the version given by Cooperstein [3]. Fixing $\infty, 0$ gives a subgroup S_6 which has orbits of sizes $6+6+15$ on the 27 remaining objects. From this we now show how to derive Dickson's cubic form for E_6.

The subsystem E_6 corresponding to $h_0 = \{\infty, 0\}$ consists of all the roots perpendicular to $\pm(0,1,1,0,1,0,0)$. Thus it is clear that the stabilizer of the quaternionic 1-space $\langle h_0 \rangle$ corresponding to this pair of vectors is a copy of (the simply-connected compact real form of) E_6, extended by a 1-dimensional torus, and by a duality map induced by the central involution in the Weyl group. To see this in a 'classical' way, consider all the terms in the quadrilinear form which involve h_0 only once, and remove this factor h_0 from them. We obtain a symmetric trilinear form in 27 variables, which is the polarized form of Dickson's cubic form for E_6. An explicit correspondence between our coordinates and Dickson's is given by the following table, where each quaternionic coordinate q is split into its complex and imaginary parts as $q = q' + q''j$ with $q', q'' \in \mathbb{C}$. The entries in the body of the table are $z_{rs} = -z_{sr}$.

	r	1	2	3	4	5	6
s	$y_s \backslash x_r$	H_1''	H_2''	H_3''	H_4''	H_5''	H_6''
1	K_3''		K_4'	I_0'	$-J_2'$	J_6'	$-I_5'$
2	I_6''	$-K_4'$		K_5'	I_1'	$-J_3'$	J_0'
3	$-J_1''$	$-I_0'$	$-K_5'$		K_6'	I_2'	$-J_4'$
4	J_5''	J_2'	$-I_1'$	$-K_6'$		K_0'	I_3'
5	$-I_4''$	$-J_6'$	J_3'	$-I_2'$	$-K_0'$		K_1'
6	$-K_2''$	I_5'	$-J_0'$	J_4'	$-I_3'$	$-K_1'$	

The 45 terms of Dickson's cubic form are $x_r y_s z_{rs}$ and $z_{rs} z_{tu} z_{vw}$, where $rstuvw$ is an even permutation of 123456. In our notation, the latter are $I_0' J_0' K_0'$, $I_3' J_6' K_5'$, $-I_5' J_3' K_6'$ and the images under the group of order 3 generated by the triality element $(1,2,4)(3,6,5)(I,J,K)$ of

$$I_0' I_1' K_1', I_0' I_3' J_3', J_2' K_1' K_5', -I_1' I_2' I_5'.$$

The former are the images under triality of the following 10 terms:

$$I_0' H_1'' J_1'', I_0' H_3'' K_3'', I_3' H_6'' J_5'', -I_5' H_6'' K_3'',$$
$$-I_1' H_2'' J_5'', -I_2' H_5'' J_1'', -I_5' H_1'' K_2'', K_1' H_5'' K_2'', J_2' H_1'' J_5'', K_5' H_2'' J_1''.$$

Finally we remark that E_6 has a subgroup of type F_4, which in our representation fixes a quaternionic 2-space, such as $\langle h_0, i_0 + j_0 + k_0\rangle$.

6. The split real form and finite groups of type E_7

To construct the split real form we take the complexification $E_7(\mathbb{C})$ as defined in Section 4 and simply restrict the matrix entries to lie in $\mathbb{R}$, so that the root groups become $\mathrm{SL}(2,\mathbb{R})$. This now defines an action of the Weyl group and a root $\mathrm{SL}(2,\mathbb{R})$ on a 56-dimensional real vector space, where the typical vector is written as

$$(H_0', H_0'', I_0', \ldots, K_6', K_6'').$$

Hence we have generators for the (simply-connected) split real form of E_7. To be precise, the action of generators for our standard root $\mathrm{SL}(2,\mathbb{R})$ is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

acting on the real 2-vectors

$$(H_1', I_1''), (I_1', H_1''), (H_2', J_2''), (J_2', H_2''), (H_4', K_4''), (K_4', H_4''),$$

and as

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

acting on the real 2-vectors

$$(J_1', K_1''), (K_1', J_1''), (K_2', I_2''), (I_2', K_2''), (I_4', J_4''), (J_4', I_4'').$$

The same matrices can be interpreted over any field F whatsoever, and give generators for the (simply-connected) split form of $E_7(F)$. (Over finite fields all forms of E_7 are split.) Of course, this identification of the groups relies on the theory of algebraic groups, and it would be nice to have a self-contained derivation of the basic properties in the finite case, such as the group order, and simplicity. In the remainder of this section we outline how such a proof might be obtained.

First we define the bilinear and quadrilinear forms in finite characteristic. The symplectic form is defined by $(x_t, jx_t) = -(jx_t, x_t) = 1$, for each $x \in \{h, i, j, k\}$ and $t \in \mathbb{F}_7$, and $(x, y) = 0$ for all other pairs of basis vectors x, y. The quadrilinear form needs no change to its definition in any odd characteristic. In characteristic 2, as usual, the situation is more complicated, and we refer to Aschbacher [1] for a full treatment. Essentially, Aschbacher's method is to replace the symmetric quadrilinear form by a form which is invariant only under even permutations of the four variables. This allows us to replace the definition $[h_0, jh_0, i_0, ji_0] = 1/2$ by two definitions, $[h_0, jh_0, i_0, ji_0] = 1$ and $[h_0, i_0, jh_0, ji_0] = 0$, while defining $[h_0, i_0, k_0, j_0] = 1$ so that the other terms remain the same. It is then possible to modify the calculations to show that this version of the form is invariant in arbitrary characteristic. For example the proof given above of the invariance of $[h_1, jh_1, i_1, ji_1]$ must now be replaced by two similar arguments, as follows:

$$\begin{aligned}
[h_1, jh_1, i_1, ji_1] &\mapsto [ah_1 + bji_1, djh_1 + ci_1, ai_1 + bjh_1, dji_1 + ch_1] \\
&= (ad)^2[h_1, jh_1, i_1, ji_1] + (bc)^2[ji_1, i_1, jh_1, h_1] \\
&\quad + abcd([h_1, i_1, jh_1, ji_1] + [ji_1, jh_1, i_1, h_1] \\
&\qquad + [h_1, jh_1, jh_1, h_1] + [ji_1, i_1, i_1, ji_1]) \\
&= (ad)^2 + (bc)^2 + abcd(0 + 0 - 1 - 1) \\
&= (ad - bc)^2 = 1,
\end{aligned}$$

$$\begin{array}{rcl}[h_1,i_1,jh_1,ji_1] & \mapsto & [ah_1+bji_1,ai_1+bjh_1,djh_1+ci_1,dji_1+ch_1]\\ & = & (ad)^2[h_1,i_1,jh_1,ji_1]+(bc)^2[ji_1,jh_1,i_1,h_1]\\ & & \quad +abcd([h_1,jh_1,i_1,ji_1]+[ji_1,,i_1,jh_1,h_1]\\ & & \qquad +[h_1,jh_1,jh_1,h_1]+[ji_1,i_1,i_1,ji_1])\\ & = & 0+0+abcd(1+1-1-1)=0.\end{array}$$

Next we determine the order of the stabilizer of $\langle h_0\rangle$. The same argument as in the complex case shows that, at least if the characteristic is odd, the centralizer of the 2-space $\langle h_0, jh_0\rangle$ is (the simply-connected version of) $E_6(q)$. The whole stabilizer of the 1-space $\langle h_0\rangle$, therefore, is generated by this $E_6(q)$, together with one more dimension of torus, acting as $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ on (h_0, jh_0), and 27 root groups which add multiples of 27 coordinates (these are exactly the coordinates corresponding to the 27 variables of Dickson's cubic form, as specified above) onto jh_0. (Modifications to this argument are needed if the characteristic is 2, but from now on the argument is essentially independent of the characteristic.)

Thus, now writing $E_6(q)$ for the simple group, the stabilizer of $\langle h_0\rangle$ (a 1-space over $\mathbb{F}_q$) is a group of shape $q^{27}{:}(C_{q-1}\times E_6(q))$, in the case when $q\equiv 2 \bmod 3$, or $q^{27}{:}3.(C_{(q-1)/3}\times E_6(q)).3$ if $q\equiv 1 \bmod 3$. The order of this group is

$$q^{63}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)(q-1)$$

and can be obtained by elementary means, for example as in Dickson [5], or as in [8], or as sketched on page 168 of [7], if one wishes to avoid any Lie theory.

In order to complete the calculation of the order of the group G, therefore, we now need to count the images of $\langle h_0\rangle$. First we observe that any element of the Weyl group maps $\langle h_0\rangle$ to one of the 56 coordinate 1-spaces. Under the action of the stabilizer of $\langle h_0\rangle$ in the Weyl group, these fall into four orbits, which may be distinguished by the inner product of the corresponding vector of E_7^* with $h_0 = (0,1,1,0,1,0,0)$, which is either 3, 1, -1 or -3. These orbits may be represented by $\langle h_0\rangle$, $\langle ji_0\rangle$, $\langle i_0\rangle$, and $\langle jh_0\rangle$ respectively. Now Dickson [5] already shows that the number of images of $\langle i_0\rangle$ under $E_6(q)$ is $(q^8+q^4+1)(q^9-1)/(q-1)$, and it is straightforward to show that the normal subgroup of order q^{27} adds a factor of q in the case $\langle ji_0\rangle$ and a factor of q^{10} in the case $\langle i_0\rangle$. Hence under the action of the full stabilizer of $\langle h_0\rangle$ the orbit sizes are respectively

$$1, q(q^8+q^4+1)(q^9-1)/(q-1), q^{10}(q^8+q^4+1)(q^9-1)/(q-1), q^{27}.$$

Therefore the sum of these four orbit lengths is

$$(1+q^5)(1+q^9)(q^{14}-1)/(q-1).$$

In fact, expanding out this expression as a sum of powers of q, we find the following 56 exponents:

$$\begin{array}{l}0,1,2,3,4,5,6,7,8,9,10,11,12,13,\\ 5,6,7,8,9,10,11,12,13,14,15,16,17,18,\\ 9,10,11,12,13,14,15,16,17,18,19,20,21,22,\\ 14,15,16,17,18,19,20,21,22,23,24,25,26,27,\end{array}$$

which correspond to the 56 orbits of the Borel subgroup on images of $\langle h_0\rangle$.

Finally, it is necessary to show that this set of 1-spaces is closed under the action of the Weyl group. (It suffices, of course, to consider a single element outside the Weyl group of E_6.) Once this is done, it follows at once that the order of the group G is

$$q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1).$$

Simplicity of the group modulo scalars can then be proved from Iwasawa's Lemma. The action on the given orbit of 1-spaces is clearly primitive, and the kernel of the action consists only of scalars. The point stabilizer has a normal abelian subgroup of order q^{27} which contains root elements, and it is not hard to show that these root elements generate the group. Clearly they lie in the derived subgroup, so the group is perfect. Hence simplicity follows. In odd characteristic, there is a centre of order 2, so the order of the simple group $E_7(q)$ is half the above expression. In characteristic 2, the group is already simple as the centre is trivial.

7. Further remarks

The reason for labelling the 7 sets of four quaternionic coordinates with the letters H, I, J, K is that these behave in many ways like the quaternions $1, i, j, k$. Thus we define products of these as in the quaternion group: $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$ and $I^2 = J^2 = K^2 = -H$, with H acting as the identity element. Analogous to the element $\omega = \frac{1}{2}(-1+i+j+k)$ of order 3 we have $\Omega = \frac{1}{2}(-H+I+J+K)$.

Now consider our 28-dimensional quaternionic space as a 7-dimensional 'space' over the tensor product of these two copies of the quaternions. A 'vector' may be represented as $(X_0, X_1, X_2, X_3, X_4, X_5, X_6)$, and the actions of the elements (1)–(6) given in Section 3 are as follows:

(1) $(X_t) \mapsto (X_t j)$;
(2) $(X_t) \mapsto -(X_0, IX_1K, JX_2I, JX_3I, KX_4J, IX_5K, KX_6J)$;
(3) $(X_t) \mapsto (X_0, -KX_1Jj, -IX_2Kj, X_3, -JX_4Ij, X_5, X_6)$;
(4) $(X_t) \mapsto (X_{t+1})$;
(5) $(X_t) \mapsto (X_{2t}^{\Omega})$;
(6) $(X_t) \mapsto (-\overline{X_0}^{I-J}, X_2^{\Omega}, X_1^{\overline{\Omega}}, \overline{X_6}^{I-K}, X_4, \overline{X_5}^{I-K}, \overline{X_3}^{I-K})$.

(The minus signs can be removed from (2) by multiplying by the central involution, and from (3) by taking the inverse.)

In order to express the action of our typical element of a fundamental SU(2), we write $z = a+bi$, $wj = cj+dk$, where $a, b, c, d \in \mathbb{R}$. The element $z+wj$ now acts as

$$\begin{array}{rcl} X_1 & \mapsto & X_1a - IX_1Ibi - KX_1Jcj + JX_1Kdk, \\ X_2 & \mapsto & X_2a - JX_2Jbi - IX_2Kcj + KX_2Idk, \\ X_4 & \mapsto & X_4a - KX_4Kbi - JX_4Icj + IX_4Jdk. \end{array}$$

In effect, the quaternion group generated by right-multiplication by i and j has been replaced by a different quaternion group in each of the three coordinates. The non-trivial element of the Weyl group in this SU(2) may be taken as the case $a = b = d = 0$, $c = -1$, that is, the negative of the element (3), which we can write more succinctly as

$$(X_1, X_2, X_4) \mapsto (KX_1Jj, IX_2Kj, JX_4Ij).$$

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