THERE IS NO STRICTLY SINGULAR CENTRALIZER ON $L_p$

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Abstract. We prove that if $\Phi$ is a centralizer on $L_p$, where $0 < p < \infty$, then there is a copy of $\ell_2$ inside $L_p$ where $\Phi$ is bounded. If $\Phi$ is symmetric, then it is also bounded on a copy of $\ell_q$, provided $0 < p < q < 2$. This shows that for a wide class of exact sequences $0 \to L_p \to Z \to L_p \to 0$ the quotient map is not strictly singular, which generalizes a recent result of Jesús Suárez.

1. Introduction

An operator acting between Banach or quasi-Banach spaces is said to be strictly singular if it is not an isomorphism on any infinite dimensional subspace of its domain.

Exact sequences of Banach or quasi-Banach spaces $0 \to Y \to Z \to X \to 0$ in which the quotient map $\pi : Z \to X$ is strictly singular has spurred moderate interest since the early studies on the ‘three space problem’. Let us call them ‘strictly singular sequences’. In some sense, if one has a strictly singular sequence in which the spaces $X$ and $Y$ are ‘nice’, the middle space $Z$ must be ‘exotic’.

Amongst the most striking examples of this phenomenon one finds that for each $p \in (0, \infty)$ there is a strictly singular sequence

\[ 0 \longrightarrow \ell_p \longrightarrow Z_p \longrightarrow \ell_p \longrightarrow 0. \]

These were constructed by Kalton and Peck in [9]; see also [3].

More often than not the construction of strictly singular sequences is achieved by means of a quasilinear map from $X$ to $Y$, and this is certainly the case for the Kalton-Peck sequences, whose associated quasilinear maps are centralizers (a special type of quasilinear map; see Section 1.2). There is a function space analogue of (1),

\[ 0 \longrightarrow L_p \longrightarrow ZF_p \longrightarrow L_p \longrightarrow 0, \]

whose associated quasilinear map is the ‘classical’ centralizer

\[ \Omega(f) = f \log \left( \frac{|f|}{\|f\|} \right). \]
The space $ZF_p$ was introduced in [5], although it arises quite naturally in interpolation theory; see [10] Section 3D.

Very recently Jesús Suárez proved the following remarkable results on the behaviour of $\Omega$ on $L_p$:

(a) For every $0 < p < \infty$, there is a copy of $\ell_2$ in $L_p$ where the restriction of $\Omega$ is bounded.  
(b) If $0 < p < q < 2$, then $\Omega$ is bounded on a copy of $\ell_q$ inside $L_p$.

See [12] Propositions 3.1 and 4.1. Roughly this means that the sequence $[2]$ is not strictly singular because the quotient map is invertible on an isomorphic copy of $\ell_2$ (or $\ell_q$) inside $L_p$.

The purpose of this short note is to prove that (a) holds for all centralizers and (b) holds (at least) for symmetric centralizers. Our approach is based on a result by Kalton that describes centralizers as differentials of interpolation scales of Köthe function spaces from [7]. We also use results from [6] and a recent result of the author on the behaviour of centralizers acting between two different Lebesgue spaces [2].

1.1. Function spaces. Let $L_0$ denote the space of all real or complex measurable functions on the unit interval $\mathbb{I}$, where we identify two functions if they agree almost everywhere with respect to Lebesgue measure. A function space $X$ is a linear subspace of $L_0$, together with a quasinorm $\| \cdot \|$ having the following properties:

- The unit ball $B_X = \{ f \in X : \| f \| \leq 1 \}$ is closed in $L_0$ for the topology of convergence in measure.
- If $f, g \in X, g \in L_0$ and $|g| \leq |f|$, then $g \in X$ and $\| g \| \leq \| f \|$.

Important examples of function spaces are the spaces $L_p$ for $0 < p \leq \infty$. Given a function space $X$ and $A \subset \mathbb{I}$ we write $X(A)$ for the space of those functions in $X$ vanishing outside $A$.

We consider Köthe function spaces in the sense of [7]. Thus they are Banach function spaces whose norm satisfies the inequalities $\|hx\|_1 \leq \|x\|_X \leq \|hx\|_\infty$ for some everywhere positive functions $h, k$ and for every $x \in X$.

1.2. Centralizers and extensions. Let $X$ and $Y$ be function spaces. A centralizer from $X$ to $Y$ is a homogeneous mapping $\Phi : X \to L_0$ satisfying the following condition: there is a constant $C$ such that, for every $a \in L_\infty$ and for every $f \in X$, the difference $\Phi(af) - a\Phi(f)$ belongs to $Y$ and

$$\|\Phi(af) - a\Phi(f)\|_Y \leq C\|a\|_\infty\|f\|_X.$$ 

When $Y = X$ we say that $\Phi$ is a centralizer on $X$.

Although we will not use it, we remark that every centralizer is quasilinear; that is, there is a constant $Q$ such that for every $f, g \in X$ the difference $\Phi(f + g) - \Phi f - \Phi g$ falls in $Y$ and one has $\|\Phi(f + g) - \Phi f - \Phi g\|_Y \leq Q(\|f\|_X + \|g\|_X)$.

A centralizer from $X$ to $Y$ gives rise to an exact sequence

$$0 \longrightarrow Y \overset{1}{\longrightarrow} Y \oplus_\Phi X \overset{\pi}{\longrightarrow} X \longrightarrow 0$$

as follows:

- The middle space is $Y \oplus_\Phi X = \{(g, f) \in L_0 \times X : g - \Phi(f) \in Y\}$ with the quasinorm given by $\|(g, f)\|_\Phi = \|g - \Phi f\|_Y + \|f\|_X$.
- $\iota(g) = (g, 0)$ and $\pi(g, f) = f$. 


Actually only quasilinearity of Φ is required here.

We say that two centralizers Φ and Ψ are equivalent, and we write Φ ≈ Ψ if the difference takes values in Y and is bounded in the sense that \( \|\Phi(f) - \Psi(f)\|_Y \leq B\|f\|_X \) for some B and every \( f \in X \).

Let \( U \) be a subspace of \( X \) and suppose \( \Phi \) is bounded on \( U \) in the sense that \( \Phi \) maps \( U \) into \( Y \) (not \( L_0 \)) and \( \|\Phi(u)\|_Y \leq B\|u\|_X \) for some constant \( B \) and every \( u \in U \). Then the map \( s : U \to Y \oplus_\Phi X \) defined by \( s(u) = (0, u) \) is a bounded linear operator and \( \pi \circ s = I_U \). Thus \( \pi \) cannot be strictly singular if \( \Phi \) is bounded on some infinite dimensional subspace of \( X \).

Important examples of centralizers are the following (see [6], Section 3 and especially Theorem 3.1). Let \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be a Lipschitz function vanishing at the origin. Then the map \( L_p \to L_0 \) given by

\[
  f \mapsto f \varphi \left( \log \frac{|f|}{\|f\|_p}, \log \frac{|r_f|}{\|f\|_p} \right)
\]

is a (real, symmetric) centralizer on \( L_p \). Here \( r_g \) is the so-called rank-function of \( g \in L_0 \) defined by

\[
  r_g(t) = \lambda\{ s \in \mathbb{R}^+ : |g(s)| > |g(t)| \text{ or } s \leq t \text{ and } |g(s)| = |g(t)| \},
\]

which arises in real interpolation.

1.3. Real centralizers. Let \( X \) be a complex function space and let \( X^\Re = \Re(X) \) be a corresponding real function space. A centralizer on \( X \) is said to be real if it sends real functions into real functions. Clearly, every real centralizer on \( X \) induces a centralizer on \( X^\Re \) by restriction. On the other hand each centralizer \( \Phi \) on \( X^\Re \) extends to a real centralizer on \( X \) by the formula \( \Phi^R f = \Phi(u) + i\Phi(v) \), where \( u = \Re f \) and \( v = \Im f \). These processes are each inverse of the other, up to equivalence.

Moreover, if \( \Phi \) is any centralizer on \( X \), there are real centralizers \( \Phi_1 \) and \( \Phi_2 \) such that \( \Phi \approx \Phi_1 + i\Phi_2 \); see [7] Lemma 7.1.

2. Results

Let \( A \) be a Borel subset of \( \mathbb{I} \). A Rademacher sequence in \( A \) is a sequence \( (r_n) \) in \( L_0(A) \) such that \( \lambda\{ t \in A : r_n(t) = 1 \} = \lambda\{ t \in A : r_n(t) = -1 \} = \frac{1}{2}\lambda(A) \) for all \( n \) and \( \mathbb{E}[r_n r_m | A] = 0 \) for \( n \neq m \).

Khintchine’s inequality states that if \( (r_n) \) is a Rademacher sequence and \( (t_n) \) is in \( \ell_2 \), then \( f = \sum_n t_n r_n \) belongs to \( L_s \) for every \( s \in (0, \infty) \) and, moreover, there is a constant \( M \) depending only on \( s \) and \( \lambda(A) \) such that

\[
  M^{-1} \|t_n\|_{\ell_2} \leq \|f\|_s \leq M\|t_n\|_{\ell_2}.
\]

Thus a Rademacher sequence spans a subspace isomorphic to \( \ell_2 \) in \( L_s \) for any \( s \in (0, \infty) \).

Our first result is based on certain ideas from complex interpolation. Let us indicate the minimal background one needs to understand the proof.

Let \( X_0 \) and \( X_1 \) be (complex) Köthe function spaces on the unit interval. Consider the closed strip \( \mathbb{S} = \{ z \in \mathbb{C} : 0 \leq \Re(z) \leq 1 \} \) and let \( \mathcal{F}(X_0, X_1) \) denote the space of bounded, continuous functions \( F : \mathbb{S} \to X_0 + X_1 \) having the following properties:

- \( F \) is analytic on the interior of \( \mathbb{S} \).
- \( F(k + it) \in X_k \) for each \( k = 0, 1 \) and all \( t \in \mathbb{R} \).
- For \( k = 0, 1 \), the map \( t \in \mathbb{R} \mapsto F(k + it) \in X_k \) is bounded and continuous.
Then $\mathcal{F} = \mathcal{F}(X_0, X_1)$ is a Banach space under the norm

$$\|F\|_{\mathcal{F}} = \sup\{\|F(k + it)\|_{X_k} : t \in \mathbb{R}, k = 0, 1\}.$$ 

For $\theta \in [0, 1]$ we define the interpolation space

$$X_\theta = [X_0, X_1]_\theta = \{f \in L_0 : f = F(\theta) \text{ for some } F \in \mathcal{F}\}$$

with the (quotient) norm $\|f\|_{X_\theta} = \inf\{\|F\|_{\mathcal{F}} : f = F(\theta)\}$.

The equation $[X_0, X_1]_\theta = X$ induces a ‘derivation’ on $X$ as follows. We fix a small $\epsilon > 0$ and for each $f \in X$ we choose $F \in \mathcal{F}(X_0, X_1)$ such that $F(\theta) = f$, with $\|F\|_{\mathcal{F}} \leq (1 + \epsilon)\|f\|_X$. Then we put $\Omega(f) = F'(\theta)$. The map $\Omega : X \to L_0$ is a centralizer on $X$, and two centralizers obtained with different choices of $F$ are equivalent.

An important result of Kalton [7, Theorem 7.6] states that if $\Phi$ is a real centralizer on the complex reflexive K"{o}the spaces.

**Proposition 1.** Let $\Phi$ be a centralizer on $L_p$, with $p > 1$, then there is a constant $c > 0$ and a couple of K"{o}the functions such that $L_p = [X_0, X_1]_{\theta = 1/2}$ with equivalent norms, in the sense that both spaces contain the same functions and there is $M$ such that

$$M^{-1}\|f\|_p \leq \inf_{f = F(\theta)} \|F\|_{\mathcal{F}} \leq M\|f\|_p$$

for all $f \in L_p$, and $\Phi \approx c\Omega$, where $\Omega$ is the corresponding derivation on $X_{1/2} = L_p$.

**Proof.** It should be clear from the remarks in Section 1.3 that it suffices to prove the proposition assuming that $\Phi$ is a real centralizer on the complex $L_p$.

First suppose $p > 1$. Then by the result of Kalton quoted above, we know that there are a couple of K"{o}the spaces $(X_0, X_1)$ and $c > 0$ such that $L_p = [X_0, X_1]_{1/2}$ and $\Phi \approx c\Omega$.

Let us take a look at $\Omega$. First, by iteration, we have $L_p = [X_{1/4}, X_{3/4}]_{1/2}$ where $X_{k/4} = [X_0, X_{1/4}]_{k/4}$ for $k = 1, 3$ and both $X_{1/4}$ and $X_{3/4}$ are super-reflexive by [8, Theorem 5.8]. On the other hand, if $F \in \mathcal{F}(X_0, X_1)$, then the function $G$ defined by $G(z) = F'(\frac{1}{2}(z + \frac{1}{2}))$ belongs to $\mathcal{F}(X_{1/4}, X_{3/4})$ and one has $\|G\|_{\mathcal{F}} \leq \|F\|_{\mathcal{F}}$, $G(\frac{1}{2}) = F(\frac{1}{2})$ and $G'(\frac{1}{2}) = \frac{1}{2}F'(\frac{1}{2})$.

Thus replacing the couple $(X_0, X_1)$ by $(X_{1/4}, X_{3/4})$ preserves the induced centralizer, up to a constant factor, and so we may assume $X_0$ and $X_1$ are super-reflexive K"{o}the spaces.

Now, for $i = 0, 1$, take everywhere positive functions $h_i$ and $k_i$ so that $\|h_if\|_1 \leq \|f\|_{X_i} \leq \|k_if\|_\infty$ for all $f \in X_i$ and observe that for fixed $\delta > 0$ there is $M$ large enough and a subset $B \subset \mathbb{H}$ with $\lambda(B) > 1 - \delta$ where $k_i \leq M$ and $h_i \geq 1/M$ for $i = 0, 1$.

It follows that $L_\infty(B) \subset X_i(B) \subset L_1(B)$, with continuous inclusions, and since $X_i$ is super-reflexive it is also $s_i$-concave for some finite $s_i$, and so we have a continuous inclusion $L_{s_i}(B) \subset X_i(B)$ (see [4, p. 14]). Now taking $s = \max s_i$ we conclude that $L_s(B)$ embeds continuously into $X_i$, and so there is a constant $M$ such that $\|f\|_{X_i} \leq M\|f\|_s$ for every $f \in L_s(B)$ and $i = 0, 1$.

Now, let $(r_n)$ be a Rademacher sequence in $L_s(A)$, where $A \subset B$, and let $R$ be the closed linear span of $(r_n)$ in $L_s(A)$. Then, for $(\lambda_n) \in \ell_2$ the sum $\sum_n \lambda_n r_n$ is in...
Let \( \Phi \) where \( \Phi \) is a centralizer on \( L_p \) and \( \Phi \) is a symmetric centralizer on \( L_p \) and \( \Phi \) is a symmetric centralizer on \( L_p \).

A centralizer \( \Phi \) on \( L_p \) is said to be symmetric if there is a constant \( S \) such that
\[
\| \Phi(f) - \Phi(g) \|_p \leq S \| f - g \|_p
\]
for every \( f, g \in L_p \) and every measure preserving Borel automorphism \( \sigma \) of \( \mathbb{I} \).

The decreasing rearrangement of a real-valued \( f \in L_0 \) is defined by the formula
\[
f^*(t) = \inf_{\lambda(B) = t} \sup_{s \in A \setminus B} f(s) \quad (0 \leq t \leq 1)
\]
where \( B \) runs over the Borel subsets of \( \mathbb{I} \). That is, \( f^* \) is the only decreasing, right-continuous function having the same distribution as \( f \). It is a basic fact from measure theory that for each \( f \in L_0 \), there is a measure preserving Borel automorphism \( \sigma \) of \( \mathbb{I} \) (depending on \( f \)) such that \( f^* = f \circ \sigma \) (almost everywhere), and so \( f^* \) is a true rearrangement of \( f \); see [11, Lemma 2].

Note that if \( \Phi \) is a symmetric centralizer on \( L_p \) and \( f^* = f \circ \sigma \), then \( \| \Phi(f) - (\Phi(f^*) \circ \sigma^{-1}) \|_p \leq S \| f \|_p \) and so the map \( \Phi_s(f) = (\Phi(f^*)) \circ \sigma^{-1} \) is a symmetric centralizer equivalent to \( \Phi \) with the additional property that the distribution of \( \Phi_s(f) \) depends only on the distribution of \( f \).

We emphasize that, in general, centralizers take values in \( L_0 \). For symmetric centralizers we have, however, the following.

**Lemma 1.** Suppose \( 0 < p < r < \infty \) and let \( \Phi \) be a symmetric centralizer on \( L_p \). If \( f \in L_r \), then \( \Phi f \in L_p \).

**Proof.** It suffices to prove the lemma for real spaces. Let \( \Phi_r : L_r \to L_0 \) be the restriction of \( \Phi \) to \( L_r \). This is a centralizer from \( L_r \) to \( L_p \), so by the main result in [12] \( \Phi_r \) must be trivial and there is \( \phi \in L_0 \) and a constant \( M \) such that
\[
\| \Phi_r(f) - \phi f \|_p \leq M \| f \|_r \quad (f \in L_r).
\]
We claim that \( \phi \in L_s \), where \( s^{-1} + r^{-1} = p^{-1} \). By the Hölder inequality this implies that \( \phi f \in L_p \), and the same occurs for \( \Phi(f) = \Phi_r(f) \). To see this, observe that since \( f \mapsto \phi f \) is equivalent to \( \Phi_r \) it is a symmetric centralizer from \( L_r \) to \( L_p \), and so there is a constant \( S \) such that
\[
\| (\phi \circ \sigma)(f \circ \sigma) - \phi(f \circ \sigma) \|_p \leq S \| f \|_r \quad (f \in L_r)
\]
whenever $\sigma$ is a measure preserving automorphism of the unit interval. Now, since for every $g \in L_s$ one has $\|g\|_s = \sup_{\|f\|_p \leq 1} \|gf\|_p$, we see that $\|\phi \circ \sigma - \phi\|_s \leq M'$ for some $M'$ independent on $\sigma$. By symmetry one also has $\|\phi^* \circ \sigma - \phi^*\|_s \leq M'$, where $\phi^*$ is the decreasing arrangement of $\phi$ and $\sigma$ is as before. In particular $\|\phi^* \circ \sigma - \phi^*\|_s$ is finite when $\sigma(t) = 1 - t$. Let $m = \phi^*(\frac{1}{2})$ be the median of $\phi$. Now, since $\phi^*$ is decreasing, $\phi^* \circ \sigma$ is increasing and both agree with $m$ at $t = \frac{1}{2}$, we see that

$$\|\phi - m1\|_s = \|\phi^* - m1\|_s \leq \|\phi^* \circ \sigma - \phi^*\|_s$$

is finite and $\phi \in L_s$.

We are now ready to prove the following.

**Proposition 2.** Let $0 < p < q < 2$. There is a subspace $U$ of $L_p$ isomorphic to $\ell_q$ where the restriction of any symmetric centralizer is bounded.

**Proof.** It suffices to prove the result for real spaces. Moreover, we may and do assume that the distribution of $\Phi(f)$ depends only on that of $f$.

We proceed as in [12, Proof of Proposition 4.1]. For fixed $q \in (p, 2)$ we consider a $q$-stable random variable $\vartheta \in L_p$ and a sequence of independent copies $(\vartheta_n)$. We recall that a random variable is said to be $q$-stable if its characteristic function (Fourier transform) is $e^{-|t|^q/q}$. We refer the reader to [1, Chapter 6, Section 4] for basic information on stable variables. Here we use the following facts:

- If $\vartheta$ is $q$-stable, then $E[|\vartheta|^r] < \infty$ for $p < r < q$.
- If $(\vartheta_n)$ is a sequence on independent copies of a $q$-stable random variable $\vartheta$ and $(\lambda_n)$ is a sequence normalized in $\ell_q$, then $\sum_n \lambda_n \vartheta_n$ has the same distribution as $\vartheta$.

Therefore the map $(\lambda_n) \in \ell_q \mapsto \sum_n \lambda_n \vartheta_n \in L_p$ is well defined and it is an isometric embedding whose image we denote by $U$. Moreover, by the lemma, $\Phi(\vartheta)$ belongs to $L_p$, and then for $(\lambda_n)$ normalized $\ell_q$ we have $\|\Phi(\sum \lambda_n \vartheta_n)\|_p = \|\Phi(\vartheta)\|_p$, and so $\Phi$ is bounded on $U$.

**Problem.** Is there a strictly singular sequence $0 \to L_p \to Z \to L_p \to 0$ for $0 < p < 2$? (See [3, Theorem 2(c)] for the case $2 \leq p < \infty$.)

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**References**


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