DECOMPOSING THE $C^*$-ALGEBRAS
OF GROUPOID EXTENSIONS

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Abstract. We decompose the full and reduced $C^*$-algebras of an extension of a groupoid by the circle into a direct sum of twisted groupoid $C^*$-algebras.

1. Introduction

Let $H$ be a compact group and denote by $\hat{H}$ the collection of equivalence classes of the irreducible unitary representations of $H$. The Peter-Weyl Theorem implies, first, that every irreducible unitary representation of $H$ is finite-dimensional and, second, that the left-regular representation $\lambda$ of $H$ on $L^2(H)$ is unitarily equivalent to the direct sum $\bigoplus_{U \in \hat{H}} U \cdot U$, where $d_U$ is the dimension of $U$. The $C^*$-algebra $C^*(H)$ of $H$ is universal for the unitary representations of $H$, which means roughly that the unitary representations $U$ of $H$ are in one-to-one correspondence with the nondegenerate representations $\pi_U$ of $C^*(H)$ on the Hilbert space of $U$. Since $H$ is compact, the left-regular representation $\pi_{\lambda}$ is an isomorphism, and the reduced $C^*$-algebra $C^*_r(H) := \pi_{\lambda}(C^*(H))$ coincides with $C^*(H)$. So in the $C^*$-setting, the Peter-Weyl theorem says that $C^*(H) = C^*_r(H)$ is a direct sum $\bigoplus_{U \in \hat{H}} M_{d(U)}(\mathbb{C})$ of matrix algebras.

A similar result holds for extensions of locally compact groups $H$ by the circle $\mathbb{T}$. Let $\omega : H \times H \to \mathbb{T}$ be a continuous 2-cocycle. We associate two $C^*$-algebras to the pair $(H, \omega)$. The first is the twisted group $C^*$-algebra $C^*_\omega(H)$, which is universal for the $\omega$-representations of $H$. For the second, equip $H^\omega := T \times H$ with the product topology and multiplication $(s, \eta)(t, \gamma) = (st\omega(\eta, \gamma), \eta \gamma)$; then $H^\omega$ is a locally compact group and has a $C^*$-algebra. It follows from $[20]$, for example, that $C^*(H^\omega)$ is isomorphic to the direct sum $\bigoplus_{n \in \mathbb{Z}} C^*(H, \omega^n)$ of twisted group $C^*$-algebras (see also $[14]$ Corollary 3]). In this paper we generalize this latter result to locally compact Hausdorff groupoids.

Let $G$ be a locally compact Hausdorff groupoid and $\omega : G^{(2)} \to \mathbb{T}$ a continuous 2-cocycle on the set $G^{(2)}$ of composable pairs in $G$. We show that the $C^*$-algebra of the extension $G^\omega$ is isomorphic to the direct sum $\bigoplus_{n \in \mathbb{Z}} C^*(G, \omega^n)$ of twisted groupoid $C^*$-algebras and that this isomorphism factors through to the reduced $C^*$-algebras (see Theorems 3.2 and 4.1). The full twisted groupoid $C^*$-algebras have been used in $[13]$, $[10]$ and $[4]$ to characterize when groupoid $C^*$-algebras

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have continuous trace or bounded trace, and their non-selfadjoint subalgebras have been studied in [11]. The reduced twisted groupoid $C^*$-algebras appear as the $C^*$-algebras with diagonal subalgebras in $[9]$ and as the $C^*$-algebras with Cartan subalgebras in $[18]$. For example, if $A$ is a $C^*$-algebra with diagonal subalgebra $B$, then Kumjian’s Theorem 3.1 of $[9]$ implies that there exists a principal étale groupoid $G$ and an extension of $G$ by $\mathbb{T}$ implemented by a (possibly Borel) cocycle $\omega$ such that $A$ is isomorphic to $C^*_r(G, \mathbb{Z})$, and the isomorphism maps the diagonal $B$ to a diagonal in $C^*_r(G, \mathbb{Z})$. A similar result holds for Cartan subalgebras, except there the groupoid $G$ may be only topologically principal $[18]$ §5.

The main theorems of this paper provide a general framework for investigating twisted groupoid $C^*$-algebras using the literature on the non-twisted case. For example, suppose that $G$ is principal. Then we deduce in Proposition 3.9 that $C^*(G)$ has continuous trace if and only if $C^*(G^\omega)$ has continuous trace, and if $C^*(G)$ has continuous trace, then so does $C^*(G, \omega)$. See Proposition 8.10 for more results along these lines. We also deduce in Corollary 13.3 that if a groupoid $G$ is amenable, then $C^*(G, \omega)$ and $C^*_r(G, \omega)$ are isomorphic.

2. Preliminaries

Throughout, $G$ is a second-countable, locally compact, Hausdorff groupoid with Haar system $\{\lambda^u\}_{u \in G^{(0)}}$. We denote by $\lambda^u$ the image of $\lambda^u$ under inversion. We write $G^{(0)}$ for the unit space of $G$, $r = r_G, s = s_G : G \to G^{(0)}$ for the range and source maps $r_G(\gamma) = \gamma \gamma^{-1}$ and $s_G(\gamma) = \gamma^{-1} \gamma$, respectively, and $G^{(2)} := \{(\gamma, \eta) : s_G(\gamma) = r_G(\eta)\}$ for the set of composable pairs.

2.1. Groupoid extensions. Let $\omega : G^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle, so that $\omega$ satisfies the cocycle identity

$$\omega(\gamma, \eta)\omega(\eta, \xi) = \omega(\eta, \xi)\omega(\gamma, \eta\xi).$$

We will assume throughout that $\omega$ is normalized in the sense that

$$\omega(r_G(\gamma), \gamma) = 1 = \omega(\gamma, s_G(\gamma)) \text{ and } \gamma \in G;$$

since every 2-cocycle is cohomologous to a normalized one and because the associated $C^*$-algebras depend only on the class of the 2-cocycle (see [16 Proposition II.1.2]), there is no loss of generality. Following [16, page 73] we denote by $G^\omega$ the extension of $G$ by $\mathbb{T}$ defined by $\omega$: thus $G^\omega$ is the groupoid $\mathbb{T} \times G$ with the product topology, with range and source maps $r_G^\omega(t, \gamma) = (1, r_G(\gamma))$ and $s_G^\omega(t, \gamma) = (1, s_G(\gamma))$, multiplication $(s, \eta)(t, \gamma) = (st\omega(\eta, \gamma), \eta\gamma)$ and inverse $(t, \gamma)^{-1} = (t^{-1}\omega(\gamma, \gamma^{-1})^{-1}, \gamma^{-1})$. We identify the unit space of $G^\omega$ with $G^{(0)}$ via $(1, u) \mapsto u$. We say that $G^\omega$ is the groupoid extension associated to $(G, \omega)$. When we want to emphasize the product nature of $G^\omega$ we will denote it by $\mathbb{T} \times_{\omega} G$.

In order to reconcile our work with the literature, suppose that

$$G^{(0)} \to \mathbb{T} \times G^{(0)} \xrightarrow{i} E \xrightarrow{j} G := E/\mathbb{T} \to G^{(0)}$$

is an extension of topological groupoids such that $i$ induces a free action of $\mathbb{T}$ on $E$ by $t \cdot \gamma = i(t, r_E(\gamma))\gamma$ for $\gamma \in E$ and $t \in \mathbb{T}$. In [13, page 131], Muhly and Williams discuss a correspondence between extensions $E$ and Borel 2-cocycles defined using a Borel cross-section of $j$. They show that the 2-cocycle $\omega$ associated to an extension

\footnote{The coboundary implementing the equivalence is the image under the boundary map of the function $b(\gamma) = \omega(r_G(\gamma), \gamma)$.}
E is continuous if and only if there exists a continuous section of j, and then E is topologically isomorphic to \( G^\omega \). (If the cocycle \( \omega \) is not continuous, then \( G^\omega \) is a Borel groupoid and the topological groupoid \( E \) is Borel isomorphic to \( G^\omega \).) When \( E = G^\omega \) for a continuous \( \omega \), the inclusion \( i \) in (2.1) is \( i(t, u) = (t, u) \) and \( j \) is the projection onto \( G \). Furthermore, since \( \omega \) is normalized, the action of \( T \) on \( G^\omega \) induced by \( i \) is \( s \cdot (t, \gamma) = (st, \gamma) \). Our main reason for restricting our attention to extensions associated to continuous cocycles is that we use the continuity in an apparently essential way in Lemma 3.6.

**Example 2.1.** It is quite easy to construct groupoids \( G \) with non-trivial continuous 2-cocycles \( \omega \), and hence there are many non-trivial extensions \( G^\omega \) as described above. For example, take \( X = S^3 \) and recall that the Čech cohomology group \( H^3(X, \mathbb{Z}) \) is non-trivial. Since \( H^3(X, \mathbb{Z}) \) is isomorphic to the second sheaf cohomology group \( H^2(X, S) \) (see, for example, [19, Theorem 4.42]), there exists a non-trivial cocycle \( \lambda := \{ \lambda_{ijk} : U_{ijk} \to \mathbb{T} \} \), where the \( U_{ijk} \) are the triple overlaps of an open cover \( \{ U_i \} \) of \( X \). Define

\[
\Psi : \bigcup_i U_i \times \{ i \} \to X \text{ by } \Psi(x, i) = x \text{ for } x \in U_i.
\]

Then \( R(\Psi) := \{ ((x, i), (x, j)) : x \in U_i \cap U_j \} \) becomes a groupoid with range and source maps given by \( r_{R(\Psi)}((x, i), (x, j)) = (x, i) \) and \( s_{R(\Psi)}((x, i), (x, j)) = (x, j) \), multiplication defined by \( ((x, i), (x, j))(x, k)) = ((x, i), (x, k)) \) and inverse \( ((x, i), (x, j))^{-1} = ((x, j), (x, i)) \). Now define \( \omega_\lambda : R(\Psi)^{(2)} \to \mathbb{T} \) by

\[
\omega_\lambda(((x, i), (x, j)), ((x, j), (x, k))) = \lambda_{ijk}(x)
\]

for \( x \in U_{ijk} \). It is straightforward to check that \( \omega_\lambda \) is a non-trivial continuous 2-cocycle (the \( \lambda_{ijk} : U_{ijk} \to \mathbb{T} \) are continuous by definition, and \( \omega_\lambda \) is a coboundary if and only if \( \lambda \) is).

Recall that a groupoid \( G \) is **principal** if the map \( \Phi : \gamma \mapsto (r_G(\gamma), s_G(\gamma)) \) is injective and is **proper** if \( \Phi \) is proper. We say \( G \) is **transitive** if given \( u, v \in G^{(0)} \) there exists \( \gamma \in G \) such that \( r_G(\gamma) = u \) and \( s_G(\gamma) = v \).

**Remark 2.2.** Although Example 2.1 shows that there are many examples of non-trivial continuous 2-cocycles, principal transitive groupoids have only trivial ones. To see this, let \( G \) be a principal and transitive groupoid, and let \( \omega \) be a normalized 2-cocycle on \( G \). Pick \( u \in G^{(0)} \), and let \( b : \gamma \mapsto \omega(\gamma, \alpha_\gamma) \), where \( \alpha_\gamma \) is the unique element such that \( r_G(\alpha_\gamma) = s_G(\gamma) \) and \( s_G(\alpha_\gamma) = u \). Then \( \omega(\gamma, \eta) = b(\gamma)b(\eta)b(\gamma\eta) \) and thus \( \omega \) is a coboundary.

### 2.2. The \( C^* \)-algebras

Let \( E \) be a second-countable, locally compact groupoid with left Haar system \( \beta \), and \( \sigma : E^{(2)} \to \mathbb{T} \) a continuous, normalized 2-cocycle. For \( f, g \in C_c(E) \), the formulas

\[
f * g(\gamma) = \int_E f(\eta)g(\eta^{-1}\gamma)\sigma(\eta, \eta^{-1}\gamma) \, d\beta^{e\sigma}(\gamma)(\eta) \quad \text{and} \quad f^*(\gamma) = f(\gamma^{-1})\sigma(\gamma, \gamma^{-1})
\]

define a convolution and involution on \( C_c(E) \). These operations make \( C_c(E) \) into a *-algebra, denoted by \( C_c(E, \sigma) \). We denote by \( \text{Rep}(C_c(E, \sigma)) \) the set of Hilbert-space representations \( \rho : C_c(E, \sigma) \to B(\mathcal{H}) \) that are continuous for the inductive limit topology on \( C_c(E) \) and the weak operator topology on \( B(\mathcal{H}) \). Then

\[
\|f\| = \sup\{\|\rho(f)\| : \rho \in \text{Rep}(C_c(E, \sigma))\}
\]
is finite and defines a pre-$C^*$-norm on $C_c(E, \sigma)$; the twisted groupoid $C^*$-algebra $C^*(E, \sigma)$ is defined as the completion of $C_c(E, \sigma)$ in this norm. (All of this is non-trivial. If $\rho \in \text{Rep}(C_c(E, \sigma))$, then $\rho$ is the integrated form of a unitary representation of $E$ by [17, Théorème 4.1(ii)], and then $\rho$ is bounded in Renault’s $I$-norm [16, Proposition II.1.17].) Representations bounded by the $I$-norm are continuous in the inductive limit topology. It now follows that (2.2) defines a norm by [16, Definition II.1.12].) If the cocycle is identically 1, then we write $C^*(E)$ for $C^*(E, 1)$ and call it the groupoid $C^*$-algebra of $E$.

Let $\tau$ be normalized left Haar measure on $\mathbb{T}$; we will denote $d\tau(t)$ by $dt$. Let $\omega$ be a continuous 2-cocycle on $G$ and let $G^\omega$ be the associated groupoid extension. Since $G^\omega = \mathbb{T} \times_\omega G$ has the product topology, the product measures $\{\tau \times \lambda^u : u \in G^{(0)}\}$ define a Haar system on the extension $G^\omega$. For fixed $n \in \mathbb{Z}$, let

$$C_c(G^\omega, n) = \{f \in C_c(G^\omega) : f(s \cdot (t, \gamma)) = s^{-n} f(t, \gamma)\}.$$ 

As above, we denote by $\text{Rep}(C_c(G^\omega, n))$ the set of Hilbert-space representations $\rho : C_c(G^\omega, n) \to B(\mathcal{H})$ that are continuous for the inductive limit topology on $C_c(G^\omega, n)$ and the weak operator topology on $B(\mathcal{H})$. Then $C_c(G^\omega, n)$ is a *-subalgebra of $C_c(G^\omega)$, and, as in [17, §5] and [13, page 130], the $C^*$-algebra $C^*(G^\omega, n)$ is the completion of $C_c(G^\omega, n)$ in the norm $\|f\| = \sup\{\|\rho(f)\| : \rho \in \text{Rep} C_c(G^\omega, n)\}$. (Again, this is non-trivial: Corollaire 4.8 of [17] implies that this indeed defines a norm bounded by the $I$-norm.) The $C^*$-algebra $C^*(G^\omega, n)$ was studied in [17, §1], and, when $n = -1$, in [13].

**Remark 2.3.** Let $f, g \in C_c(G^\omega, n)$. Since the Haar system $\{\tau \times \lambda^u\}$ on $G^\omega$ is pulled back from the one on $G$ and $\tau$ is normalized, the convolution $f * g$ can be written as an integral over $G$: a direct calculation shows that for any $s \in \mathbb{T}$,

$$f * g(t, \gamma) := \int_G \int_{\mathbb{T}} f(r, \eta) g((r, \eta)^{-1}(t, \gamma)) \, dr \, d\lambda^{\omega \sigma(\gamma)}(\eta)$$

$$= \int_G \int_{\mathbb{T}} r^{-n} r^n f(1, \gamma) g(t \omega(\eta, \eta^{-1})^{-1} \omega(\eta^{-1}, \gamma), \eta^{-1}\gamma) \, dr \, d\lambda^{\omega \sigma(\gamma)}(\eta)$$

$$= \int_G f(s, \eta) g((s, \eta)^{-1}(t, \gamma)) \, d\lambda^{\omega \sigma(\gamma)}(\eta).$$

### 3. Decomposing the $C^*$-algebra of a groupoid extension

Throughout $\omega : G^{(2)} \to \mathbb{T}$ is a continuous normalized 2-cocycle, and $G^\omega$ is the groupoid extension associated to $(G, \omega)$. Note that $\omega^n$ is also a continuous 2-cocycle. The goal of this section is to prove that $C^*(G^\omega)$ is isomorphic to a direct sum of twisted groupoid $C^*$-algebras $C^*(G, \omega^n)$. We start by proving that $C^*(G^\omega, n)$ is a quotient of $C^*(G^\omega)$ and is isomorphic to $C^*(G, \omega^n)$.

**Lemma 3.1.** Let $G^\omega$ be the groupoid extension associated to $(G, \omega)$. Fix $n \in \mathbb{Z}$.

(a) [17, Lemma 3.3] Define $\chi_n : C_c(G^\omega) \to C_c(G^\omega, n)$ by

$$\chi_n(f)(t, \gamma) := \int_{\mathbb{T}} f(s \cdot (t, \gamma)) s^n \, ds = \int_{\mathbb{T}} f(st, \gamma) s^n \, ds.$$
Then $\chi_n$ is a $*$-homomorphism continuous with respect to the inductive limit topologies and extends to a $*$-homomorphism $\chi_n : C^*(G^\omega) \to C^*(G^\omega, n)$ such that $\chi_n(f) = f$ for $f \in C_c(G^\omega, n)$. In particular, $\chi_n$ is a quotient map.

(b) Let $\phi_n : C_c(G^\omega, n) \to C_c(G, \omega^n)$ be the map $\phi_n(f)(\gamma) = f(1, \gamma)$ for $\gamma \in G$. Then $\phi_n$ extends to a $*$-isomorphism of $C^*(G^\omega, n)$ onto $C^*(G, \omega^n)$.

Proof. Part (a) is [17, Lemma 3.3] (see also [16, Proposition II.1.22] for a detailed proof of the case $n = 1$).

It suffices to show that $\phi_n : C_c(G^\omega, n) \to C_c(G, \omega^n)$ is a continuous bijective $*$-homomorphism with a continuous inverse. For then $\phi_n$ and $\phi_n^{-1}$ extend to $*$-homomorphisms $\phi_n : C^*(G^\omega, n) \to C^*(G, \omega^n)$ and $\phi_n^{-1} : C^*(G, \omega^n) \to C^*(G^\omega, n)$, and by continuity $\phi_n \circ \phi_n^{-1} = id$ and $\phi_n^{-1} \circ \phi_n = id$, giving that $\phi_n$ is an isomorphism.

Let $\eta, \gamma \in G$ with $r_G(\gamma) = r_G(\eta)$. Since $\omega$ is normalized, we have

\[1 = \omega(r_G(\eta^{-1}), \eta^{-1}) = \omega(s_G(\eta), \eta^{-1}) = \omega(\eta^{-1}, \eta^{-1})\]

and

\[\omega(\eta, \eta^{-1}) = \omega(\eta^{-1}, \eta) = \omega(\eta^{-1}, \eta)\omega(\eta, \eta^{-1})^{-1} = \omega(\eta^{-1}, \eta)\omega(\eta^{-1}, \eta)\]

Thus

\[\omega(\eta, \eta^{-1}) = \omega(\eta^{-1}, \eta)\omega(\eta^{-1}, \eta) = \omega(\eta^{-1}, \eta)\omega(\eta^{-1}, \eta)\]

and it follows that

\[\omega(\eta, \eta^{-1})\omega(\eta^{-1}, \gamma) = \omega(\eta, \eta^{-1})\gamma.

So, for $f, g \in C_c(G^\omega, n)$,

\[\phi_n(f \ast g)(\gamma) = f \ast g(1, \gamma) = \int_G f(1, \eta)g((1, \eta)^{-1}(1, \gamma)) \, d\lambda^{r_G(\gamma)}(\eta)\]

\[= \int_G f(1, \eta)g(\omega(\eta, \eta^{-1})\omega(\eta^{-1}, \gamma) \omega(\eta^{-1}, \eta)) \, d\lambda^{r_G(\gamma)}(\eta)\]

\[= \int_G f(1, \eta)g(\omega(\eta, \eta^{-1}), \eta^{-1}) \, d\lambda^{r_G(\gamma)}(\eta) \quad \text{(using (3.1))}\]

\[= \int_G f(1, \eta)g(1, \eta^{-1})\omega(\eta, \eta^{-1}) \, d\lambda^{r_G(\gamma)}(\eta)\]

\[= \phi_n(f) \ast \phi_n(g)\]

and

\[\phi_n(f^\ast)(\gamma) = f^\ast((1, \gamma)^{-1}) = f(\omega(\gamma, \gamma^{-1})^{-1}, \gamma^{-1})\]

\[= f(1, \gamma^{-1})\omega(\gamma, \gamma^{-1})^{-n} = \phi_n(f^\ast)(\gamma)\]

So $\phi_n$ is a $*$-homomorphism. To see that $\phi_n$ is injective on $C_c(G^\omega, n)$, suppose $f(1, \gamma) = g(1, \gamma)$ for all $\gamma \in G$. Then for all $t \in \mathbb{T}$,

\[f(t, \gamma) = t^{-n} f(1, \gamma) = t^{-n} g(1, \gamma) = g(t, \gamma)\]

and thus $f = g$. To see that $\phi_n$ is onto $C_c(G, \omega^n)$, let $f \in C_c(G)$ and note that $(t, \gamma) \mapsto t^{-n} f(\gamma)$ is in $C_c(G^\omega, n)$, and $\phi_n$ sends it back to $f$. So $\phi_n : C_c(G^\omega, n) \to C_c(G, \omega^n)$ is a bijection.

If $F_i \to F$ in the inductive limit topology on $C_c(G^\omega)$, then $F_i(1, \cdot) \to F(1, \cdot)$ uniformly on a fixed compact set as well. Thus $\phi_n$ is continuous for the inductive limit topology on $C_c(G^\omega, n)$ and extends to a $*$-homomorphism of the $C^*$-algebras.
Similarly, if \( f_i \to f \in C_c(G) \) in the inductive limit topology, then \( |t^{-n}f_i(\gamma) - t^{-n}f(\gamma)| \leq |f_i(\gamma) - f(\gamma)| \) is eventually small, so that \( \phi_n^{-1}(f_i) \to \phi_n^{-1}(f) \) in the inductive limit topology as well. As outlined at the beginning of the proof, this implies that \( \phi_n \) extends to an isomorphism of \( C^*(G^\omega, n) \) onto \( C^*(G, \omega^n) \).

Define \( \Upsilon_n := \phi_n \circ \chi_n : C^*(G^\omega) \to C^*(G, \omega^n) \); then

\[
\Upsilon_n(F)(\gamma) = \int_T F(t, \gamma)t^n \, dt \quad \text{for } F \in C_c(G^\omega).
\]

**Theorem 3.2.** Let \( G \) be a second-countable, locally compact Hausdorff groupoid with a Haar system \( \lambda \). Let \( \omega : G^{(2)} \to \mathbb{T} \) be a continuous 2-cocycle and let \( G^\omega = G \times_\omega \mathbb{T} \) be the groupoid extension associated to \( (G, \omega) \). Then the map \( \Upsilon : C_c(G^\omega) \to \bigoplus_{n \in \mathbb{Z}} C_c(G, \omega^n) \) defined by \( F \mapsto (\Upsilon_n(F)) \) extends to an isomorphism of \( C^*(G^\omega) \) onto \( \bigoplus_{n \in \mathbb{Z}} C^*(G, \omega^n) \).

To prove Theorem 3.2 we first prove that the subalgebra \( I_n := \overline{C_c(G^\omega, n)} \| \cdot \|_{C^*(G^\omega)} \) is an ideal of \( C^*(G^\omega) \) which is isomorphic to \( C^*(G^\omega, n) \) and, second, that \( C^*(G^\omega) \) is the (internal) direct sum of the \( I_n \).

**Lemma 3.3.** Let \( G^\omega \) be the groupoid extension associated to \( (G, \omega) \).

(a) For \( f \in C_c(G^\omega, n) \), \( \|f\|_{C^*(G^\omega)} = \|f\|_{C^*(G^\omega, n)} \)

(b) The map \( \chi_n : C_c(G^\omega, n) \subset C_c(G^\omega) \to C_c(G^\omega, n) \) extends to an isometry \( \chi_n \) of the subalgebra \( I_n \) of \( C^*(G^\omega) \) onto \( C^*(G^\omega, n) \).

(c) The quotient map \( \chi_n : C^*(G^\omega) \to C^*(G^\omega, n) \) is identically zero on \( I_m \) if \( n \neq m \).

**Proof.** (a) Fix \( f \in C_c(G^\omega, n) \). If \( \pi \in \text{Rep}(C_c(G^\omega, n)) \), then by Lemma 3.1(b), \( \pi \circ \chi_n \in \text{Rep}(C_c(G^\omega)) \). Since \( f = \chi_n(f) \) we have

\[
\|f\|_{C_c(G^\omega, n)} = \sup\{\|\pi(f)\| : \pi \in \text{Rep}(C_c(G^\omega, n))\} = \sup\{\|\pi \circ \chi_n(f)\| : \pi \in \text{Rep}(C_c(G^\omega, n))\} \leq \|f\|_{C^*(G^\omega)}.
\]

Conversely, if \( \rho \in \text{Rep}(C_c(G^\omega)) \), then \( \rho|_{C_c(G^\omega, n)} \) is also continuous in the inductive limit topology on \( C_c(G^\omega, n) \). Fix \( \epsilon > 0 \). Pick a representation \( \rho \) of \( C_c(G^\omega) \) such that \( \|f\|_{C^*(G^\omega)} < \|\rho(f)\| + \epsilon \). Then

\[
\|f\|_{C^*(G^\omega)} < \|\rho(f)\| + \epsilon = \|\rho|_{C_c(G^\omega, n)}(f)\| + \epsilon \leq \|f\|_{C^*(G^\omega, n)} + \epsilon.
\]

Thus \( \|f\|_{C^*(G^\omega)} \leq \|f\|_{C^*(G^\omega, n)} \) and \( \|f\|_{C^*(G^\omega)} = \|f\|_{C^*(G^\omega, n)} \), as desired.

(b) Fix \( g \in I_n \). Let \( \{f_i\} \subset C_c(G^\omega, n) \) be a sequence converging to \( g \). By (a), \( \|f_i\|_{C^*(G^\omega)} = \|\chi_n(f_i)\|_{C^*(G^\omega, n)} \), and hence \( \|g\|_{C^*(G^\omega)} = \|\chi_n(g)\|_{C^*(G^\omega, n)} \). So \( \chi_n \) is isometric on the subalgebra \( I_n \) of \( C^*(G^\omega) \). Furthermore, \( \chi_n|_{I_n} \) is onto since \( C_c(G^\omega, n) \) is dense in \( C^*(G^\omega, n) \). So \( \chi_n|_{I_n} \) is an isomorphism.

(c) This is a direct calculation.

**Lemma 3.4.** Let \( G^\omega \) be the groupoid extension associated to \( (G, \omega) \). For each \( n \in \mathbb{Z} \), \( I_n \) is an ideal in \( C^*(G^\omega) \). Furthermore, \( I_n I_m = \{0\} \) if \( n \neq m \).
Proof. Let \( f \in C_c(G^\omega) \) and \( g \in C_c(G^\omega, n) \). Then
\[
  f * g(s \cdot (t, \gamma)) = \int_G \int_T f(r, \eta) g((r, \eta)^{-1}(st, \gamma)) \, dr \, d\lambda^{rc}(\gamma)(\eta)
\]
\[
  = \int_G \int_T f(r, \eta) g(s^{-1}t(\eta, \eta^{-1})\omega(\eta^{-1}, \gamma), \eta^{-1} \gamma) \, dr \, d\lambda^{rc}(\gamma)(\eta)
\]
\[
  = \int_G \int_T f(r, \eta) g(s \cdot ((r, \eta)^{-1}(t, \gamma))) \, dr \, d\lambda^{rc}(\gamma)(\eta)
\]
\[
  = s^{-n} \int_G \int_T f(r, \eta) g((r, \eta)^{-1}(t, \gamma)) \, dr \, d\lambda^{rc}(\gamma)(\eta)
\]
\[
  = s^{-n} f * g(t, \gamma).
\]
Thus \( f * g \in C_c(G^\omega, n) \subset I_n \). Since \( C_c(G^\omega, n) \) is closed under involution \( g * f \in I_n \) as well. Since \( I_n \) is closed the above calculations show that \( I_n \) is an ideal in \( C^*(G^\omega) \).

To see that \( I_m I_n = \{0\} \) unless \( n = m \), let \( f \in C_c(G^\omega, m) \), \( g \in C_c(G^\omega, n) \). Then
\[
  f * g(t, \gamma) = \int_G \int_T f(r, \eta) g(r^{-1}t(\eta, \eta^{-1})\omega(\eta^{-1}, \gamma), \eta^{-1} \gamma) \, dr \, d\lambda^{rc}(\gamma)(\eta)
\]
\[
  = \int_G \int_T r^{-m} f(1, \eta) r^n g(t(\eta, \eta^{-1})\omega(\eta^{-1}, \gamma), \eta^{-1} \gamma) \, dr \, d\lambda^{rc}(\gamma)(\eta)
\]
\[
  = \int_G f(1, \eta) g(t(\eta, \eta^{-1})\omega(\eta^{-1}, \gamma), \eta^{-1} \gamma) \lambda^{rc}(\gamma)(\eta) \int_T r^{m-n} \, dr.
\]
\[
  = \begin{cases} 
    \int_G f(1, \eta) g(t(\eta, \eta^{-1})\omega(\eta^{-1}, \gamma), \eta^{-1} \gamma) \lambda^{rc}(\gamma)(\eta) & \text{if } m = n \\
    0 & \text{otherwise}
  \end{cases}
\]
\( \square \)

Notation 3.5. For \( f \in C_c(G) \), \( \psi \in C(T) \), denote by \( \psi \otimes f \) the function \( (t, \gamma) \mapsto \psi(t) f(\gamma) \). In particular, for fixed \( m \), we write \( s^m \otimes f \) for the function \( (t, \gamma) \mapsto t^m f(\gamma) \) in \( I_{-m} \).

Lemma 3.6. Let \( G^\omega \) be the groupoid extension associated to \((G, \omega)\). The span\( \{s^m \otimes f : m \in \mathbb{Z}, f \in C_c(G)\} \) is dense in \( C_c(G^\omega) \) in the inductive limit topology.

Proof. Fix \( F \in C_c(G^\omega) \) and \( \varepsilon > 0 \). Let \( U_1 \) and \( U_2 \) be open, relatively compact neighborhoods in \( T \) and \( G \), respectively, such that \( \text{supp} F \subset U_1 \times U_2 \). Because \( \omega \) is continuous, \( G^\omega = \mathbb{T} \times \omega G \) has the product topology, and the map \( t \mapsto F(t, \cdot) \) is in \( C_c(T, C_c(G)) \). So the support of \( t \mapsto F(t, \cdot) \) is contained in \( U_1 \). For each \( t \in U_1 \) let
\[
  W_t := \{ s \in T : \| F(s, \cdot) - F(t, \cdot) \|_\infty < \varepsilon / 2 \} \cap U_1.
\]
Then \( W_t \) is an open cover of the compact set \( \text{supp}(t \mapsto F(t, \cdot)) \), so there exists a finite subcover \( W_{t_1}, \ldots, W_{t_N} \). Let \{\( \psi_i \}_{i=1}^N \} be a partition of unity subordinate to this cover. Since \( \sum \psi_i(t) \leq 1 \) for all \( t \in T \),
\[
  \left\| \sum_{i=1}^N \psi_i(t) F(t_i, \cdot) - F(t, \cdot) \right\|_\infty = \left\| \sum_{i=1}^N \psi_i(t) F(t_i, \cdot) - \sum_{i=1}^N \psi_i(t) F(t, \cdot) \right\|_\infty
\]
\[
  \leq \sum_{i=1}^N \psi_i(t) \left\| F(t_i, \cdot) - F(t, \cdot) \right\|_\infty < \frac{\varepsilon}{2}.
\]
For $\gamma \in \bigcup_{i=1}^{N} \text{supp}(F(t_{i}, \cdot)) \subset U_{2}$, let $U_{\gamma}$ be the open set

$$U_{\gamma} := \{ \eta \in G : |F(t_{i}, \gamma) - F(t_{i}, \eta)| < \epsilon/2 \text{ for } 1 \leq i \leq N \} \cap U_{2}. \tag{1}$$

Since $\bigcup_{i=1}^{N} \text{supp}(F(t_{i}, \cdot))$ is compact there exists a finite subcover $U_{\gamma_{1}}, \ldots, U_{\gamma_{M}}$. Let $\{f_{j}\}_{j=1}^{M}$ be a partition of unity subordinate to this subcover. For $\gamma \in G$ and each $i \in \{1, \ldots, N\}$ we have

$$|\sum_{j=1}^{M} f_{j}(\gamma)F(t_{i}, \gamma_{j}) - F(t_{i}, \gamma)| = |\sum_{j=1}^{M} f_{j}(\gamma)F(t_{i}, \gamma_{j}) - \sum_{j=1}^{M} f_{j}(\gamma)F(t_{i}, \gamma)| \leq \sum_{j=1}^{M} f_{j}(\gamma)|F(t_{i}, \gamma_{j}) - F(t_{i}, \gamma)| < \frac{\epsilon}{2}. \tag{2}$$

Now set $F_{\epsilon} := \sum_{i,j=1}^{M,N} F(t_{i}, \gamma_{j})\psi_{i} \otimes f_{j}$, and note that $\text{supp} F_{\epsilon}$ is contained in $U_{1} \times U_{2}$ by construction. We have

$$\|F_{\epsilon} - F\|_{\infty} = \sup_{(t,\gamma)} \left\{ |\sum_{i,j} F(t_{i}, \gamma_{j})\psi_{i}(t)f_{j}(\gamma) - F(t, \gamma)| \right\} \leq \sup_{(t,\gamma)} \left\{ |\sum_{i,j} \psi_{i}(t)f_{j}(\gamma)F(t_{i}, \gamma_{j}) - \sum_{i} \psi_{i}(t)F(t_{i}, \gamma) + \sum_{i} \psi_{i}(t)F(t_{i}, \gamma) - F(t, \gamma)| \right\} \leq \sum_{i} \psi_{i}(t) \left\{ |\sum_{j} f_{j}(\gamma)F(t_{i}, \gamma_{j}) - F(t_{i}, \gamma)| + \frac{\epsilon}{2} \right\} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3}$$

We have now shown that $\text{span}\{\psi \otimes f : \psi \in C(\mathbb{T}), f \in C_{c}(G)\}$ is dense in $C_{c}(G^{\omega})$ in the inductive limit topology. Thus it follows from the Stone-Weierstrass theorem that $\text{span}\{s^{m} \otimes f : m \in \mathbb{Z}, f \in C_{c}(G)\}$ is dense in $C_{c}(G^{\omega})$. \hfill \Box

Lemmas 3.4 and 3.6 give:

**Proposition 3.7.** Let $G^{\omega}$ be the groupoid extension associated to $(G, \omega)$. Then $C^{*}(G^{\omega}) = \bigoplus_{n \in \mathbb{Z}} I_{n}$. \hfill \Box

**Proof of Theorem 3.2** Both $\chi_{n} : I_{n} \to C^{*}(G^{\omega}, n)$ and $\phi_{n} : C^{*}(G^{\omega}, n) \to C^{*}(G, \omega^{n})$ are isomorphisms by Lemmas 3.3 and 3.1. However, $\Upsilon_{n}|_{I_{n}} = \phi_{n} \circ \chi_{n}|_{I_{n}}$ is an isomorphism of $I_{n}$ onto $C^{*}(G, \omega^{n})$. But by Lemma 3.3(c), $\chi_{n}(I_{m}) = \{0\}$ if $n \neq m$, so Theorem 3.2 follows from Proposition 3.7. \hfill \Box

We now show that Theorem 3.2 leads to a general framework for deducing results about twisted groupoid $C^{*}$-algebras from untwisted ones. The basic idea is that many properties of groupoids are shared with their extensions by $\mathbb{T}$. We start with a general lemma. The **stabilizer subgroupoid** of a groupoid $G$ is $\{ \gamma \in G : r(\gamma) = s(\gamma) \}$ and $A_{u} := \{ \gamma \in G : r(\gamma) = u = s(\gamma) \}$ is the **stability subgroup** at $u$. \hfill \Box

**Lemma 3.8.** Let $G$ be a groupoid and $G^{\omega}$ be the extension associated to $(G, \omega)$. Let $A$ and $A^{\omega}$ be the respective stabilizer subgroupoids of $G$ and $G^{\omega}$. Then the map $(t, \gamma) \mapsto [\gamma]$ induces a homeomorphism and isomorphism of $G^{\omega}/A^{\omega}$ onto $G/A$. \hfill \Box

**Proof.** The stability subgroups of $G^{\omega}$ are $\mathbb{T} \times_{\omega} A_{u}$ where $A_{u}$ is the stability subgroup of $G$ at $u$. Thus $A^{\omega} = \bigcup_{u \in G(0)} \mathbb{T} \times_{\omega} A_{u} = \mathbb{T} \times_{\omega} A$. \hfill \Box
We will show that the map \( f : (t, \gamma) \mapsto [\gamma] \) induces a homeomorphism and isomorphism of \( G^\omega/A^\omega \) onto \( G/A \). Certainly \( f \) is a groupoid morphism and is continuous and surjective. If \( f(t, \gamma) = f(s, \delta) \), then there exists \( \alpha \in A_{s,\gamma}(\alpha) \) such that \( \gamma = \delta \alpha \). Then \( (t, \gamma) = (s, \delta)(s^{-1}t\omega(\delta, \alpha), \alpha) \) and \( (s^{-1}t\omega(\delta, \alpha), \alpha) \in A^\omega \). Hence \( [(t, \gamma)] = [(s, \delta)] \). So \( f \) induces a continuous bijection \( \tilde{f} : G^\omega/A^\omega \to G/A \). Similarly, the function \( g : G \to G^\omega/A^\omega \) defined by \( g(\gamma) = [(1, \gamma)] \) induces a continuous bijection \( \tilde{g} : G/A \to G^\omega/A^\omega \), and it is easy to check that \( \tilde{g} \) is the inverse of \( \tilde{f} \). Thus \( \tilde{f} \) is a homeomorphism.

\[ \square \]

Proposition 3.9. Let \( G \) be a principal groupoid and let \( G^\omega \) be the extension associated to a continuous 2-cocycle \( \omega : G^{(2)} \to \mathbb{T} \). Then

(a) \( C^*(G) \) has continuous trace if and only if \( C^*(G^\omega) \) has continuous trace; and

(b) if \( C^*(G) \) has continuous trace, then so does \( C^*(G, \omega) \).

Proof. \[ \square \] First suppose that \( C^*(G) \) has continuous trace. Since \( G \) is principal, [12] Theorem 2.3 implies that \( G \) is a proper groupoid. Now consider \( G^\omega \): since \( G \) is principal the stability subgroups of \( G^\omega \) are \( \mathbb{T} \times \{u\} \) where \( u \in G^{(0)} \), and the stabilizer subgroupoid is \( A^\omega = \mathbb{T} \times G^{(0)} \). In particular, the stability subgroups of \( G^\omega \) are all abelian, and \( u \mapsto \mathbb{T} \times \{u\} \) is continuous in the Fell topology on the set of closed subgroups of \( G \). By Lemma [3.8] the quotient groupoid \( G^\omega/A^\omega \) and \( G \) are homeomorphic, and hence \( G^\omega/A^\omega \) is proper. Now \( C^*(G^\omega) \) has continuous trace by [10] Theorem 1.1.

Conversely, suppose that \( C^*(G^\omega) \) has continuous trace. By Theorem [3.2] \( C^*(G) = C^*(G, \omega^0) \) is a direct summand of \( C^*(G^\omega) \). Hence \( C^*(G) \) has continuous trace by [15] Proposition 6.2.10).

Suppose that \( C^*(G) \) has continuous trace. Then \( C^*(G^\omega) \) has continuous trace by [3.3]. By Theorem [3.2] \( C^*(G, \omega) \) is a direct summand of \( C^*(G^\omega) \), and hence \( C^*(G, \omega) \) has continuous trace as well. \[ \square \]

Many properties are shared by \( G \) and \( G^\omega \): having a Haar system; being Cartan, proper or integrable, and any topological property of the orbit spaces. This gives the proposition below. Example [3.11] below shows that we cannot expect to extend Propositions [3.9] and [3.10] to non-principal groupoids \( G \).

Proposition 3.10. Let \( G \) be a principal groupoid and let \( G^\omega \) be the extension associated to a continuous 2-cocycle \( \omega : G^{(2)} \to \mathbb{T} \).

(a) \( C^*(G) \) is a Fell algebra if and only if \( C^*(G^\omega) \) is a Fell algebra. If \( C^*(G) \) is a Fell algebra, then so is \( C^*(G, \omega) \).

(b) \( C^*(G) \) has bounded trace if and only if \( C^*(G^\omega) \) has bounded trace. If \( C^*(G) \) has bounded trace, then so does \( C^*(G, \omega) \).

(c) \( C^*(G) \) is liminal if and only if \( C^*(G^\omega) \) is liminal. If \( C^*(G) \) is liminal, then so is \( C^*(G, \omega) \).

(d) \( C^*(G) \) is postliminal if and only if \( C^*(G^\omega) \) is postliminal. If \( C^*(G) \) is postliminal, then so is \( C^*(G, \omega) \).

Proof. Since \( G \) is principal, the stability subgroups of \( G^\omega \) are \( \mathbb{T} \times \{u\} \) where \( u \in G^{(0)} \); in particular they are abelian and vary continuously.

\[ \square \] and \[ \square \] We can proceed as in the proof of Proposition [3.9] replacing [10] Theorem 1.1] with [3.1] Theorem 6.5] and [4] Theorem 6.4], respectively.
First suppose that $C^*(G)$ is liminal. Since the stability subgroups of $G$ are trivial, the orbit space $G^{(0)}/G$ is $T_1$ by \cite{3} Theorem 6.1. But the orbit space of $G^\omega$ is homeomorphic to the orbit space of $G$ via $[(1, u)] \mapsto [u]$, hence is $T_1$ as well. Since the stability subgroups of $G^\omega$ are amenable and liminal, $C^*(G^\omega)$ is liminal by \cite{3} Theorem 6.1.

Second, suppose that $C^*(G^\omega)$ is liminal. By Theorem \ref{thm:decomposition}, $C^*(G) = C^*(G, \omega^0)$ is a direct summand of $C^*(G^\omega)$. Hence $C^*(G)$ is liminal by \cite{15} Proposition 6.2.9.

This gives the first statement of (c).

Finally, suppose that $C^*(G)$ is liminal. Then $C^*(G^\omega)$ is liminal. By Theorem \ref{thm:decomposition}, $C^*(G, \omega)$ is a direct summand of $C^*(G^\omega)$, and hence $C^*(G, \omega)$ must be liminal as well. This gives the second statement of (c).

\cite{3} Theorem 7.1 of \cite{3} says that the groupoid $C^*$-algebra of a groupoid with amenable stability subgroups is postliminal if and only if the orbit space is $T_0$ and the stability subgroups are postliminal. So (c) follows as above using \cite{3} Theorem 7.1 in place of \cite{3} Theorem 6.1.

\hfill $\square$

\textbf{Example 3.11.} When the groupoid $G$ is not principal, the stability subgroups of $G^\omega$ can easily fail to be abelian, liminal or postliminal even if the stability subgroups of $G$ are abelian, liminal or postliminal, respectively. Thus the theorems used to prove Propositions \ref{prop:liminal} and \ref{prop:postliminal}, such as \cite{10} Theorem 1.1 and \cite{3} Theorem 6.1, do not apply. The following is an example of a group $G$ and a 2-cocycle $\omega$ such that $C^*(G)$ has continuous trace but $C^*(G, \omega)$ and $C^*(G^\omega)$ are not even postliminal. Thus we cannot expect an analog of Proposition \ref{prop:liminal} when the groupoid $G$ is not principal.

Let $\theta \in (0, 1)$ be irrational and define $\omega : \mathbb{Z}^2 \times \mathbb{Z}^2 \to T$ by $\omega((m_1, m_2), (n_1, n_2)) = e^{-2\pi i m_1 n_2 \theta}$. The twisted group $C^*$-algebra $C^*(\mathbb{Z}^2, \omega)$ is isomorphic to the irrational rotation algebra $A_\theta = C(T) \rtimes \mathbb{Z}$ (see, for example, \cite{5} pp. 21-22). Since $\theta$ is irrational the orbit space $T/\mathbb{Z}$ is not $T_0$, and hence $A_\theta$ is not postliminal by \cite{8} Theorem 3.3. Thus $C^*(\mathbb{Z}^2, \omega)$ is not postliminal. By Theorem \ref{thm:decomposition}, $C^*(\mathbb{Z}^2, \omega)$ is a summand of $C^*((\mathbb{Z}^2)^\omega)$, so $C^*((\mathbb{Z}^2)^\omega)$ is not postliminal either. Thus $C^*(\mathbb{Z}^2, \omega)$ and $C^*((\mathbb{Z}^2)^\omega)$ are not postliminal even though $C^*(\mathbb{Z}^2) \cong C(T^2)$ has continuous trace.

\section{A reduced version of the decomposition theorem}

The goal of this section is to prove a version of Theorem \ref{thm:decomposition} for reduced crossed products. Let $E$ be a second-countable locally compact Hausdorff groupoid with a Haar system $\beta$, and $\sigma : E^{(2)} \to T$ be a continuous 2-cocycle. Let $u \in E^{(0)}$. The \textit{left-regular representation} $\Pi_u$ is the representation of $C_c(E, \sigma)$ on $L^2(E, \beta_u)$ characterized by

\begin{equation}
\langle \Pi_u(f)\xi, \zeta \rangle = \int_E \int_E f(\gamma \eta)\xi(\eta^{-1}) \overline{\zeta(\gamma)} \sigma(\gamma \eta, \eta^{-1}) \, d\beta_u(\gamma) \, d\beta_u(\eta)
\end{equation}

for $f \in C_c(E, \sigma)$ and $\xi, \zeta \in L^2(E, \beta_u)$. Since $\Pi_u$ is continuous in the inductive limit topology, it extends to a representation of $C^*(E, \sigma)$. The reduced $C^*$-algebra $C^*_r(E, \sigma)$ of $(E, \sigma)$ is the completion of $C_c(E, \sigma)$ with respect to the norm $\|f\|_r = \sup_{u \in E^{(0)}} \{\|\Pi_u(f)\|\}$. Alternatively, $C^*_r(E, \sigma) = C^*(E, \sigma)/I$, where $I = \bigcap_{u \in E^{(0)}} \ker(\Pi_u)$; we write $q = q_e$ for the quotient map.
Theorem 4.1. Let $G$ be a second-countable, locally compact Hausdorff groupoid with a Haar system $\lambda$. Let $\omega : G^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle and $G^\omega$ be the extension associated to $(G, \omega)$. Let $\Upsilon : C^*(G^\omega) \to \bigoplus_{n \in \mathbb{Z}} C^*(G, \omega^n)$ and $\Upsilon_n : C^*(G^\omega) \to C^*(G, \omega^n)$ be as in Theorem 3.2. Then there exists a homomorphism $\Omega_n : C^*_r(G^\omega) \to C^*_r(G, \omega^n)$ such that the diagram

\begin{equation}
\begin{array}{ccc}
C^*(G^\omega) & \xrightarrow{\Upsilon_n} & C^*(G, \omega^n) \\
\downarrow q_{G^\omega} & & \downarrow q_{G, n} \\
C^*_r(G^\omega) & \xrightarrow{\Omega_n} & C^*_r(G, \omega^n)
\end{array}
\end{equation}

commutes. Furthermore, the map $\Omega : C^*_r(G^\omega) \to \bigoplus_{n \in \mathbb{Z}} C^*_r(G, \omega^n)$, defined by $a \mapsto (\Omega_n(a))$, is an isomorphism.

For $n \in \mathbb{Z}$ and $u \in G^{(0)}$, we write $L^u_n$ for the left-regular representation of $C^*(G, \omega^n)$ on $L^2(G, \lambda_u)$ and $R^u_n$ for the left-regular representation of $C^*(G^\omega)$ on $L^2(G^\omega, \tau \times \lambda_u)$; both are characterized by (4.1).

Lemma 4.2. Let $G^\omega$ be the extension associated to $(G, \omega)$. Let $u \in G^{(0)}$. For $n \in \mathbb{Z}$ define

\[ \mathcal{H}^u_n := \overline{\text{span}}\{ s^{-n} \otimes \xi : \xi \in C_c(G) \} \subset L^2(G^\omega, \tau \times \lambda_u). \]

(a) If $m \neq n$, then $\mathcal{H}^u_m$ is orthogonal to $\mathcal{H}^u_n$.

(b) There is a unitary $V_n : L^2(G, \lambda_u) \to \mathcal{H}^u_n$ such that

\begin{equation}
V_n(\xi) = s^{-n} \otimes \xi \quad \text{for} \quad \xi \in C_c(G).
\end{equation}

(c) There is a unitary $V : \bigoplus_{n \in \mathbb{Z}} L^2(G, \lambda_u) \to L^2(G^\omega, \tau \times \lambda_u)$ characterized by

\[ V((\xi_n)) = \bigoplus_{n \in \mathbb{Z}} V_n(\xi_n) \quad \text{for} \quad \xi_n \in C_c(G). \]

(d) For $n \in \mathbb{Z}$ let $L^u_n : C^*(G, \omega^n) \to B(L^2(G, \lambda_u))$ and $R^u_n : C^*(G^\omega) \to B(L^2(G^\omega, \tau \times \lambda_u))$ be the respective left-regular representations, and set $L^u = \bigoplus_{n \in \mathbb{Z}} L^u_n$. Then

\[ V(L^u \circ \Upsilon(a))V^* = R^u(a) \quad \text{for all} \quad a \in C^*(G^\omega). \]

Proof. We compute:

\begin{align*}
\langle r^{-m} \otimes \xi, r^{-n} \otimes \zeta \rangle_{L^2(G^\omega)} &= \int_G \int_T r^{-m} \otimes \xi(t, \gamma) \overline{r^{-n} \otimes \zeta(t, \gamma)} \, dt \, d\lambda_u(\gamma) \\
&= \int_G \int_T t^{-m} \xi(\gamma) \overline{\zeta(\gamma)} \, dt \, d\lambda_u(\gamma) \\
&= \langle \xi, \zeta \rangle_{L^2(G)} \delta_{m,n}.
\end{align*}

Now (4.4) implies, first, that $\mathcal{H}^u_m$ is orthogonal to $\mathcal{H}^u_n$ and, second, that there is an isometry $V_n$ satisfying (4.3). By definition of $\mathcal{H}^u_n$, $V_n$ is onto and hence is unitary. This gives (a) and (b).

By Lemma 3.6, $\text{span}\{ r^m \otimes \xi : m \in \mathbb{Z}, \xi \in C_c(G) \}$ is dense in $C_c(G^\omega)$ in the inductive limit topology, and hence it is dense in $L^2(G^\omega, \tau \times \lambda_u)$ as well. Now (c) follows from (a) and (b).
For (d), let \( m, n \in \mathbb{Z} \), \( \xi, \zeta \in C_c(G) \) and \( F \in C_c(G^\omega) \). Then, using Fubini’s Theorem several times,
\[
\langle R^u(F)(r^{-m} \otimes \xi), r^{-n} \otimes \zeta \rangle_{L^2(G^\omega)} = \int_G \int_T \int_G \int_T F((t, \gamma)(s, \eta))(r^{-m} \otimes \xi)((s, \eta)^{-1})r^{-n} \otimes \zeta(t, \gamma) \, ds \, d\lambda^u(\eta) \, dt \, d\lambda_\eta(\gamma)
\]
\[
= \int_G \int_G \int_T \int_T F(st\omega(\gamma, \eta), \gamma, \eta)s^m m^{\omega(\gamma, \eta)} m^{\omega(\gamma, \eta)^{-1}} m^{\xi(\eta^{-1})} t^n \zeta(\gamma) \, ds \, dt \, d\lambda^u(\eta) \, d\lambda_\eta(\gamma),
\]
and replacing \( s \) with \( st^{-1}(\omega(\gamma, \eta) \rangle \) give
\[
\int_G \int_G \int_T \int_T F(s, \gamma s^m t^{-m} m^{\omega(\gamma, \eta)} m^{\omega(\gamma, \eta)^{-1}} m^{\xi(\eta^{-1})} t^n \zeta(\gamma) \, ds \, dt \, d\lambda^u(\eta) \, d\lambda_\eta(\gamma),
\]
which, because \( \omega(\gamma, \eta) \omega(\gamma, \eta^{-1}) = \omega(\gamma, \eta^{-1}) \omega(\gamma, \eta^{-1}) = \omega(\gamma, \eta^{-1}) \), becomes
\[
\int_G \int_G \left( \int_T F(s, \gamma s^m) ds \right) \left( \int_T t^{-m} dt \right) \omega(\gamma, \eta^{-1}) m^{\xi(\eta^{-1})} t^n \zeta(\gamma) \, d\lambda^u(\eta) \, d\lambda_\eta(\gamma)
\]
\[
= \delta_{m,n} \int_G \int_G \mathcal{Y}_m(F) \xi(\eta^{-1}) \zeta(\gamma) \omega(\gamma, \eta^{-1}) m^{\lambda^u(\eta)} \, d\lambda_\eta(\gamma)
\]
\[
= \delta_{m,n} \langle L^u_m(\mathcal{Y}_m(F)) \xi, \zeta \rangle_{L^2(G^\omega)}.
\]
Since \( C_c(G^\omega) \) is dense in \( C^*(G^\omega) \), it follows that for \( a \in C^*(G^\omega) \),
\[
\langle R^u(a) \left( \sum r^{-m} \otimes \xi_m \right), \sum r^{-n} \otimes \zeta_n \rangle_{L^2(G^\omega)} = \sum_{m,n} \langle L^u_m(\mathcal{Y}_m(a)) \xi_m, \zeta_n \rangle_{L^2(G^\omega)} = \sum_{n} \langle L^u_n(\mathcal{Y}_n(a)) \xi_n, \zeta_n \rangle_{L^2(G^\omega)}.
\]
So for \( x = \sum s^m \otimes \xi_m, y = \sum s^n \otimes \zeta_n \) we have
\[
\langle R^u(a)x, y \rangle_{L^2(G^\omega)} = \sum_{n} \langle L^u_n(\mathcal{Y}_n(a)) \xi_n, \zeta_n \rangle_{L^2(G)} = \langle L^u(\mathcal{Y}(a))V^x, V^y \rangle_{\oplus_{n \in \mathbb{Z}} L^2(G^\omega)},
\]
and then (d) follows because the set of such \( x, y \) is dense in \( L^2(G^\omega, \tau \times \lambda_n) \). \( \square \)

**Proof of Theorem 4.1** By Lemma 4.2, we have \( \ker(R^u) \subset \ker(L^u_n \circ \mathcal{Y}_n) \) for all \( n \). Since this holds for all \( u \in G^{(0)} \), \( \ker(q_{G,\omega}) \subset \ker(q_{G,n}) \). Thus the map \( q_{G,\omega} \circ \mathcal{Y}_n \) induces a homomorphism \( \Omega_n \) such that the diagram (d) commutes.

To see that \( \Omega = (\Omega_n) \) is isometric, recall from Proposition 3.7 that \( C^*(G^\omega) = \bigoplus_{m \in \mathbb{Z}} I_m \) and let \( a = (a_n) \in C^*(G^\omega) \), where \( a_n \in I_n \). Using first Lemma 4.2 and, second, \( \mathcal{Y}_n = \mathcal{Y}|_{I_n} \), we get \( \| R^u(a) \| = \| L^u(\mathcal{Y}(a)) \| = \max_{n} \| L^u_n(\mathcal{Y}_n(a_n)) \|  \). Since this holds for all \( u \in G^{(0)} \),
\[
\| q_{G,\omega}(a) \|_{C^*_\tau(G^\omega)} = \max_{n} \| q_{G,n}(\mathcal{Y}_n(a_n)) \|_{C^*_\tau(G^\omega)} = \max_{n} \| \Omega_n(q_{G,\omega}(a_n)) \|_{C^*_\tau(G^\omega)} = \| \Omega(q_{G,\omega}(a)) \|_{C^*_\tau(G^\omega)}.
\]
Hence \( \Omega \) is isometric. That \( \Omega \) is surjective follows from the commutativity of the diagram since \( \mathcal{Y} = (\mathcal{Y}_n) \) and the quotient maps are surjective. Thus \( \Omega \) is an isomorphism. \( \square \)

**Corollary 4.3.** Let \( G^\omega \) be the extension associated to \( (G, \omega) \). If \( G \) is amenable, then \( C^*(G, \omega) = C^*_\tau(G, \omega) \).
Proof. Let $j : G^\omega \to G$ be the quotient map. Then $\ker j = T \times G^{(0)}$ is amenable. Since $G$ is amenable, Proposition 5.1.2 of [11] implies that $G^\omega$ is amenable. By Theorems 3.21 and 4.11 we have
\[ \bigoplus_{n \in \mathbb{Z}} C^*_r(G, \omega^n) \cong C^*_r(G^\omega) = C^*(G^\omega) \cong \bigoplus_{n \in \mathbb{Z}} C^*(G, \omega^n). \]
By the commutativity of [12] the summands corresponding to $n = 1$ match up, so the result follows. \qed

5. ACTIONS OF PROPER GROUPOIDS AND FIXED-POINT ALGEBRAS

Let $G$ be a principal proper groupoid. Then $G^\omega = T \times \omega G$ is also proper. There is an action $\text{lt}$ of $G^\omega$ on $C_0(G^{(0)})$ defined by
\[ \text{lt}_\gamma(f)(v) = f(s_{G^\omega}(\gamma)) \quad \text{for } f \in C_0(G^{(0)}) \text{ and } \gamma \in G^\omega \text{ with } r_{G^\omega}(\gamma) = v. \]
Since $G^\omega \\backslash G^{(0)} = G \\backslash G^{(0)}$, [12] Proposition 2.2 implies that $C_0(G^\omega \\backslash G^{(0)})$ is Morita equivalent to $C^*(G)$. Theorem 3.9 of [2] implies that $C_0(G^\omega \\backslash G^{(0)})$ is Morita equivalent to an ideal $I$ of $C^*(G^\omega)$. In the following proposition we reconcile these two results by using the decomposition of $C^*(G^\omega)$ into the direct sum $\bigoplus_{n \in \mathbb{Z}} C^*(G, \omega^n)$ to identify the ideal $I$ with the summand corresponding to $n = 0$.

**Proposition 5.1.** Let $G$ be a principal proper groupoid. Then the generalized fixed-point algebra $C_0(G^{(0)})^{\text{lt}} = C_0(G^\omega \\backslash G^{(0)})$ is Morita equivalent to the direct summand $C^*(G) = C^*(G, \omega^0)$ of $C^*(G^\omega)$.

**Proof.** Theorem 3.9 of [2] says that there is a $C^*$-subalgebra $I$ of the reduced groupoid crossed product $C_0(G^{(0)}) \times_{\text{lt}, r} G^\omega$ that is Morita equivalent to a generalized fixed-point algebra $C_0(G^{(0)})^{\text{lt}}$ of $(C_0(G^{(0)}), G^\omega, \text{lt})$. In our special case where the groupoid acts properly on its unit space, $I$ is an ideal by Remark 4.14 of [2]. By Proposition 4.1 of [2], $C_0(G^{(0)})^{\text{lt}} = C_0(G^\omega \\backslash G^{(0)})$. Combining [1] Corollary 2.1.7 and Proposition 3.3.5 gives that $G^\omega$ is measurewise amenable, and hence $C_0(G^{(0)}) \times_{\text{lt}, r} G^\omega = C_0(G^{(0)}) \times_{\text{lt}} G^\omega$ by [1] Proposition 6.1.10. By [7] Remark 4.22, $C_0(G^{(0)})^{\text{lt}}$ is isomorphic to $C^*(G^\omega)$. So $I$ is an ideal in $C^*(G^\omega)$ that is Morita equivalent to $C_0(G^\omega \\backslash G^{(0)})$; it remains to identify the ideal $I$.

The imprimitivity bimodule implementing the Morita equivalence is a completion of $C_c(G^{(0)})$ with respect to the left inner product given by
\[ \langle f, g \rangle (t, \gamma) = f(r_{G^\omega}(t, \gamma))g(s_{G^\omega}(t, \gamma)) = f(r_{G}(\gamma))g(s_{G}(\gamma)) \]
for $f, g \in C_c(G^{(0)})$ and $(t, \gamma) \in T \times \omega G$. The point is that the inner product is independent of $t$. Thus $I$ is an ideal of $C^*(G^\omega, 0) \cong C^*(G)$. When we apply Theorem 3.9 of [2] to the action $\text{lt}$ of $G$ on $C_0(G^{(0)})$ we obtain a Morita equivalence based on $C_c(G^{(0)})$ between an ideal $J$ of $C^*_r(G)$ and $C_0(G^{(0)})$, with left inner product given by
\[ \langle f', g' \rangle (\gamma) = f'(r_{G}(\gamma))g'(s_{G}(\gamma)) \]
for $f', g' \in C_c(G^{(0)})$ and $\gamma \in G$. Note that $C^*(G) = C^*_r(G)$ by amenability. Comparing (5.1) and (5.2) shows that $I = J$. Finally, by [2] Theorem 5.9, $J = C^*(G)$. \qed

The action $\text{lt}$ is called saturated if the ideal $I$ of $C_0(G^{(0)}) \times_{\text{lt}, r} G^\omega$ is in fact $C_0(G^{(0)}) \times_{\text{lt}, r} G^\omega$. We note that in the situation of Proposition 5.1 the action is very far away from being saturated since it is just one summand in $\bigoplus_{n \in \mathbb{Z}} C^*(G, \omega^n)$. 

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