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AUTOEQUIVALENCES OF TORIC SURFACES

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Abstract. We show that the autoequivalence group of the derived category of any smooth projective toric surface is generated by the standard equivalences and spherical twists obtained from -2-curves. In many cases we give all relations between these generators. We also prove a close link between spherical objects and certain pairs of exceptional objects.

In this article, we study the derived category of any smooth, projective, toric surface or rather its group of autoequivalences. We give generators for each such group and in great, although not full, generality we are able to go further and write an explicit description of the group; see Theorem 1. Similar descriptions are only known for abelian varieties [O1] and for varieties with ample or anti-ample canonical bundle [BO].

We follow the philosophy established in the work of Bondal, Mukai, Orlov and others that the group of autoequivalences of a variety is highly influenced by the positivity of the (anti-)canonical bundle. In particular, varieties with a trivial canonical bundle possess the richest autoequivalences, while the group of autoequivalences for varieties at both ends of the spectrum (i.e. K_X ample or anti-ample) are minimal by the famous result of Bondal and Orlov [BO].

Our surfaces have a rather negative canonical bundle as $-K_X$ is big. Thus we expect rather few autoequivalences beyond the standard ones. However, toric surfaces can contain smooth rational curves of self-intersection -2, a simple example being the second Hirzebruch surface, and such curves give rise to spherical twists. We prove that these twists are essentially the only new autoequivalences which can occur: Theorem 1 is the general result and Theorem 2 gives the application to toric surfaces. At the opposite end of the surface classification, Ishii and Uehara [IU, Theorem 1.3] have already proved a corresponding statement: the only non-standard autoequivalences for smooth, projective surfaces of general type whose canonical model has at most A_n-singularities come from spherical twists associated to -2-curves. We draw heavily upon their results in this article. In many cases, however, we go further and describe all relations between the generators.

In Theorem 6, we prove a relationship between exceptional and spherical objects on a smooth, projective, toric surface. It is well-known that such surfaces come with an abundance of exceptional objects, including, for example, all line bundles. We

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link the rather few spherical objects to the wealth of exceptional ones by discussing exceptional presentations of spherical objects, i.e. exact triangles $E' \to E \to S$ with E', E exceptional and S spherical.

While toric surfaces form the main class of examples, our results actually hold in greater generality. In Section 5, we give some examples of non-toric rational surfaces where the group of autoequivalences can also be described.

1. Setup and results

Let X be a smooth, projective surface over an algebraically closed field $\mathbf{k}$. Denote its derived category by $\mathcal{D}(X) := D^b(\mathrm{Coh}(X))$; this is a $\mathbf{k}$-linear, triangulated category. See the textbook [Hu] for background on derived categories of varieties. For any two objects $A, B \in \mathcal{D}(X)$, we set $\mathrm{Hom}^\bullet(A,B) = \bigoplus_i \mathrm{Hom}(A, B[i])[-i]$; this is a complex of $\mathbf{k}$-vector spaces with trivial differential. Note that by our assumptions on X, the dimension of $\mathrm{Hom}^\bullet(A,B)$ is finite. Let ω_X denote the canonical bundle on X. Then $- \otimes \omega_X[2] \colon \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(X)$ is a Serre functor; i.e. there are canonical isomorphisms, bifunctorially in $A, B \in \mathcal{D}(X)$:

$$\mathrm{Hom}(A,B) \cong \mathrm{Hom}(B, A \otimes \omega_X[2])^*.$$

The *standard autoequivalences* form the subgroup of $\mathrm{Aut}(\mathcal{D}(X))$,

$$A(X) := (\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1],$$

where $\mathrm{Pic}(X)$ are the line bundle twists, $\mathrm{Aut}(X)$ surface automorphisms and $\mathbb{Z}[1]$ the shifts of complexes.

Sometimes, $\mathrm{Aut}(\mathcal{D}(X))$ is strictly larger than $A(X)$. For example, when X is an abelian surface there will always be the non-standard original Fourier-Mukai transform (see [Mu]). Another source for non-standard equivalences are the *spherical twists* T_S introduced in [ST]; we refer to [Hu, §8] for a concise presentation. These are built from *spherical objects* in $\mathcal{D}(X)$, i.e. objects $S \in \mathcal{D}(X)$ such that $\mathrm{Hom}^\bullet(S,S) = \mathbf{k} \oplus \mathbf{k}[-2]$ and $S \otimes \omega_X \cong S$. A crucial example is given by $S = \mathcal{O}_C$, where $C \subset X$ is a smooth, rational curve with self-intersection number $C^2 = -2$.

Let us introduce some notation:

$$\begin{aligned}
\Delta(X) &:= \{C \subset X \text{ irreducible } -2\text{-curve}\}, && \text{a possibly infinite set;}\\
\mathrm{Pic}_\Delta(X) &:= \langle \mathcal{O}_X(C) \mid C \in \Delta \rangle, && \text{as a subgroup of } \mathrm{Pic}(X);\\
B(X) &:= \langle \mathsf{T}_S \mid S \in \mathcal{D}(X) \text{ spherical} \rangle, && \text{a normal subgroup of } \mathrm{Aut}(\mathcal{D}(X)).
\end{aligned}$$

In [IU], Ishii and Uehara prove that for a smooth, projective surface of general type whose canonical model has at worst A_n-singularities, the autoequivalences are generated by $B(X)$ and the standard autoequivalences. The following theorem is a counterpart to this in the case where $-K_X$ is big; i.e. a sufficiently high power of $-K_X$ gives a birational map from X to a surface in projective space (see [La, Definition 2.2.1]). Under certain conditions, we can go further and describe the structure of the group of autoequivalences.

Theorem 1. *Let X be a smooth, projective surface and consider the following conditions:*

(1) *The anti-canonical bundle is big.*
(2) *The -2-curves on X form disjoint chains of type A.*
(3) $\mathrm{Pic}(X) \cong \mathrm{Pic}_\Delta(X) \oplus P$ *where P is an* $\mathrm{Aut}(X)$*-invariant complement.*

If X satisfies (1) *and* (2)*, then* $\mathrm{Aut}(\mathcal{D}(X))$ *is generated by* $\mathrm{Pic}(X)$*,* $\mathrm{Aut}(X)$*,* $\mathbb{Z}[1]$ *and* $B(X)$*. If X satisfies* (1)–(3)*, then there is the following decomposition of* $\mathrm{Aut}(\mathcal{D}(X))$*:*

$$\mathrm{Aut}(\mathcal{D}(X)) = B(X) \rtimes (P \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1].$$

Conditions (1)–(3) are satisfied by broad classes of surfaces, as the next two results show. To state Theorem 2, we need to introduce one further piece of terminology (see Section 4 for details): $\mathrm{Aut}(\Sigma(X))$ is the group of automorphisms of a fan Σ giving the toric surface X; this is a finite subgroup of $\mathrm{Aut}(X)$.

Theorem 2. *If X is a smooth, projective, toric surface, then conditions* (1) *and* (2) *of Theorem* 1 *are satisfied. All but three such surfaces admit a splitting of* $\mathrm{Pic}_\Delta(X) \subset \mathrm{Pic}(X)$*. An* $\mathrm{Aut}(X)$*-invariant complement exists if and only if an* $\mathrm{Aut}(\Sigma(X))$*-invariant complement exists.*

An $\mathrm{Aut}(\Sigma(X))$-invariant complement tautologically exists whenever there are no non-trivial fan automorphisms (the 'generic' case), yielding the full structure of $\mathrm{Aut}(\mathcal{D}(X))$ from Theorem 1. When $\mathrm{Aut}(\Sigma(X))$ is non-trivial, an invariant complement may or may not exist; in Section 4 we give examples of both possibilities.

Theorem 1 also applies to some non-toric surfaces. The result below is proved Section 5, where we also give examples of such surfaces meeting condition (3).

Theorem 3. *If X is a smooth, projective, rational surface with a* $\mathbf{k}^*$*-action such that all isotropy groups are either* 0 *or* $\mathbf{k}^*$*, then conditions* (1) *and* (2) *of Theorem* 1 *are satisfied.*

We end this section with a couple of remarks about Theorem 1.

Remark 4. On any smooth, projective surface X with $K_X \neq 0$, spherical objects are necessarily supported on curves. The relations $\mathrm{Pic}(X) \cap B(X) = \mathrm{Pic}_\Delta(X)$ and $\mathrm{Aut}(X) \cap B(X) = 1$ then hold [IU, §4]. Generally, $B(X)$ is a normal subgroup of $\mathrm{Aut}(\mathcal{D}(X))$. These properties hint at the semi-direct decomposition of $\mathrm{Aut}(\mathcal{D}(X))$ in Theorem 1, but there are two obstacles: our choice of $B(X)$ as the normal factor of $\mathrm{Aut}(\mathcal{D}(X))$, together with $\mathrm{Pic}_\Delta(X) \subset B(X)$, demands a splitting $\mathrm{Pic}(X) = P \oplus \mathrm{Pic}_\Delta(X)$. Next, the action of $\mathrm{Aut}(X)$ on $\mathrm{Pic}(X)$ forces P to be $\mathrm{Aut}(X)$-invariant. These two conditions are exactly the content of (3).

Remark 5. We state what is known about $B(X)$. If X is rational with $-K_X$ big, as it will be in the examples of Sections 4 and 5, then Δ is a finite set as follows, for example, from the fact that X is a Mori dream space [TVV, §2].

Let $\mathcal{C} = \bigcup_{C \in \Delta} C$ be the union of all -2-curves on X and $\mathcal{C} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_r$ be its decomposition into connected components. Let $B(X)|_{\mathcal{C}_l} \subset B(X)$ be the subgroup obtained from spherical objects supported on $\mathcal{C}_l$. Then one has $B(X) = B(X)|_{\mathcal{C}_1} \times \cdots \times B(X)|_{\mathcal{C}_r}$ since spherical twists corresponding to fully orthogonal objects commute. Ishii and Uehara [IU] give a minimal set of $|\mathcal{C}_l| + 1$ generators,

$$B(X)|_{\mathcal{C}_\ell} = \langle \mathsf{T}_{\mathcal{O}_C(-1)}, \mathsf{T}_{\omega_{\mathcal{C}_\ell}} \mid C \in \Delta_\ell \rangle = \langle \mathsf{T}_{\mathcal{O}_C(-1)}, \mathsf{T}_{\mathcal{O}_C} \mid C \in \Delta_\ell \rangle,$$

so that the second set of $2|\mathcal{C}_l|$ twists also generates; here $\Delta_\ell := \{C \in \Delta \mid C \subset \mathcal{C}_\ell\}$. We point out that the results of [IU] apply only to chains $\mathcal{C}_\ell$ of type A. Finally, by [IUU, Corollary 37], $B(X)|_{\mathcal{C}_l}$ is an affine braid group on $|\mathcal{C}_l|$ strands.

2. Exceptional and spherical objects

An object $E \in \mathcal{D}(X)$ is *exceptional* if $\mathrm{Hom}^\bullet(E,E) = \mathbf{k}$; i.e. it is as simple as possible from the point of view of the derived category. Toric and, more generally, rational surfaces carry many exceptional objects — enough to form full exceptional collections; see [HP]. On the contrary, spherical objects are rarer on the surfaces under study because they have to be supported on configurations of -2-curves. Therefore, it seems natural to wonder whether spherical objects can be expressed via exceptional ones.

Before stating the theorem, we recall that an *exceptional pair* consists of two exceptional objects $E', E \in \mathcal{D}(X)$ such that $\mathrm{Hom}^\bullet(E,E') = 0$. We will call (E',E) a *special exceptional pair* if it is an exceptional pair with $\mathrm{Hom}^\bullet(E',E) = \mathbf{k} \oplus \mathbf{k}[-1]$.

Theorem 6. *Let X be a smooth, projective surface and suppose $E', E, S \in \mathcal{D}(X)$ are objects which fit into an exact triangle $E' \to E \to S$.*

(i) *If (E',E) is a special exceptional pair, then* $\mathrm{Hom}^\bullet(S,S) = \mathbf{k} \oplus \mathbf{k}[-2]$.
(ii) *If E is exceptional, S is spherical and* $\mathrm{Hom}^\bullet(E,S) = \mathbf{k}$*, then (E',E) is a special exceptional pair.*
(iii) *If S is spherical and X is a rational surface satisfying conditions* (1) *and* (2) *of Theorem* 1*, then (E',E) can be chosen to be a special exceptional pair.*

Note that by Theorem 2, smooth, projective toric surfaces satisfy the conditions of (iii) for the above theorem.

Proof. For (i) and (ii), use the following diagram in $\mathcal{D}(\mathbf{k})$:

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(E',E') \arrow[r] & \mathrm{Hom}^\bullet(E',E) \arrow[r] & \mathrm{Hom}^\bullet(E',S) \\
\mathrm{Hom}^\bullet(E,E') \arrow[r] \arrow[u] & \mathrm{Hom}^\bullet(E,E) \arrow[r] \arrow[u] & \mathrm{Hom}^\bullet(E,S) \arrow[u] \\
\mathrm{Hom}^\bullet(S,E') \arrow[r] \arrow[u] & \mathrm{Hom}^\bullet(S,E) \arrow[r] \arrow[u] & \mathrm{Hom}^\bullet(S,S) \arrow[u]
\end{tikzcd}$$

A diagram chase around this diagram implies (i), and also (ii), using the assumption of sphericality on S to invoke Serre duality.

For claim (iii) we use [IU, Proposition 1.6], which states that the spherical twists of objects supported on a chain of -2-curves act transitively on these spherical objects. Using this, together with Theorem 1, we see that for any spherical object $S \in \mathcal{D}(X)$, there exists $\varphi \in \mathrm{Aut}(\mathcal{D}(X))$ such that $\varphi(S) \cong \mathcal{O}_C(a)$ for some $C \in \Delta(X)$ and $a \in \mathbb{Z}$. (As in Remark 5, knowing this property for a single A_n-chain is enough in order to apply it to X.)

Since X is assumed to be rational, line bundles are exceptional objects, and we get the exceptional presentation $\mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C$ for the sheaf $\mathcal{O}_C$. We may contract $\mathcal{O}_C$ in X to obtain a surface with a rational singularity. Choosing a smooth curve which goes through the singular point, its strict transform H in X will have $H.C = 1$. Then $\varphi^{-1}(\mathcal{O}_X(-C+aH)) \to \varphi^{-1}(\mathcal{O}_X(aH)) \to S$ is a presentation for S by exceptional objects.

Finally, the assertions that $\mathrm{Hom}^\bullet(E', E) = \mathbf{k} \oplus \mathbf{k}[-1]$ and $\mathrm{Hom}^\bullet(E, E') = 0$ follow at once from $\mathrm{Hom}^\bullet(\mathcal{O}_X, \mathcal{O}_X(-C)) = H^\bullet(\mathcal{O}_X(-C)) = 0$, together with $\mathrm{Hom}^\bullet(\mathcal{O}_X(-C), \mathcal{O}_X) = H^\bullet(\mathcal{O}_X(C)) = \mathbf{k} \oplus \mathbf{k}[-1]$. □

Example 7. Part (i) of the theorem states that S satisfies the Ext-condition for spherical objects. However, it can happen that $S \ncong S \otimes \omega_X$, and so S is not spherical. As a specific example, consider F_2, the second Hirzebruch surface. It contains a (unique) -2-curve $C \subset F_2$; hence the object $\mathcal{O}_C \in \mathcal{D}^b(F_2)$ is spherical. Let $\pi\colon X \to F_2$ be the blow up of F_2 in one (of the two) torus-invariant points on C. We denote by D the exceptional curve and by C' again the strict transform of C. Thus, $\pi^{-1}(C) = C' + D$.

The functor $\mathsf{L}\pi^*\colon \mathcal{D}^b(F_2) \to \mathcal{D}^b(X)$ is fully faithful, as follows from adjunction, the projection formula, and $\mathsf{R}\pi_*\mathcal{O}_X = \mathcal{O}_{F_2}$ (or see [Hu, Proposition 11.13]). Consider the triangle $\mathcal{O}_{F_2}(-C) \to \mathcal{O}_{F_2} \to \mathcal{O}_C$ in $\mathcal{D}^b(F_2)$. Here, $(\mathcal{O}_{F_2}(-C), \mathcal{O}_{F_2})$ is a special exceptional pair. Pulling back this triangle under $\mathsf{L}\pi^*$ yields $\mathcal{O}_X(-C' - D) \to \mathcal{O}_X \to \mathcal{O}_{C'+D}$. Since the pullback functor is fully faithful, $(\mathcal{O}_X(-C' - D), \mathcal{O}_X)$ is also a special exceptional pair. We have

$$\mathrm{Hom}^\bullet(\mathcal{O}_{C'+D}, \mathcal{O}_{C'+D}) = \mathbf{k} \oplus \mathbf{k}[-2],$$

from part (i) of Theorem 6 or from the full faithfulness of $\mathsf{L}\pi^*$. However, the sheaf $\mathcal{O}_{C'+D}$ is not invariant under twisting with ω_X: the curves C' and D on X are smooth and rational but of self-intersection -3 and -1, respectively.

3. Proof of Theorem 1

Before giving an outline of the proof, we recall the assumptions on the surface X:

(1) The anti-canonical bundle is big.
(2) The -2-curves on X form disjoint chains of type A.
(3) $\mathrm{Pic}(X) \cong \mathrm{Pic}_\Delta(X) \oplus P$, where P is an $\mathrm{Aut}(X)$-invariant complement.

Let $\mathcal{C} = \bigcup_{C \in \Delta} C$ be the union of all -2-curves on X and let $\mathcal{C} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_r$ be its partition into connected components. By assumption (2), each $\mathcal{C}_i$ is a chain of type A. Given any autoequivalence $\varphi \in \mathrm{Aut}(\mathcal{D}(X))$, we modify it in three steps until we arrive at a standard autoequivalence; for this, we need only conditions (1) and (2):

Step 1: Modify φ using $\mathrm{Aut}(X)$ and $\mathbb{Z}[1]$ such that points outside $\mathcal{C}$ are fixed.
Step 2: Show that the subcategory of objects supported on a chain $\mathcal{C}_i$ is preserved.
Step 3: Invoke the braid group action of Ishii and Uehara [IU] to modify further by spherical twists until all points are fixed.

At this stage, the resulting autoequivalence is standard, i.e. an element of $A(X)$. This proves that $\mathrm{Aut}(\mathcal{D}(X))$ is generated by $A(X)$ and $B(X)$. Finally we address the relations. It is here that we make use of condition (3):

Step 4: Prove the decomposition $\mathrm{Aut}(\mathcal{D}(X)) = B(X) \rtimes (P \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1]$.

Step 1. By a well-known result of Orlov [O1] there is a unique Fourier-Mukai kernel $P \in \mathcal{D}(X \times X)$ so that $\varphi \cong \mathsf{FM}_P$. As the anti-canonical sheaf is big by assumption, the conditions required for [Ka, Theorem 2.3(2)] hold. In particular, looking at the proof of this theorem we see that there exists an irreducible component $Z \subset \mathrm{supp}(P) \subset X \times X$ such that the restrictions to Z of the natural projections

$\pi_1, \pi_2 \colon X \times X \to X$ are surjective and birational. Also see [Hu, §6] for this. Following [IU], we set

$$q := \pi_2|_Z \circ \pi_1|_Z^{-1} \colon X \dashrightarrow X.$$

As X is a smooth surface, using [Ka, Lemma 4.2], we note that this birational map is a genuine isomorphism — any birational map between smooth surfaces is a sequence of blowups and blowdowns, but Kawamata's lemma shows that the birational map in question is an isomorphism in codimension 1.

Now we show that for any point $x \in X$, the support of $\varphi(k(x))$ is either the point $q(x)$ or a connected subset of $\mathcal{C}$. Note that $\operatorname{supp}(\varphi(k(x))$ must be connected, as the map $\operatorname{Hom}_{\mathcal{D}(X)}(k(x), k(x)) \to \operatorname{Hom}_{\mathcal{D}(X)}(\varphi(k(x)), \varphi(k(x)))$ is bijective. It is a general property of equivalences to commute with Serre functors, in particular $\varphi(k(x)) = \varphi(k(x) \otimes \omega_X) \cong \varphi(k(x)) \otimes \omega_X$, for any point x. As ω_X is a non-trivial line bundle, $\varphi(k(x))$ must have proper support, i.e. $\dim \varphi(k(x)) < 2$. Therefore $\varphi(k(x))$ is either supported at a point or supported on a union of curves. Suppose $C \subset \operatorname{supp}(\varphi(k(x)))$ is any irreducible curve contained in the support. Since $\omega_X|_C \otimes \varphi(k(x))|_C = (\omega_X \otimes \varphi(k(x)))|_C = \varphi(k(x))|_C$ and $\varphi(k(x))$ is supported on the curve C, we get $\omega_X|_C = \mathcal{O}_C$. Hence, $C \subset X$ is a curve with $K_X.C = 0$. Since $-K_X$ is big, it follows from Lemma 8 below that C is a smooth, rational curve with $C^2 = -2$. Now looking at the FM transform at the level of its support, we observe that

$$q(x) = \pi_2(Z \cap (\{x\} \times X)) \subseteq \pi_2(\operatorname{supp}(P) \cap (\{x\} \times X)) = \operatorname{supp}(\varphi(k(x))).$$

If $\varphi(k(x))$ is supported at a point, then this point must be $q(x)$. Otherwise we have shown that all components of $\operatorname{supp}(\varphi(k(x)))$ are -2-curves and so $q(x)$ is contained in some -2-curve C. As q is a surface automorphism, we find $x \in q^{-1}(C)$, another -2-curve. In particular this implies that if $x \in X \setminus \mathcal{C}$, then $\varphi(k(x))$ is supported at the point $q(x)$ and is therefore a shifted skyscraper sheaf of length 1:

$$\varphi(k(x)) = k(q(x))[i] = q_*(k(x))[i].$$

The integer i does not depend on x: for an equivalence between derived categories of smooth, projective schemes, mapping a skyscraper sheaf to a skyscraper sheaf is an open property; see [Hu, Corollary 6.14]. Hence, $\psi := q^* \circ \varphi[-i]$ is an autoequivalence of $\mathcal{D}(X)$ which fixes all skyscraper sheaves $k(x)$ for $x \in X \setminus \mathcal{C}$.

Step 2. We claim that ψ preserves $\mathcal{C}$, i.e. induces an autoequivalence of $\mathcal{D}_{\mathcal{C}}(X)$. Here, $\mathcal{D}_{\mathcal{C}}(X)$ is the full subcategory of $\mathcal{D}(X)$ consisting of objects whose support is contained in $\mathcal{C}$. In order to prove the claim, suppose that $A \in \mathcal{D}_{\mathcal{C}}(X)$. We need to show that $\operatorname{supp}(\psi(A)) \subseteq \mathcal{C}$. If there was $y \in \operatorname{supp}(\psi(A))$, $y \notin \mathcal{C}$, there would be a non-zero morphism $\psi(A) \to k(y)$. However, this would imply a non-zero map $A \to \psi^{-1}(k(y)) = k(y)$, in contradiction to the assumption $\operatorname{supp}(A) \subset \mathcal{C}$.

In fact we can see that ψ preserves each connected component $\mathcal{C}_i$. For this, consider a curve B whose self-intersection number is not -2; in particular, B is not contained in $\mathcal{C}$. If B does not meet the component $\mathcal{C}_i$, then the same is true for the transform; i.e. $\operatorname{supp}(\psi(\mathcal{O}_B))$ does not intersect $\mathcal{C}_i$, using same reasoning as in the previous paragraph. More generally, if B does not meet several of the components, then the same will be true for the transform. So if we can find enough curves B to separate the components of $\mathcal{C}$, then ψ has to preserve each of them. See Lemma 9 below for a proof of this fact.

Therefore we are in a position to use the 'Key Proposition' of Ishii and Uehara [IU] repeatedly on each chain of -2-curves: there exist an integer j and an autoequivalence $\Psi \in B(X)$ such that $\Psi \circ \psi$ sends every skyscraper sheaf $k(x)$ for $x \in \mathcal{C}$ to $k(y)[j]$ for some $y \in \mathcal{C}$. In [IU], only globally defined autoequivalences are used, so that the presence of several chains does not pose an obstacle.

Step 3. A well-known lemma of Bridgeland and Maciocia ([BM, 3.3]; see also [Hu, Corollary 5.23]) states that an autoequivalence permuting skyscraper sheaves of length 1 must be in $\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)$. Thus we get

$$\Psi[-j] \circ \psi = \Psi \circ q^* \circ \varphi[-i-j] \in \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X).$$

Hence $\mathrm{Aut}(\mathcal{D}(X))$ is indeed generated by $\mathrm{Aut}(X)$, $\mathrm{Pic}(X)$, $B(X)$ and $\mathbb{Z}[1]$.

Step 4. The relations $\mathrm{Aut}(X) \cap B(X) = 1$ and $\mathrm{Pic}(X) \cap B(X) = \mathrm{Pic}_\Delta(X)$ are proved in Lemma 4.14 and Proposition 4.18 of [IU]; note that we can treat each chain individually using Remark 5.

Now we assume that the embedding $\mathrm{Pic}_\Delta(X) \subset \mathrm{Pic}(X)$ splits and that there is a complement P fixed by $\mathrm{Aut}(X)$; this is condition (3) of Theorem 1. We get

$$\begin{aligned} A(X) &= \mathbb{Z}[1] \times (\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)) \\ &\cong \mathbb{Z}[1] \times \big((\mathrm{Pic}_\Delta(X) \rtimes \mathrm{Aut}(X)) \oplus (P \rtimes \mathrm{Aut}(X))\big). \end{aligned}$$

We thus have two subgroups of $\mathrm{Aut}(\mathcal{D}(X))$, namely $\mathbb{Z}[1] \times (P \rtimes \mathrm{Aut}(X))$ and the normal subgroup $B(X)$, which together generate $\mathrm{Aut}(\mathcal{D}(X))$ and whose intersection is trivial. Hence we obtain the desired semi-direct product decomposition, and the proof of Theorem 1 is finished, apart from the following lemmas.

Lemma 8. *Let X be a smooth, projective surface with $-K_X$ big. If C is an irreducible, reduced curve on X with $K_X.C = 0$, then C is a -2-curve, i.e. smooth and rational with $C^2 = -2$.*

Proof. A big divisor is pseudo-effective [La, Theorem 2.2.26]. Hence we can use Zariski's decomposition $-K_X = P + N$, where P and N are $\mathbb{Q}$-divisors with P nef, N effective and where P has zero intersection number with every prime divisor of N; furthermore N is also negative definite [La, Theorem 2.3.19]. The positive part P carries all the sections of $-K_X$ and is therefore big as well [La, Proposition 2.3.21]. Since P is big and nef, we get $P^2 > 0$ [La, Theorem 2.2.16].

Our next claim is that $K_X.C = 0$ implies $P.C = 0$: If C is a component of N, this is obvious from the Zariski decomposition. Otherwise we have $C.N \geq 0$ as N is effective. We also find $P.C \geq 0$ as P is nef. From $0 = (-K_X).C = (P+N).C \geq 0$ we deduce $P.C = 0$.

The Hodge index theorem yields $C^2 < 0$ since $P^2 > 0$ and $P.C = 0$. Finally, applying the adjunction formula with $K_X.C = 0$ and $C^2 < 0$ gives $\deg(K_C) = (K_X + C).C = C^2 < 0$. Riemann-Roch and duality imply $g(C) = 1 - \chi(\mathcal{O}_C) = 1 + \chi(\omega_C) = 1 + \deg(K_C)/2 \leq 0$; hence $g(C) = 0$. It follows that C is rational and smooth; see [BHPV, §II.11] for details. Using the adjunction formula again shows $C^2 = -2$. □

Lemma 9. *Let X be a projective surface such that all -2-curves appear in ADE-chains. Then for any two such chains, there exists a curve meeting one chain transversally and avoiding the other.*

Proof. Fix two different chains $\mathcal{C}$, $\mathcal{C}'$ of -2-curves. By assumption, these are disjoint. We contract $\mathcal{C}$ and $\mathcal{C}'$ to obtain a surface Y with two rational singularities y, y'. This is possible; i.e. Y is algebraic, since we are dealing with chains of -2-curves of type ADE (see [Ar, Theorem 2.7]). In fact, Y is projective since X was. Choosing an ample divisor of sufficiently large degree, we find a curve $B \subset Y$ going through y but missing y'. Its preimage under the contraction $X \to Y$ then has the desired property. $\square$

4. Toric surfaces and proof of Theorem 2

In this section, we will work with a smooth, projective, toric surface X. We start by fixing some notation and gathering a few well-known properties of toric surfaces that we will use later. As a general reference for the exposition below, we refer the reader to [Fu].

Let N be a rank 2 lattice and define $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. A toric surface X is specified by a fan Σ of (strongly convex rational polyhedral) cones in $N_{\mathbb{R}}$. We denote by $\Sigma(1)$ the set of rays (one dimensional cones) in Σ, by $\{v_i\}_{i\in\Sigma(1)} \subset N$ the set of primitive generators of the rays and by $\{D_i\}_{i\in\Sigma(1)}$ the set of torus invariant divisors corresponding to the rays; each D_i is an irreducible, torus-invariant curve.

We assume that the fan is complete (the support of Σ is $N_{\mathbb{R}}$), which (in the surface case) is equivalent to the property that X is projective. The variety X is smooth, and this is equivalent to the condition on the fan that for each two-dimensional cone σ, the generators of the rays of σ form a basis for N. Ordering the generators cyclicly, it follows that

$$\alpha_i v_i = v_{i-1} + v_{i+1} \qquad \forall i = 1, \ldots, |\Sigma(1)|$$

for some integers α_i. It can be shown that $-\alpha_i$ is the self-intersection number of D_i for each $i \in \Sigma(1)$. Since X is smooth, there is an exact sequence [Fu, §3.4]

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X) \to 0, \tag{1}$$

where we denote by $M := N^\vee$ the dual lattice of N. $\mathrm{Pic}(X)$ is a free abelian group, so $\mathrm{Pic}_\Delta(X)$ is the free abelian subgroup generated by $\Delta(X)$.

Lemma 10. *$\Delta(X)$ consists of a finite number of chains of type A.*

Proof. Let C be a curve in $\Delta(X)$. Using the exact sequence (1) we observe that C is linearly equivalent to a sum $\sum_{i\in\Sigma(1)} a_i D_i$ of torus invariant divisors indexed by the rays in the fan Σ of X. Since C is effective, we may choose this Weil divisor in such a way that it is also effective, so $a_i \geq 0$ for each $i \in \Sigma(1)$. Then

$$-2 = C.C = C.\Big(\sum_i a_i D_i\Big) = \sum_i a_i (C.D_i),$$

so there exists some $i \in \Sigma(1)$ such that $C.D_i < 0$. Since C and D_i are both irreducible curves, we conclude that $C = D_i$. Thus all curves in $\Delta(X)$ are torus invariant curves corresponding to rays of Σ. Such curves intersect if and only if the corresponding rays span a cone (see for example [Fu, §5.1]). By looking at the fan Σ which is supported on $N_{\mathbb{R}} \cong \mathbb{R}^2$ we see that the only possible configurations are a finite number of chains of type A or a single closed chain of type $\tilde{A}_{|\Sigma(1)|}$.

In order to see that this final possibility doesn't occur, note that if $D_i^2 = -2$, then $2v_i = v_{i-1} + v_{i+1}$, which in turn means that v_i lies on the line in $N_{\mathbb{R}}$ through

v_{i-1} and v_{i+1}. It is clear, however, that the generators of the rays of a complete fan cannot all be collinear. $\square$

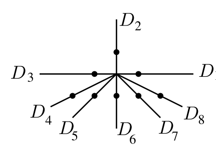
Lemma 11. *If X is a smooth, projective, toric variety (not necessarily a surface), then $-K_X$ is big.*

Proof. As is well-known (see [Fu, §4.3]), $-K_X$ is linearly equivalent to $\sum_{i\in\Sigma(1)} D_i$, the sum of all torus invariant prime divisors. Picking an ample divisor $H = \sum_i a_i D_i$, we can assume that all $a_i > 0$. Then $H + mK_X$ is effective for some $m > 0$ or, in other words, $-mK_X$ is the sum of an ample and an effective divisor, hence big. $\square$

Lemma 12. *If X is a smooth, projective, toric surface containing two divisors D_i, D_{i+1} corresponding to adjacent rays $i, i+1 \in \Sigma(1)$ with $D_i^2 \neq -2$ and $D_{i+1}^2 \neq -2$, then the group embedding $\operatorname{Pic}_\Delta(X) \subset \operatorname{Pic}(X)$ splits.*

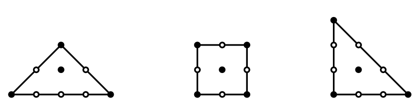
Proof. We use the standard exact sequence $0 \to M \xrightarrow{\iota} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\pi} \operatorname{Pic}(X) \to 0$. Since X is smooth, the generators v_i and v_{i+1} of the rays $i, i+1 \in \Sigma(1)$ form a basis of $N \cong \mathbb{Z}^2$. Using the dual basis for M and considering the map ι, it is easy to see that the free abelian group $\operatorname{Pic}(X)$ has a basis $\{\pi(D_j) \mid j \in \Sigma, j \neq i, i+1\}$. Furthermore, since $D_i^2 \neq -2$ and $D_{i+1}^2 \neq -2$, the subgroup spanned by classes of -2-curves is generated by elements of this basis and so is primitive in $\operatorname{Pic}(X)$. Hence, the quotient $\operatorname{Pic}(X)/\operatorname{Pic}_\Delta(X)$ is free and there exists a splitting. $\square$

Example 13. We now give an example of a smooth, toric surface X such that the embedding $\operatorname{Pic}_\Delta(X) \subset \operatorname{Pic}(X)$ of abelian groups does not split. Consider the toric surface given by the fan in the following picture:



It can be obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ four times. The self-intersection numbers are $D_2^2 = 0$, $D_4^2 = D_8^2 = -1$, and $D_1^2 = D_3^2 = D_5^2 = D_6^2 = D_7^2 = -2$. Choosing v_1 and v_2 as a basis of N, we see that the map $M \to \mathbb{Z}^{\Sigma(1)}$ is given by the transpose of the matrix $\left(\begin{smallmatrix} 1 & 0 & -1 & -2 & -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \end{smallmatrix}\right)$. In particular, the classes of $D_3, \ldots, D_8$ form a basis of $\operatorname{Pic}(X)$. All of these are -2-classes except for D_4 and D_8. Writing the -2-curve D_1 in terms of this basis, we have $D_1 = 2(D_8 - D_4) + D_3 + D_5 + D_7$ in $\operatorname{Pic}(X)$. Therefore $0 = 2(D_8 - D_4)$ in $\operatorname{Pic}(X)/\operatorname{Pic}_\Delta(X)$, so there is torsion. This implies that the embedding of $\operatorname{Pic}_\Delta(X)$ into $\operatorname{Pic}(X)$ is not primitive.

In fact, it is an easy combinatorial exercise to show that there are only three smooth, projective, toric surfaces which do not have such a splitting. They are given by the smooth fans over the following polygons — here and in the following, the vertices on the boundary of the polygon are the generators of the rays of the fan. Circular dots ($\circ$) indicate -2-curves.



Lemma 14. *If X is not one of the three surfaces in Example* 13*, then there exists an* $\mathrm{Aut}(X)$*-invariant complement P for the subgroup* $\mathrm{Pic}_\Delta(X)$ *in* $\mathrm{Pic}(X)$ *if and only if there exists an* $\mathrm{Aut}(\Sigma(X))$*-invariant complement.*

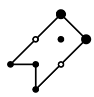
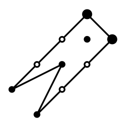
Proof. This follows at once from two geometric facts about toric varieties:

First, $\mathrm{Aut}(X)$ is generated by its identity component $\mathrm{Aut}^0(X)$ and the subgroup $\mathrm{Aut}(\Sigma(X))$ of fan automorphisms (the latter is by definition the subgroup of lattice automorphisms of N fixing the fan Σ). This statement is a corollary of Demazure's Structure Theorem [Oda, §3.4].

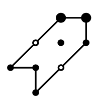
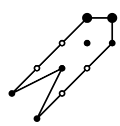
Second, $\mathrm{Aut}^0(X)$ acts trivially on all of $\mathrm{Pic}(X)$ because the Picard group of a toric variety is discrete, i.e. $\mathrm{Pic}^0(X) = 0$. □

Together, Lemmas 10, 12 and 14 prove all parts of Theorem 2. It remains to investigate when an $\mathrm{Aut}(\Sigma(X))$-invariant complement exists. For trivial reasons, this is always true if $\mathrm{Aut}(\Sigma(X)) = 1$. For more symmetric toric surfaces, both answers are possible, as the next two examples show.

Example 15. Suppose $\mathrm{Aut}(\Sigma(X)) = \mathbb{Z}/2$ and the action exchanges two rays which do not correspond to -2-curves and whose generators form a $\mathbb{Z}$-basis for N, for example the toric surfaces given by fans over the following polygons:

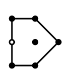



Excluding the two marked curves (•), the remaining torus invariant divisors form a basis for $\mathrm{Pic}(X)$, and the subset of these divisors which are not -2-curves generate an $\mathrm{Aut}(\Sigma(X))$-invariant complement to $\mathrm{Pic}_\Delta(X)$. Similarly, and again in the case $\mathrm{Aut}(\Sigma(X)) = \mathbb{Z}/2$, suppose there exists a $\mathbb{Z}$-basis for N coming from a ray which is fixed by the action and has odd self-intersection number and another ray which doesn't correspond to a -2-curve. For example, consider fans over the following polygons, where the basis for N is again marked:

It is then possible to show that there exists an invariant linear combination of the fixed divisor and the two divisors in the $\mathrm{Aut}(\Sigma(X))$-orbit of the non-fixed marked ray, which, together with all the remaining torus invariant divisors, forms a basis for $\mathrm{Pic}(X)$. Again, the subset of these divisors which are not -2-curves generates an $\mathrm{Aut}(\Sigma(X))$-invariant complement to $\mathrm{Pic}_\Delta(X)$.

Example 16. For the following example, computer algebra was used to make sure that no invariant complement exists. Note that the rays fixed by the $\mathrm{Aut}(\Sigma(X))$ action correspond to curves with even self-intersection number, so the argument in the previous example doesn't apply:



We conclude this section with a few general observations about, and on the construction of, some classes of examples. As a straightforward consequence of

Theorem 1, we see that any smooth, projective, toric surface without -2-curves has no autoequivalences beyond the standard ones, i.e. $\mathrm{Aut}(\mathcal{D}(X)) = A(X)$. We note that there are infinitely many examples of such surfaces including, for example, all Hirzebruch surfaces F_n for $n > 2$. It is not hard to check that $-K_X$ is ample if and only if there are no torus invariant curves of self-intersection -2 or lower. In fact, there are famously just five smooth toric Fano surfaces [Oda, Proposition 2.21]. Therefore, there are infinitely many smooth, projective, toric surfaces where $-K_X$ is not ample (and so are not covered by the theorem of Bondal and Orlov [BO]) but for which $\mathrm{Aut}(\mathcal{D}(X)) = A(X)$.

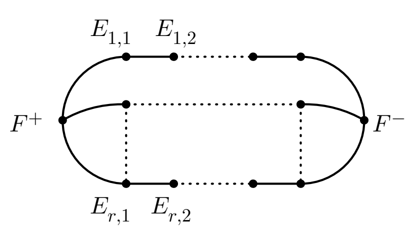
On the other hand, it is easy to construct examples with more interesting groups of autoequivalences. If v_0, v_1 form a basis for a rank two lattice N, then we can define inductively $v_{s+1} = 2v_s - v_{s-1}$ for $s = 1, \ldots, \ell$. Taking these as generators of rays of a fan, we can produce a complete smooth fan by adding extra rays (with generators $v_{\ell+2}, \ldots, v_{d-1}$), making sure we do not subdivide any of the existing cones. This doesn't affect the self-intersection numbers of $D_1, \ldots, D_\ell$, which are by construction -2. Indeed by making an appropriate choice of $v_{\ell+2}, \ldots, v_{d-1}$, we can ensure that D_0 and $D_{\ell+1}$ do not have self-intersection number -2. Therefore it is possible to construct a smooth, projective, toric surface with a chain of -2-curves of arbitary length. Blowing up the intersection point of two torus invariant curves D_s and D_{s+1} (which corresponds to subdividing the cone spanned by v_s and v_{s+1}) has the effect of reducing the self-intersection numbers of the strict transforms $\widetilde{D}_s$ and $\widetilde{D}_{s+1}$ by 1 and inserting an exceptional -1-curve. In this way we can split up a chain of -2-curves into pieces. In fact, we can produce any number of chains of -2-curves of any length.

5. Surfaces with $\mathbf{k}^*$-action and proof of Theorem 3

In this final section, we present some non-toric surfaces to which our results apply. These examples will be certain rational surfaces with $\mathbf{k}^*$-action. As references on such surfaces we use mainly the classical [OW] and also [PS].

Start with the trivially ruled surface $C \times \mathbb{P}^1$, where C is a smooth, projective curve of genus g. This surface inherits a $\mathbf{k}^*$-action from the natural action on $\mathbb{P}^1$, and the fixed points make up the two curves $F^+ := C \times \{0\}$ and $F^- := C \times \{\infty\}$.

Blowing up a fixed point produces another surface with $\mathbf{k}^*$-action. The exceptional divisor consists of fixed points, so that the process can be iterated. Likewise, all negative curves consist of fixed points and can thus be contracted to a surface with $\mathbf{k}^*$-action. By [OW, Theorem 2.5], all smooth surfaces obtained in this fashion have the following configuration of fixed curves, made up of of r arms, where we denote the curves in the ℓ-th arm by $E_{\ell,1}, E_{\ell,2}, \ldots$, starting from F^+:



In fact, all smooth, projective surfaces with an effective $\mathbf{k}^*$-action can be obtained in this way, where we allow F^+ or F^- to be contracted in case they are -1-curves, [OW, Proposition 2.6]. Let X be such a surface with associated graph as above.

By construction, the Néron-Severi group of X is generated by the $E_{\ell,i}$, F^+, F^- and D, the closure of a generic $\mathbf{k}^*$-orbit. Thus $D^2 = 0$, $D.F^{\pm} = 1$ and $D.E_{\ell,i} = 0$. Then, by [PS, Theorem 3.2.1], the anti-canonical divisor has the form $-K_X = F^+ + F^- + (2-2g-r)D + \sum_{\ell,i} E_{\ell,i}$.

We need to impose two conditions on X: First, all isotropy groups are connected; i.e. we exclude non-zero cyclic groups. Second, the surface is rational, i.e. $g = 0$ (but note that $\mathbf{k}^*$ still acts only on one factor of the original surface $\mathbb{P}^1 \times \mathbb{P}^1$).

Rationality implies $\mathrm{Pic}(X) = \mathrm{NS}(X)$. Furthermore, this is a free abelian group (see e.g. [PS, 3.15]). We proceed to verify the assumptions of Theorem 1.

Lemma 17. *If all isotropy groups are trivial, then -2-curves occur in chains of type A.*

Proof. [OW, 3.5] describes the isotropy groups from the intersection graph via continued fractions. The isotropy groups being trivial forces the sequence of self-intersection numbers of each arm to be $-1, -2, \ldots, -2, -1$. In particular, -2-curves can occur only in chains of type A (note that F^+ or F^- can also be -2-curves). □

Lemma 18. *Let X be rational with trivial isotropy groups. Then $-K_X$ is big.*

Proof. We start by showing that $D = \sum_i E_{\ell,i}$ in $\mathrm{Pic}(X)$, where ℓ is fixed; i.e. the divisor is given by the curves on any arm of the above graph. The curves intersect as follows: If $E_{\ell,i}$ meets $F \in \{F^+, F^-\}$, then $E_{\ell,i}.F = 1$ and $E_{\ell,i}^2 = -1$; otherwise $E_{\ell,i}.F^{\pm} = 0$ and $E_{\ell,i}^2 = -2$. Further, $D.F^{\pm} = E_{\ell,i}.E_{\ell,i+1} = 1$ and all other intersection products vanish. This implies $C.D = C.\sum_i E_{\ell,i}$ for C any of the curves $F^+, F^-, D, E_{k,j}$. Since those curves generate the Picard group, the divisors D and $\sum_i E_{\ell,i}$ are numerically equivalent. They are then also linearly equivalent, as there are no non-trivial line bundles of degree 0.

Thus we know $-K_X = F^+ + F^- + 2D$. This already implies that $-K_X$ is in the pseudo-effective cone of X. Furthermore, the relation also shows that all of $-mK_X - F^-$, $-mK_X - F^+$, $-mK_X - D$ and $-mK_X - E_{\ell,i}$ lie in the pseudo-effective cone if $m \gg 0$. Hence $-K_X$ sits in the interior of the pseudo-effective cone and is therefore big; see [La, Theorem 2.2.26]. □

Observe that a toric surface always has big $-K_X$ (see Lemma 11), whereas blowing up $\mathbb{P}^2$ in nine general points produces a rational surface on which $-K_X$ is not big anymore. Surfaces with $\mathbf{k}^*$-action do not necessarily have big anti-canonical class, and we need to restrict ourselves to examples with trivial isotropy in order to have this property.

Lemma 19. *If X is rational with trivial isotropy groups and such that not both F^+ and F^- are -2-curves, then the inclusion $\mathrm{Pic}_\Delta(X) \subset \mathrm{Pic}(X)$ splits.*

Proof. $\mathrm{Pic}(X)$ is the quotient of the free abelian group generated by F^+, F^-, D and the exceptional curves $E_{\ell,i}$, subject to the relations $F^+ - F^- + \sum_{\ell,i}(i-1)E_{\ell,i} + (F^-)^2 D = 0$ and $D = \sum_i E_{\ell,i}$ for all ℓ (see [PS, Corollary 3.5]).

In the quotient $\mathrm{Pic}(X)/\mathrm{Pic}_\Delta(X)$ we observe that $E_{\ell,i} = 0$ for all exceptional curves not adjacent to F^+ or F^-, as these are all -2-curves. Also, by assumption, at least one of F^+ and F^- will survive in the quotient.

Therefore, for each arm we have a relation in which the two remaining classes have coefficient 1 and one further relation in which the classes of F^+ and F^- have coefficient 1. It follows that the quotient $\mathrm{Pic}(X)/\mathrm{Pic}_\Delta(X)$ has a basis consisting of the curves $E_{\ell,1}$ for all ℓ, together with D and either F^+ or F^- (or neither if one of them is a -2-curve). $\square$

Remark 20. The numbers $(F^+)^2$ and $(F^-)^2$ are not arbitrary: assuming trivial isotropy, they can attain any values subject to the restriction $(F^+)^2 + (F^-)^2 = 2 - \mathrm{rk}(\mathrm{Pic}(X))$; see [OW, Theorem 2.5(iv)]. In particular, $(F^+)^2 = (F^-)^2 = -2$ forces $\mathrm{rk}(\mathrm{Pic}(X)) = 6$.

We give examples of surfaces with $\mathbf{k}^*$ which meet all conditions of Theorem 1:

Lemma 21. *Let X be rational with trivial isotropy groups and either $(F^+)^2 < -2$ or $(F^-)^2 < -2$. If all arms of the intersection graph have different lengths, then $\mathrm{Aut}(X)$ fixes $\mathrm{Pic}(X)$ element-wise.*

Proof. Without loss of generality, we may assume $(F^+)^2 < -2$. Since F^+ is a negative curve of minimal intersection number, it is fixed by all automorphisms. By the assumption on arm lengths, all other negative curves are also fixed. The remaining curve F^- is then likewise fixed. $\square$

We finish with a comment on the relationship between the two types of examples: a surface with $\mathbf{k}^*$-action as presented here will be toric (i.e. admit an action of $(\mathbf{k}^*)^2$ compatible with the original action) only if $r \leq 2$; cf. [OW, §4.2]. This leads to a circular intersection graph corresponding to the rays in the fan of a toric surface.

The first toric surface of Example 13 is a surface with $\mathbf{k}^*$-action which has no cyclic isotropy groups. This surface has $r = 1$, and both F^- and F^+ are -2-curves. The divisor D_2 of that example is the closure D of a generic $\mathbf{k}^*$-orbit mentioned in this section. By contrast, the surface given by the square polygon of Example 13 comes from a $\mathbf{k}^*$-surface with $r = 2$. By Remark 20, the third example does not lead to a surface with $\mathbf{k}^*$-action of trivial isotropy, as it has Picard rank 7.

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