ON THE CATEGORY OF COFINITE MODULES WHICH IS ABELIAN

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Abstract. Let $R$ denote a commutative Noetherian (not necessarily local) ring and $I$ an ideal of $R$ of dimension one. The main purpose of this paper is to generalize, and to provide a short proof of, K. I. Kawasaki’s theorem that the category $\mathcal{M}(R, I)_{cof}$ of $I$-cofinite modules over a commutative Noetherian local ring $R$ forms an Abelian subcategory of the category of all $R$-modules. Consequently, this assertion answers affirmatively the question raised by R. Hartshorne in his article Affine duality and cofiniteness [Invent. Math. 9 (1970), 145-164] for an ideal of dimension one in a commutative Noetherian ring $R$.

1. Introduction

Let $R$ denote a commutative Noetherian ring, and let $I$ be an ideal of $R$. In [4], Hartshorne defined an $R$-module $L$ to be $I$-cofinite if $\text{Supp}(L) \subseteq V(I)$ and $\text{Ext}_R^i(R/I, L)$ is finitely generated module for all $i$. He posed the following question:

Does the category $\mathcal{M}(R, I)_{cof}$ of $I$-cofinite modules form an Abelian subcategory of the category of all $R$-modules? That is, if $f : M \to N$ is an $R$-module homomorphism of $I$-cofinite modules, are $\ker f$ and $\text{coker } f$ $I$-cofinite?

Hartshorne gave the following counterexample (see [3]): Let $k$ be a field and let $R = k[[x, y, z, u]]/(xy - uz)$. Set $M = R/(xy - uv)R$ and $I = (x, u)R$. Then applying functor $H^0_I(-)$ to the exact sequence

$$0 \to R \xrightarrow{x y - u v} R \to M \to 0,$$

we obtain the exact sequence

$$\cdots \to H^2_I(R) \xrightarrow{f} H^2_I(R) \to H^2_I(M) \to 0.$$

Since $H^i_I(R) = 0$ for all $i \neq 2$, one can show that

$$\text{Ext}^i_R(R/I, H^2_I(R)) \cong \text{Ext}^{i+2}_R(R/I, R),$$

for all $i$. Thus, $H^2_I(R)$ is $I$-cofinite. However, $\text{coker } f = H^2_I(M)$ is not $I$-cofinite. On the positive side, Hartshorne proved that if $I$ is a prime ideal of dimension one in a complete regular local ring $R$, then the answer to his question is yes. On the other hand, in [3], Delfino and Marley extended this result to arbitrary complete
local rings. Recently, Kawasaki [6] generalized Delfino and Marley’s result for an arbitrary ideal $I$ of dimension one in a local ring $R$. Kawasaki’s proof relies on a spectral sequence, and several pages of his paper are devoted to a proof of that theorem. See also [7] and [8].

The main purpose of this paper is to generalize and to present a much shorter proof of Kawasaki’s theorem, using somewhat more elementary methods than those used by Kawasaki. More precisely, we shall show that:

**Theorem 1.1.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$ of dimension one. Let $\mathcal{M}(R, I)_{cof}$ denote the category of $I$-cofinite modules. Then $\mathcal{M}(R, I)_{cof}$ forms an Abelian subcategory of the category of all $R$-modules.

One of our tools for proving Theorem 1.1 is the following, which is a generalization of a result of Melkersson (cf. [11] Proposition 4.3]).

**Proposition 1.2.** Let $I$ denote an ideal of a Noetherian ring $R$ and let $M$ be an $R$-module such that $\dim M \leq 1$ and $\operatorname{Supp}(M) \subseteq V(I)$. Then $M$ is $I$-cofinite if and only if the $R$-modules $\operatorname{Hom}_R(R/I, M)$ and $\operatorname{Ext}_R^1(R/I, M)$ are finitely generated.

Throughout this paper, $R$ will always be a commutative Noetherian ring with non-zero identity and $I$ will be an ideal of $R$. For an Artinian $R$-module $A$, we denote by $\operatorname{Att}_R A$ the set of attached prime ideals of $A$. For each $R$-module $L$, we denote by $\operatorname{Assh}_R L$ the set $\{p \in \operatorname{Ass}_R L : \dim R/p = \dim L\}$. We shall use $\operatorname{Max} R$ to denote the set of all maximal ideals of $R$. Also, for any ideal $a$ of $R$, we denote $\{p \in \operatorname{Spec} R : p \supseteq a\}$ by $V(a)$. Finally, for any ideal $b$ of $R$, the radical of $b$, denoted by $\operatorname{Rad}(b)$, is defined to be the set $\{x \in R : x^n \in b \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [2] and [9].

Recall that a module is called a minimax module when it has a finitely generated submodule, such that the quotient by it is an Artinian module [12].

### 2. The Results

Let us first recall the important concept of the arithmetic rank of an ideal. The arithmetic rank of an ideal $b$ in a commutative Noetherian ring $T$, denoted by $\operatorname{ara}(b)$, is the least number of elements of $T$ required to generate an ideal which has the same radical as $b$, i.e.,

$$\operatorname{ara}(b) = \min\{n \in \mathbb{N}_0 : \exists b_1, \ldots, b_n \in T \text{ with } \operatorname{Rad}(b_1, \ldots, b_n) = \operatorname{Rad}(b)\}.$$ 

Let $M$ be a $T$-module. The arithmetic rank of an ideal $b$ of $T$ with respect to $M$, denoted by $\operatorname{ara}_M(b)$, is defined by the arithmetic rank of the ideal $b + \operatorname{Ann}_T M/ \operatorname{Ann}_T M$ in the ring $T/ \operatorname{Ann}_T M$.

The main point of this note is to generalize and to provide a short proof the main result of Kawasaki [6 Theorem 1] concerning a question raised by R. Hartshorne. The following proposition plays a key role in the proof of that theorem. Before we state Proposition 2.6, we recall some lemmas that we will use in the proof of this proposition.

**Lemma 2.1.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Then, for any $R$-module $T$, the following conditions are equivalent:

(i) $\operatorname{Ext}_R^n(R/I, T)$ is finitely generated for all $n \geq 0$.

(ii) $\operatorname{Ext}_R^n(N, T)$ is finitely generated for all $n \geq 0$ and for each finitely generated $R$-module $N$ for which $\operatorname{Supp} N \subseteq V(I)$. 

Proof. See [5, Lemma 1].

**Lemma 2.2.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Let $x \in I$ and $M$ be an $R$-module such that $\text{Supp} \ M \subseteq V(I)$. If $(0 :_M x)$ and $M/xM$ are $I$-cofinite, then $M$ is also $I$-cofinite.

Proof. See [11, Corollary 3.4].

**Lemma 2.3.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Let $M$ be a minimax $R$-module such that $\text{Supp} \ M \subseteq V(I)$. Then $M$ is $I$-cofinite if and only if $(0 :_M I)$ is finitely generated.

Proof. See [10, Proposition 4.3].

**Lemma 2.4.** Let $(R, \mathfrak{m})$ be a local (Noetherian) ring and let $A$ be an Artinian $R$-module. Let $I$ be an ideal of $R$ such that the $R$-module $\text{Hom}_R(R/I, A)$ is finitely generated. Then $V(I) \cap \text{Att}_R A \subseteq V(\mathfrak{m})$.

Proof. See [1, Lemma 2.5].

**Lemma 2.5.** Let $(R, \mathfrak{m})$ and $A$ be as in Lemma 2.4. Suppose that $x$ is an element in $\mathfrak{m}$ such that $V(Rx) \cap \text{Att}_R A \subseteq \{\mathfrak{m}\}$. Then the $R$-module $A/xA$ has finite length.

Proof. See [1, Lemma 2.4].

**Proposition 2.6.** Let $I$ be an ideal of a Noetherian ring $R$ and $M$ be an $R$-module such that $\dim M \leq 1$ and $\text{Supp} \ M \subseteq V(I)$. Then the following statements are equivalent:

(i) $M$ is $I$-cofinite,

(ii) the $R$-modules $\text{Hom}_R(R/I, M)$ and $\text{Ext}^1_R(R/I, M)$ are finitely generated.

Proof. The conclusion (i) $\implies$ (ii) is obviously true. In order to prove that (ii) $\implies$ (i), since by assumption (ii) $\text{Hom}_R(R/I, M)$ is finitely generated, using Lemma 2.3 and [10, Theorem 1.3], we may assume $\dim M = 1$.

We now prove by induction on $t := \text{ara}_M(I) = \text{ara}(I + \text{Ann}_R M/\text{Ann}_R M)$ that $M$ is $I$-cofinite. If $t = 0$, then it follows from definition that $I^n \subseteq \text{Ann}_R(M)$ for some positive integer $n$, and so $M = (0 :_M I^n)$. Therefore the assertion follows from Lemma 2.1. So assume that $t > 0$, and the result has been proved for all $i \leq t - 1$. Let

$$T := \{p \in \text{Supp} \ M \mid \dim R/p = 1\}.$$ 

It is easy to see that $T = \text{Assh}_RM$. As $\text{Ass}_R \text{Hom}_R(R/I, M) = V(I) \cap \text{Ass}_RM = \text{Ass}_RM$, it follows that the set $\text{Ass}_RM$ is finite. Hence $T$ is finite. Moreover, since for each $p \in T$ the $R_p$-module $\text{Hom}_{R_p}(R_p/I_Rp, M_p)$ is finitely generated, by [9, Ex. 7.7], and $M_p$ is an $IR_p$-torsion $R_p$-module, with $\text{Supp} M_p \subseteq V(pR_p)$, it follows that the $R_p$-module $\text{Hom}_{R_p}(R_p/I_Rp, M_p)$ is Artinian. Consequently, according to Melkersson’s results [10, Theorem 1.3] and Lemma 2.3, $M_p$ is an Artinian and $IR_p$-cofinite $R_p$-module. Let

$$T := \{p_1, \ldots, p_n\}.$$ 

By Lemma 2.4, we have

$$V(IR_{p_j}) \cap \text{Att}_{R_{p_j}}(M_{p_j}) \subseteq V(p_jR_{p_j}),$$

where $V(IR_{p_j})$ is the set of prime ideals of $R$ containing $(IR_{p_j})$ and $\text{Att}_{R_{p_j}}(M_{p_j})$ is the set of associated primes of $M_{p_j}$ in $R_{p_j}$.
for all \( j = 1, 2, \ldots, n \). Next, let
\[
\mathcal{V} := \bigcup_{j=1}^{n} \{ q \in \text{Spec } R \mid qR_{p_j} \subseteq \text{Att}_{R_{p_j}}(M_{p_j}) \}.
\]
Then it is easy to see that \( \mathcal{V} \cap V(I) \subseteq \mathcal{T} \).

On the other hand, since \( t = \text{ara}_M(I) \geq 1 \), there exist elements \( y_1, \ldots, y_t \in I \) such that
\[
\text{Rad}(I + \text{Ann}_R(M)/\text{Ann}_R(M)) = \text{Rad}((y_1, \ldots, y_t) + \text{Ann}_R(M)/\text{Ann}_R(M)).
\]

Now, as \( I \not\subseteq \bigcup_{q \in \mathcal{V} \setminus V(I)} q \), it follows that \( (y_1, \ldots, y_t) + \text{Ann}_R(M) \not\subseteq \bigcup_{q \in \mathcal{V} \setminus V(I)} q \).

On the other hand, for each \( q \in \mathcal{V} \) we have
\[
qR_{p_j} \subseteq \text{Att}_{R_{p_j}}(M_{p_j}),
\]
for some integer \( 1 \leq j \leq n \). Hence
\[
\text{Ann}_R(M)R_{p_j} \subseteq \text{Ann}_{R_{p_j}}(M_{p_j}) \subseteq qR_{p_j}.
\]
Since \( q \) is prime we obtain that \( \text{Ann}_R(M) \subseteq q \). Consequently, it follows from
\[
\text{Ann}_R(M) \subseteq \bigcap_{q \in \mathcal{V} \setminus V(I)} q
\]
that \( (y_1, \ldots, y_t) \not\subseteq \bigcup_{q \in \mathcal{V} \setminus V(I)} q \). By [9, Ex. 16.8] there is \( a \in (y_2, \ldots, y_t) \) such that \( y_1 + a \not\subseteq \bigcup_{q \in \mathcal{V} \setminus V(I)} q \). Let \( x := y_1 + a \). Then \( x \in I \) and
\[
\text{Rad}(I + \text{Ann}_R(M)/\text{Ann}_R(M)) = \text{Rad}((x, y_2, \ldots, y_t) + \text{Ann}_R(M)/\text{Ann}_R(M)).
\]

Next, let \( N := (0 :_M x) \). Then, it is easy to see that
\[
\text{ara}_N(I) = \text{ara}(I + \text{Ann}_R(N)/\text{Ann}_R(N)) \leq t - 1
\]
(note that \( x \in \text{Ann}_R(N) \)), and hence
\[
\text{Rad}(I + \text{Ann}_R(N)/\text{Ann}_R(N)) = \text{Rad}((y_2, \ldots, y_t) + \text{Ann}_R(N)/\text{Ann}_R(N)).
\]
Now, the exact sequence
\[
0 \longrightarrow N \longrightarrow M \longrightarrow xM \longrightarrow 0
\]
induces an exact sequence
\[
0 \longrightarrow \text{Hom}_R(R/I, N) \longrightarrow \text{Hom}_R(R/I, M) \longrightarrow \text{Hom}_R(R/I, xM)
\]
\[(*) \quad \longrightarrow \text{Ext}^1_R(R/I, N) \longrightarrow \text{Ext}^1_R(R/I, M),
\]
which implies that the \( R \)-modules \( \text{Hom}_R(R/I, N) \) and \( \text{Ext}^1_R(R/I, N) \) are finitely generated. Consequently, by the inductive hypothesis, the \( R \)-module \( N \) is \( I \)-cofinite.

Moreover, the exact sequence
\[
0 \longrightarrow N \longrightarrow M \longrightarrow xM \longrightarrow 0
\]
induces an exact sequence
\[
\text{Ext}^1_R(R/I, M) \longrightarrow \text{Ext}^1_R(R/I, xM) \longrightarrow \text{Ext}^2_R(R/I, N),
\]
which implies that the \( R \)-module \( \text{Ext}^1_R(R/I, xM) \) is finitely generated.
Also, from the exact sequence
\[ 0 \rightarrow xM \rightarrow M \rightarrow M/xM \rightarrow 0 \]
we get the exact sequence
\[ \text{Hom}_R(R/I, M) \rightarrow \text{Hom}_R(R/I, M/xM) \rightarrow \text{Ext}^1_R(R/I, xM), \]
which implies that the \( R \)-module \( \text{Hom}_R(R/I, M/xM) \) is finitely generated.

Now, from Lemma 2.5, it is easy to see that \((M/xM)_{p_j}\) has finite length for all \( j = 1, \ldots, n \). Therefore, there exists a finitely generated submodule \( L_j \) of \( M/xM \) such that
\[ (M/xM)_{p_j} = (L_j)_{p_j}. \]
Let \( L := L_1 + \cdots + L_n \). Then \( L \) is a finitely generated submodule of \( M/xM \) such that
\[ \text{Supp}(M/xM)/L \subseteq \text{Supp}(M) \setminus \{p_1, \ldots, p_n\} \subseteq \text{Max } R. \]
The sequence
\[ 0 \rightarrow L \rightarrow M/xM \rightarrow (M/xM)/L \rightarrow 0 \]
provides the exact sequence
\[ \text{Hom}_R(R/I, M/xM) \rightarrow \text{Hom}_R(R/I, (M/xM)/L) \rightarrow \text{Ext}^1_R(R/I, L), \]
which implies that the \( R \)-module \( \text{Hom}_R(R/I, (M/xM)/L) \) is finitely generated. We now show that \( M/xM \) is a minimax \( R \)-module. To do this, since
\[ \text{Supp}(M/xM)/L \subseteq \text{Max } R \]
and \((M/xM)/L\) is \( I \)-torsion, it follows from [10, Theorem 1.3] that the \( R \)-module \((M/xM)/L\) is Artinian. Hence \( M/xM \) is a minimax \( R \)-module. Now, as
\[ \text{Hom}_R(R/I, M/xM) \]
is a finitely generated \( R \)-module, it follows from Melkersson’s theorem (see Lemma 2.3) that \( M/xM \) is \( I \)-cofinite. Also, since the \( R \)-modules \( N = (0 :_M x) \) and \( M/xM \) are \( I \)-cofinite, it follows from Lemma 2.2 that \( M \) is \( I \)-cofinite. This completes the inductive step. \( \square \)

We are now in a position to use the previous result to produce a proof of the main theorem, which is a generalization of the main result of [6, Theorem 1].

**Theorem 2.7.** Let \( I \) be an ideal of a Noetherian ring \( R \). Let \( \mathcal{C}^1(R, I)_{\text{cof}} \) denote the category of \( I \)-cofinite \( R \)-modules \( M \) with \( \dim M \leq 1 \). Then \( \mathcal{C}^1(R, I)_{\text{cof}} \) is an Abelian category.

**Proof.** Let \( M, N \in \mathcal{C}^1(R, I)_{\text{cof}} \) and let \( f : M \rightarrow N \) be an \( R \)-homomorphism. It is enough to show that the \( R \)-modules \( \ker f \) and \( \operatorname{coker} f \) are \( I \)-cofinite.

To this end, the exact sequence
\[ 0 \rightarrow \ker f \rightarrow M \rightarrow \operatorname{im} f \rightarrow 0 \]
induces an exact sequence
\[ 0 \rightarrow \text{Hom}_R(R/I, \ker f) \rightarrow \text{Hom}_R(R/I, M) \rightarrow \text{Hom}_R(R/I, \operatorname{im} f) \rightarrow \text{Ext}^1_R(R/I, \ker f) \rightarrow \text{Ext}^1_R(R/I, M) \]
that implies the \( R \)-modules \( \text{Hom}_R(R/I, \ker f) \) and \( \text{Ext}^1_R(R/I, \ker f) \) are finitely generated. Therefore it follows from Proposition 2.6 that \( \ker f \) is \( I \)-cofinite. Now, the assertion follows from the exact sequences

\[
0 \longrightarrow \ker f \longrightarrow M \longrightarrow \text{im} f \longrightarrow 0
\]

and

\[
0 \longrightarrow \text{im} f \longrightarrow N \longrightarrow \text{coker} f \longrightarrow 0.
\]

\( \square \)

As an immediate consequence of Theorem 2.7, we derive the following extension of Delfino-Marley’s result in [3] and Kawasaki’s result in [6] for an arbitrary Noetherian ring.

**Corollary 2.8.** Let \( I \) be an ideal of a commutative Noetherian ring \( R \) of dimension one. Then the category \( \mathcal{M}(R, I)_{\text{cof}} \) of \( I \)-cofinite modules forms an Abelian subcategory of the category of all \( R \)-modules.

*Proof.* As \( \text{Supp} M \subseteq \text{Supp} R/I \) for all \( M \in \mathcal{M}(R, I)_{\text{cof}}, \) and \( \dim R/I = 1, \) it follows that

\[
\dim M \leq 1.
\]

Now the assertion follows from Theorem 2.7. \( \square \)

**Corollary 2.9.** Let \( I \) be an ideal of a commutative Noetherian ring \( R \) of dimension one. Let \( \mathcal{M}(R, I)_{\text{cof}} \) denote the category of \( I \)-cofinite modules over \( R. \) Let

\[
X^\bullet : \cdots \longrightarrow X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} X^{i+2} \longrightarrow \cdots
\]

be a complex such that \( X^i \in \mathcal{M}(R, I)_{\text{cof}} \) for all \( i \in \mathbb{Z}. \) Then the \( i \)th homology module \( H^i(X^\bullet) \) is in \( \mathcal{M}(R, I)_{\text{cof}}. \)

*Proof.* The assertion follows from Corollary 2.8. \( \square \)

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