

## ON JOINT NUMERICAL RADIUS

VLADIMIR MÜLLER

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ABSTRACT. Let  $T_1, \dots, T_n$  be bounded linear operators on a complex Hilbert space  $H$ . We study the question whether it is possible to find a unit vector  $x \in H$  such that  $|\langle T_j x, x \rangle|$  is large for all  $j$ . Thus we are looking for a generalization of a well-known fact for  $n = 1$  that the numerical radius  $w(T)$  of a single operator  $T$  satisfies  $w(T) \geq \|T\|/2$ .

### 1. INTRODUCTION

Let  $H$  be a complex Hilbert space. Denote by  $B(H)$  the set of all bounded linear operators on  $H$ . The numerical range of an operator  $T \in B(H)$  is defined by

$$W(T) = \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}$$

and the numerical radius by

$$w(T) = \sup \{ |\langle Tx, x \rangle| : x \in H, \|x\| = 1 \} = \sup \{ |\lambda| : \lambda \in W(T) \}.$$

It is well known that  $W(T)$  is a convex subset of the complex plane  $\mathbb{C}$ . Moreover,

$$(1) \quad \frac{1}{2} \|T\| \leq w(T) \leq \|T\|.$$

The second inequality in (1) is trivial; the first one is less obvious and more interesting. It means that for each  $T \in B(H)$  and each  $\varepsilon > 0$  there exists a unit vector  $x \in H$  such that

$$|\langle Tx, x \rangle| \geq \frac{1}{2} \|T\| - \varepsilon$$

(if  $\dim H < \infty$ , then there exists a unit vector  $x \in H$  with  $|\langle Tx, x \rangle| \geq \frac{\|T\|}{2}$  since the numerical range  $W(T)$  is closed in this case). Note also that for real Hilbert spaces the first inequality in (1) is not true (consider the matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ).

Let  $T_1, \dots, T_n \in B(H)$  be an  $n$ -tuple of operators. The joint numerical range of  $T_1, \dots, T_n$  is the subset of  $\mathbb{C}^n$  defined by

$$W(T_1, \dots, T_n) = \{ (\langle T_1 x, x \rangle, \dots, \langle T_n x, x \rangle) : x \in H, \|x\| = 1 \}.$$

The aim of this paper is to study the following question:

**Problem 1.** Does there exist a unit vector  $x \in H$  such that  $|\langle T_j x, x \rangle|$  is “large” for all  $j = 1, \dots, n$ ?

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Since each operator  $T_j$  can be written as  $T_j = A_j + iB_j$  with selfadjoint operators  $A_j = \frac{1}{2}(T_j + T_j^*)$  and  $B_j = \frac{1}{2i}(T_j - T_j^*)$  and

$$|\langle T_j x, x \rangle| = |\langle A_j x, x \rangle + i \langle B_j x, x \rangle| \geq \max\{|\langle A_j x, x \rangle|, |\langle B_j x, x \rangle|\},$$

Problem 1 can be reduced to the case of  $n$ -tuples of selfadjoint operators. Moreover, it is possible to consider only finite-dimensional spaces, since

$$W(T_1, \dots, T_n) = \bigcup_P W(PT_1P, \dots, PT_nP),$$

where  $P$  runs over all finite-rank orthogonal projections (in fact, it is sufficient to consider only projections of rank  $\leq n + 1$ ).

If the operators  $T_j$  are not only selfadjoint but also positive semidefinite, then it is possible to reduce the problem to the corresponding question for the norms (even for infinitely many operators).

**Theorem 2.** *Let  $T_1, T_2, \dots \in B(H)$  be positive semidefinite operators and let  $c_j \geq 0$  satisfy  $\sum_{j=1}^\infty c_j < 1$ . Then there exists a unit vector  $x \in H$  such that*

$$\langle T_j x, x \rangle \geq c_j \|T_j\|$$

for all  $j \in \mathbb{N}$ .

*Proof.* By [M, p. 334], for the square roots  $T_j^{1/2}$  there exists a unit vector  $x \in H$  such that

$$\|T_j^{1/2}x\| \geq \sqrt{c_j} \|T_j^{1/2}\|$$

for all  $j$ . So

$$\langle T_j x, x \rangle = \|T_j^{1/2}x\|^2 \geq c_j \|T_j^{1/2}\|^2 = c_j \|T_j\|$$

for all  $j \in \mathbb{N}$ . □

**Corollary 3.** *Let  $T_1, \dots, T_n \in B(H)$  be positive semidefinite operators and let  $\varepsilon > 0$ . Then there exists a unit vector  $x \in H$  such that*

$$\langle T_j x, x \rangle \geq \frac{1}{n} \|T_j\| - \varepsilon$$

for all  $j = 1, \dots, n$ .

If the operators  $T_j$  are not positive semidefinite but only selfadjoint, then the situation is more complicated. We give an exact answer for  $n = 2$  and  $n = 3$ . The main result of Section 2 will be

**Theorem 4.** *Let  $T_1, T_2, T_3 \in B(H)$  be selfadjoint operators and  $\varepsilon > 0$ . Then:*

(i) *there exists a unit vector  $x \in H$  such that*

$$|\langle T_j x, x \rangle| \geq \frac{1}{3} \|T_j\| - \varepsilon \quad (j = 1, 2);$$

(ii) *there exists a unit vector  $y \in H$  such that*

$$|\langle T_j y, y \rangle| \geq \frac{1}{5} \|T_j\| - \varepsilon \quad (j = 1, 2, 3).$$

The estimates in Theorem 4 are the best possible.

For  $n \geq 4$  the situation is more complicated. Among other technical difficulties, the joint numerical range of an  $n$ -tuple of selfadjoint operators is in general not convex. For  $n \geq 4$  we give only some estimates of how large values of  $|\langle T_j x, x \rangle|$  in Problem 1 can be obtained.

The results can also be applied to other types of numerical ranges: the essential numerical range and the algebraic numerical range of  $n$ -tuples of elements in a unital Banach algebra.

2. CASES  $n = 2, 3$

Let  $T_1, T_2, T_3 \in B(H)$  be selfadjoint operators. The numerical range  $W(T_1, T_2)$  is always a convex set — it reduces to the convexity of the numerical range of a single operator  $W(T_1 + iT_2)$ . If  $\dim H \geq 3$ , then the numerical range  $W(T_1, T_2, T_3)$  is also convex; see e.g. [AT], [FT], [GJK]. The convexity may be used for solving Problem 1.

For  $u \in \mathbb{R}^n$  we write  $u = (u_1, \dots, u_n)$ .

**Lemma 5.** *Let  $K \subset [-1, 1]^2$  be a convex set and let  $u, v \in K$  satisfy  $u_1 = 1 = v_2$ . Then there exists  $w \in K$  such that  $|w_1| \geq 1/3$  and  $|w_2| \geq 1/3$ .*

*Proof.* If  $u_2 < -1/3$ , then set  $w = u$ .

If  $v_1 < -1/3$ , then set  $w = v$ .

If both  $u_2 \geq -1/3$  and  $v_1 \geq -1/3$ , then  $w := \frac{u+v}{2}$  satisfies

$$w_1 = \frac{u_1 + v_1}{2} = \frac{1 + v_1}{2} \geq 1/3$$

and similarly,

$$w_2 = \frac{u_2 + v_2}{2} = \frac{u_2 + 1}{2} \geq 1/3.$$

□

**Lemma 6.** *Let  $K \subset [-1, 1]^3$  be a convex set and let  $u, v, w \in K$  satisfy  $u_1 = v_2 = w_3 = 1$ . Then there exists  $x = (x_1, x_2, x_3) \in K$  such that  $|x_j| \geq 1/5$  ( $j = 1, 2, 3$ ).*

*Proof.* Let

$$M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_j| \geq 1/5 \quad (j = 1, 2, 3)\}.$$

Suppose on the contrary that  $K \cap M = \emptyset$ . Consider the matrix

$$(2) \quad \begin{pmatrix} 1 & u_2 & u_3 \\ v_1 & 1 & v_3 \\ w_1 & w_2 & 1 \end{pmatrix}.$$

Since  $u, v, w \notin M$ , in each row of matrix (2) there exists an entry with modulus  $< 1/5$  (we call such entries small).

We distinguish two cases:

*Case A.* There exists a column of (2) with two small entries.

Without loss of generality we may assume that  $|u_3| < 1/5$  and  $|v_3| < 1/5$ . Moreover, either  $w_1$  or  $w_2$  is small; without loss of generality we may assume that  $|w_2| < 1/5$ .

Let  $a = \frac{v+w}{2}$ . We have  $|a_2| = |\frac{1+w_2}{2}| \geq \frac{1-1/5}{2} \geq 1/5$  and  $|a_3| = |\frac{v_3+1}{2}| \geq 1/5$ . So  $|a_1| = |\frac{v_1+w_1}{2}| < 1/5$  and

$$(3) \quad |v_1 + w_1| < \frac{2}{5}.$$

Let  $b = \frac{u+v+w}{3} \in K$ . Then  $|b_3| = \left| \frac{u_3+v_3+1}{3} \right| \geq 1/5$  and, by (3),  $b_1 = \frac{1+v_1+w_1}{3} \geq \frac{1-2/5}{3} = \frac{1}{5}$ . So  $|b_2| = \left| \frac{1+u_2+w_2}{3} \right| < 1/5$  and

$$(4) \quad u_2 + w_2 < -\frac{2}{5}.$$

Finally, let  $x = \frac{2u+w}{3} \in K$ . We have  $|x_1| = \left| \frac{2+w_1}{3} \right| \geq \frac{1}{3} > \frac{1}{5}$ ,  $|x_2| = \left| \frac{2u_2+w_2}{3} \right| \geq \frac{1}{3}(|2u_2 + 2w_2| - |w_2|) \geq \frac{1}{3}(\frac{4}{5} - \frac{1}{5}) = \frac{1}{5}$  by (4), and  $|x_3| = \left| \frac{2u_3+1}{3} \right| \geq \frac{1}{5}$ .

So  $x \in M$ , a contradiction.

*Case B.* In each column there is one small entry.

Without loss of generality we may assume that  $|u_2| < 1/5$ ,  $|v_3| < 1/5$  and  $|w_1| < 1/5$ .

Consider the vector  $a = \frac{2u+v}{3} \in K$ . Then  $a_1 = \frac{2+v_1}{3} \geq \frac{2-1}{3} = \frac{1}{3} > \frac{1}{5}$  and  $a_2 = \frac{2u_2+1}{3} > \frac{1-2/5}{3} = \frac{1}{5}$ . So  $|a_3| = \left| \frac{2u_3+v_3}{3} \right| < \frac{1}{5}$  and

$$|u_3| \leq \frac{1}{2}(|2u_3 + v_3| + |v_3|) < \frac{1}{2}\left(\frac{3}{5} + \frac{1}{5}\right) = \frac{2}{5}.$$

Symmetrically,  $|v_1| < \frac{2}{5}$  and  $|w_2| < \frac{2}{5}$ .

Let  $b = \frac{u+v}{2} \in K$ . Then  $b_1 = \frac{1+v_1}{2} > \frac{1-2/5}{2} > \frac{1}{5}$  and  $b_2 = \frac{u_2+1}{2} > \frac{1}{5}$ . So  $|b_3| = \left| \frac{u_3+v_3}{2} \right| < \frac{1}{5}$  and

$$(5) \quad |u_3 + v_3| < \frac{2}{5}.$$

Symmetrically,  $|v_1 + w_1| < \frac{2}{5}$  and  $|u_2 + w_2| < \frac{2}{5}$ .

Let  $x = \frac{u+v+w}{3} \in K$ . Then  $x_1 = \frac{1+v_1+w_1}{3} > \frac{1-2/5}{3} = \frac{1}{5}$ , and similarly,  $x_2 > \frac{1}{5}$ ,  $x_3 > \frac{1}{5}$ . Hence  $x \in M$ , a contradiction. □

Lemmas 5 and 6 are particular cases of the following conjecture:

**Conjecture 7.** *Let  $n \in \mathbb{N}$  and let  $K \subset [-1, 1]^n$  be a convex set. Let  $u_j = (u_{j1}, \dots, u_{jn}) \in K$  satisfy  $u_{jj} = 1$  ( $j = 1, \dots, n$ ). Then there exists  $v \in K$ ,  $v = (v_1, \dots, v_n)$ , such that  $|v_j| \geq \frac{1}{2n-1}$  ( $j = 1, \dots, n$ ).*

Conjecture 7 is a particular case of the famous still open plank problem [B], whether a bounded convex subset of  $\mathbb{R}^n$  can be covered by a finite number of planks such that the sum of their relative widths is less than 1. For details see [Ba].

The estimate  $\frac{1}{2n-1}$  in Conjecture 7 cannot be improved, as the following example shows:

**Example 8.** Let  $n \in \mathbb{N}$  and let  $u_j = (u_{j1}, \dots, u_{jn}) \in \mathbb{R}^n$  be defined by  $u_{jj} = 1$  ( $j = 1, \dots, n$ ),  $u_{jk} = \frac{-1}{2n-1}$  ( $j, k = 1, \dots, n, j \neq k$ ). Let  $K$  be the convex hull of the vectors  $u_1, \dots, u_n$ .

Let  $v \in K$ ,  $v = (v_1, \dots, v_n)$  be an arbitrary vector. Then  $v = \sum_{j=1}^n \alpha_j u_j$  for some  $\alpha_j \geq 0$ ,  $\sum_{j=1}^n \alpha_j = 1$ . So there exists  $k \in \{1, \dots, n\}$  such that  $\alpha_k \leq \frac{1}{n}$ . Then  $v_k = \sum_{j=1}^n \alpha_j u_{jk} = \alpha_k + (1 - \alpha_k) \frac{-1}{2n-1} = \alpha_k \left(1 + \frac{1}{2n-1}\right) - \frac{1}{2n-1}$ . Therefore

$$\frac{-1}{2n-1} \leq v_k \leq \frac{1}{n} \left(1 + \frac{1}{2n-1}\right) - \frac{1}{2n-1} = \frac{1}{2n-1},$$

so  $\min_{1 \leq k \leq n} |v_k| \leq \frac{1}{2n-1}$  for each  $v \in K$ .

Lemmas 5 and 6 imply the following statement about the joint numerical radius mentioned in the Introduction.

**Theorem 9.** *Let  $\dim H < \infty$  and let  $T_1, T_2, T_3 \in B(H)$  be selfadjoint operators. Then:*

(i) *there exists a unit vector  $x \in H$  such that*

$$|\langle T_j x, x \rangle| \geq \frac{1}{3} \|T_j\| \quad (j = 1, 2);$$

(ii) *there exists a unit vector  $y \in H$  such that*

$$|\langle T_j y, y \rangle| \geq \frac{1}{5} \|T_j\| \quad (j = 1, 2, 3).$$

*Proof.* (i) If  $\|T_j\| \in \sigma(T_j)$ , then set  $A_j = \frac{T_j}{\|T_j\|}$ . If  $-\|T_j\| \in \sigma(T_j)$ , then set  $A_j = \frac{-T_j}{\|T_j\|}$ . Then  $\|A_j\| = 1$  and  $1 \in \sigma(A_j) \subset W(A_j)$ . So there exist unit vectors  $x_j \in H$  such that  $\langle A_j x_j, x_j \rangle = 1$  ( $j = 1, 2$ ). Consider the convex set  $W(A_1, A_2)$  and elements

$$(\langle A_1 x_1, x_1 \rangle, \langle A_2 x_1, x_1 \rangle), (\langle A_1 x_2, x_2 \rangle, \langle A_2 x_2, x_2 \rangle) \in W(A_1, A_2).$$

By Lemma 5, there exists a unit vector  $x \in H$  such that  $|\langle A_j x, x \rangle| \geq \frac{1}{3}$  ( $j = 1, 2$ ), and so  $|\langle T_1 x, x \rangle| \geq \frac{\|T_1\|}{3}$  ( $j = 1, 2$ ).

(ii) If  $\dim H \geq 3$ , then  $W(T_1, T_2, T_3)$  is a convex set and the statement can be proved similarly as above using Lemma 6 instead of Lemma 5. If  $\dim H = 1$ , then the statement is trivial.

Suppose that  $\dim H = 2$ . Let  $\tilde{H} = H \oplus \mathbb{C}$  and  $\tilde{T}_j = T_j \oplus 0 \in B(\tilde{H})$  ( $j = 1, 2, 3$ ).

It is easy to see that  $W(\tilde{T}_1, \tilde{T}_2, \tilde{T}_3) = \{t\mu : 0 \leq t \leq 1, \mu \in W(T_1, T_2, T_3)\}$ . We have proved that there exists  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in W(\tilde{T}_1, \tilde{T}_2, \tilde{T}_3)$  with  $|\lambda_j| \geq \frac{\|\tilde{T}_j\|}{5} = \frac{\|T_j\|}{5}$  ( $j = 1, 2, 3$ ). It is easy to see that there exists  $\mu \in W(T_1, T_2, T_3)$  with  $|\mu_j| \geq \frac{\|T_j\|}{5}$  ( $j = 1, 2, 3$ ). □

These estimates are the best possible.

**Example 10.** Let  $n \in \mathbb{N}$ , let  $\dim H = n$ , and let  $T_1, \dots, T_n \in B(H)$  be the diagonal matrices

$$\begin{aligned} T_1 &= \text{diag}\left(1, \frac{-1}{2n-1}, \dots, \frac{-1}{2n-1}\right), \\ T_2 &= \text{diag}\left(\frac{-1}{2n-1}, 1, \frac{-1}{2n-1}, \dots, \frac{-1}{2n-1}\right), \\ &\vdots \\ T_n &= \text{diag}\left(\frac{-1}{2n-1}, \dots, \frac{-1}{2n-1}, 1\right). \end{aligned}$$

Then  $T_1, \dots, T_n$  are commuting selfadjoint operators,  $\|T_j\| = 1$  and

$$W(T_1, \dots, T_n) = \text{conv}\left\{\left(1, \frac{-1}{2n-1}, \dots, \frac{-1}{2n-1}\right), \dots, \left(\frac{-1}{2n-1}, \dots, \frac{-1}{2n-1}, 1\right)\right\}.$$

By Example 8, for each  $v \in W(T_1, \dots, T_n)$  we have  $\min_{1 \leq j \leq n} |v_j| \leq \frac{1}{2n-1}$ .

**Corollary 11.** *Let  $\dim H < \infty$  and let  $T_1, T_2, T_3 \in B(H)$ . Then:*

(i) *there exists a unit vector  $x \in H$  such that*

$$|\langle T_j x, x \rangle| \geq \frac{1}{6} \|T_j\| \quad (j = 1, 2);$$

(ii) *there exists a unit vector  $y \in H$  such that*

$$|\langle T_j y, y \rangle| \geq \frac{1}{10} \|T_j\| \quad (j = 1, 2, 3).$$

*Proof.* (i) Write  $T_j = A_j + iB_j$  with selfadjoint operators  $A_j, B_j$ . Then  $\|A_j\| \geq \frac{\|T_j\|}{2}$  or  $\|B_j\| \geq \frac{\|T_j\|}{2}$ . For each  $j$  choose either  $A_j$  or  $B_j$  with bigger norm and apply Theorem 9.

(ii) can be proved similarly. □

*Remark 12.* We do not know what are the best constants in Corollary 11. For  $n = 2$  it lies between  $1/6$  and  $1/4$  as the following example shows. Let

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to show that for each unit vector  $x$  either  $|\langle T_1 x, x \rangle| \leq 1/4$  or  $|\langle T_2 x, x \rangle| \leq 1/4$ .

Similarly, one can show that for  $n = 3$  the best constant in Corollary 11 lies between  $1/10$  and  $1/6$ .

### 3. CASE $n \geq 4$

The following lemma is a weaker version of Conjecture 7.

**Lemma 13.** *Let  $n \in \mathbb{N}$  and let  $K \subset [0, 1]^n$  be a convex set. Let  $u_j \in K$ ,  $u_j = (u_{j1}, \dots, u_{jn})$  satisfy  $u_{jj} = 1$  ( $j = 1, \dots, n$ ). Then there exists  $v = (v_1, \dots, v_n) \in K$  such that  $|v_j| \geq \frac{1}{2n^2}$  ( $j = 1, \dots, n$ ).*

*Proof.* Let  $M = [0, 1]^n$ . Clearly  $M$  is a convex set with  $\text{width}(M) = 1$ , where

$$\text{width}(M) = \inf \left\{ \sup_{v \in M} \langle v, f \rangle - \inf_{v \in M} \langle v, f \rangle : f \in \mathbb{R}^n, \|f\| = 1 \right\}.$$

Indeed, for  $f = (f_1, \dots, f_n) \in \mathbb{R}^n, \|f\| = \left( \sum_{j=1}^n f_j^2 \right)^{1/2} = 1$  let  $J = \{j \in \{1, \dots, n\} : f_j \geq 0\}$ . Then  $\sup_{v \in M} \langle v, f \rangle = \sum_{j \in J} f_j$  and  $\inf_{v \in M} \langle v, f \rangle = \sum_{j \notin J} f_j$ . Hence  $\sup_{v \in M} \langle v, f \rangle - \inf_{v \in M} \langle v, f \rangle = \sum_{j=1}^n |f_j| \geq \sum_{j=1}^n |f_j|^2 = 1$  and  $\text{width } M \geq 1$ . Considering the vector  $f = (1, 0, \dots, 0)$  we get  $\text{width } M = 1$ .

For  $j = 1, \dots, n$  let  $L_j = \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n : \left| \sum_{k=1}^n t_k u_{kj} \right| < \frac{1}{2n} \right\}$ . Then  $\text{width}(L_j) = \frac{n^{-1}}{(\sum_{k=1}^n u_{kj}^2)^{1/2}} \leq \frac{1}{n}$ . So  $\sum_{j=1}^n \text{width}(L_j) \leq 1$ . By [B], there exists  $t = (t_1, \dots, t_n) \in M$  such that  $t \notin \bigcup_{j=1}^n L_j$ .

Let  $s = \frac{t}{\sum_{j=1}^n t_j}$ . Then  $\sum_{k=1}^n s_k = 1$ , and for each  $j = 1, \dots, n$  we have

$$\left| \sum_{k=1}^n s_k u_{kj} \right| = \frac{\left| \sum_{k=1}^n t_k u_{kj} \right|}{\sum_{k=1}^n t_k} \geq \frac{1}{2n^2}.$$

So  $v = \sum_{k=1}^n s_k u_k \in K$  and

$$|v_j| \geq \frac{1}{2n^2} \quad (j = 1, \dots, n). \quad \square$$

**Corollary 14.** *Let  $\dim H < \infty$  and  $T_1, \dots, T_n \in B(H)$  be commuting selfadjoint operators. Then there exists a unit vector  $x \in H$  such that*

$$|\langle T_j x, x \rangle| \geq \frac{\|T_j\|}{2n^2} \quad (j = 1, \dots, n).$$

*Proof.* The numerical range  $W(T_1, \dots, T_n) = \text{conv } \sigma(T_1, \dots, T_n)$  is a convex set. For each  $j = 1, \dots, n$  there exists a unit vector  $x_j \in H$  with  $|\langle T_j x_j, x_j \rangle| = \|T_j\|$ , so there exists  $u_j \in W(T_1, \dots, T_n)$  with  $|u_{jj}| = \|T_j\|$ .

Using Lemma 13 we can show as in the proof of Theorem 9 that there exists  $v \in W(T_1, \dots, T_n)$  with  $|v_j| \geq \frac{\|T_j\|}{2n^2}$  ( $j = 1, \dots, n$ ).  $\square$

Lemma 13 can also be applied for other types of numerical ranges.

Let  $H$  be an infinite-dimensional Hilbert space and let  $T_1, \dots, T_n \in B(H)$ . The essential numerical range  $W_e(T_1, \dots, T_n)$  is the set of all  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  such that there exists an orthonormal sequence  $(x_k) \subset H$  with

$$\lambda_j = \lim_{k \rightarrow \infty} \langle T_j x_k, x_k \rangle.$$

An important property of the essential numerical range is that it is always a closed convex set; see [LP].

For a single selfadjoint operator  $S \in B(H)$  we have  $\sup\{|\mu| : \mu \in W_e(S)\} = \|S\|_e$ , the essential norm of  $S$ . So an easy application of Lemma 13 (Lemmas 5 and 6, respectively) gives

**Theorem 15.** *Let  $H$  be an infinite-dimensional Hilbert space and let  $T_1, \dots, T_n \in B(H)$  be selfadjoint operators. Then there exists an orthonormal sequence  $(x_k) \subset H$  such that  $a_j := \lim_{k \rightarrow \infty} \langle T_j x_k, x_k \rangle$  exists and  $|a_j| \geq \frac{\|T_j\|_e}{2n^2}$  for all  $j = 1, \dots, n$ .*

*For  $n = 2$  and  $n = 3$  there exists an orthonormal sequence  $(x_k) \subset H$  such that  $|a_j| \geq \frac{\|T_j\|_e}{3}$  ( $j = 1, 2$ ), and  $|a_j| \geq \frac{\|T_j\|_e}{5}$  ( $j = 1, 2, 3$ ), respectively.*

**Corollary 16.** *Let  $n \in \mathbb{N}$ ,  $\varepsilon > 0$ , and let  $T_1, \dots, T_n \in B(H)$  be arbitrary operators. Then there exists an orthonormal sequence  $(x_k) \subset H$  such that  $a_j := \lim_{k \rightarrow \infty} \langle T_j x_k, x_k \rangle$  exists and  $|a_j| \geq \frac{\|T_j\|_e}{4n^2\varepsilon}$  for all  $j = 1, \dots, n$ .*

*For  $n = 2$  and  $n = 3$  there exists an orthonormal sequence  $(x_k) \subset H$  such that  $|a_j| \geq \frac{\|T_j\|_e}{6\varepsilon}$  ( $j = 1, 2$ ), and  $|a_j| \geq \frac{\|T_j\|_e}{10\varepsilon}$  ( $j = 1, 2, 3$ ), respectively.*

Another situation where the results can be applied is the algebraic numerical range.

Let  $\mathcal{A}$  be a unital Banach algebra and let  $a_1, \dots, a_n \in \mathcal{A}$ . The algebraic numerical range is defined by

$$V(a_1, \dots, a_n, \mathcal{A}) = \{(f(a_1), \dots, f(a_n)) : f \in \mathcal{A}^*, \|f\| = 1 = f(1_{\mathcal{A}})\},$$

where  $1_{\mathcal{A}}$  denotes the unit in  $\mathcal{A}$ .

It is well known that  $V(a_1, \dots, a_n, \mathcal{A})$  is always a closed convex subset of  $\mathbb{C}^n$ . For a single element  $a_1 \in \mathcal{A}$  we have

$$\sup\{|\mu| : \mu \in V(a_1, \mathcal{A})\} \geq \frac{\|a_1\|}{e}$$

(where  $e = 2.71\dots$ ); see [BD, p. 34].

**Corollary 17.** *Let  $\mathcal{A}$  be a unital Banach algebra and let  $a_1, \dots, a_n \in \mathcal{A}$ . Then there exists  $f \in \mathcal{A}^*$ ,  $\|f\| = 1 = f(1_{\mathcal{A}})$  such that*

$$|f(a_j)| \geq \frac{\|a_j\|}{2n^2e} \quad (j = 1, \dots, n).$$

For  $n = 2$  ( $n = 3$ ) we have

$$|f(a_j)| \geq \frac{\|a_j\|}{3e} \quad (j = 1, 2)$$

and

$$|f(a_j)| \geq \frac{\|a_j\|}{5e} \quad (j = 1, 2, 3),$$

respectively.

*Proof.* For  $j = 1, \dots, n$  there exists  $f_j \in \mathcal{A}^*$  with  $\|f_j\| = 1 = f_j(1_{\mathcal{A}})$ ,  $|f(a_j)| \geq \frac{\|a_j\|}{e}$ . Let  $\alpha_j$  be the complex unit such that  $f(\alpha_j a_j) \geq \frac{\|a_j\|}{e}$ . The numerical range  $V(\alpha_1 a_1, \dots, \alpha_n a_n, \mathcal{A})$  is a convex set, and so is the set  $K := \{(\operatorname{Re} \lambda_1, \dots, \operatorname{Re} \lambda_n) : (\lambda_1, \dots, \lambda_n) \in V(\alpha_1 a_1, \dots, \alpha_n a_n, \mathcal{A})\}$ . By Lemma 13, there exists  $\mu \in K \subset \mathbb{R}^n$  with  $|\mu_j| \geq \frac{\|a_j\|}{2n^2e}$  for all  $j$ . So there exists  $\lambda \in V(a_1, \dots, a_n, \mathcal{A})$  with  $|\lambda_j| \geq \frac{\|a_j\|}{2n^2e}$  ( $j = 1, \dots, n$ ).  $\square$

#### 4. NON-CONVEX CASE

In this section we consider the general case of a sequence  $T_1, T_2, \dots$  of operators on a Hilbert space  $H$ .

Let  $c_j \geq 0$ ,  $\sum_{j=1}^{\infty} c_j < 1$ . By [M, p. 353], there exist unit vectors  $x, y \in H$  such that

$$(6) \quad |\langle T_j x, y \rangle| \geq c_j \|T_j\|$$

for all  $j \in \mathbb{N}$ . By the polarization formula,

$$\begin{aligned} 4\langle T_j x, y \rangle &= \langle T_j(x+y), x+y \rangle - \langle T_j(x-y), x-y \rangle \\ &\quad + i\langle T_j(x+iy), x+iy \rangle - i\langle T_j(x-iy), x-iy \rangle. \end{aligned}$$

Set  $u_1 = \frac{x+y}{\|x+y\|}$ ,  $u_2 = \frac{x-y}{\|x-y\|}$ ,  $u_3 = \frac{x+iy}{\|x+iy\|}$ ,  $u_4 = \frac{x-iy}{\|x-iy\|}$ . So by (6), there are four unit vectors  $u_1, \dots, u_4 \in H$  such that

$$\max_{1 \leq k \leq 4} |\langle T_j u_k, u_k \rangle| \geq \frac{c_j}{4} \|T_j\|$$

for all  $j \in \mathbb{N}$ . However, it is much more difficult to find a single unit vector  $u \in H$  such that  $|\langle T_j u, u \rangle|$  is large for all  $j$ , as was required in Problem 1.

We give only a modest estimate in this case.

We need the following lemma.

**Lemma 18.** *Let  $b \leq 0$ ,  $0 < \varepsilon < 1$ . Then*

$$m\left(\{t \in [0, 2\pi) : b \leq \cos t \leq b + \varepsilon\}\right) \leq \pi\sqrt{2\varepsilon},$$

where  $m$  denotes the Lebesgue measure.



*Proof.* It is a matter of routine to show that the maximum of the function

$$b \mapsto m\left(\{t \in [0, 2\pi) : b \leq \cos t \leq b + \varepsilon\}\right)$$

is attained for  $b = -1$ . For  $0 \leq t < 2\pi$  we have

$$-1 \leq \cos t \leq -1 + \varepsilon \iff \pi - t_0 \leq t \leq \pi + t_0,$$

where  $0 < t_0 < \frac{\pi}{2}$  and  $\sin t_0 = \sqrt{1 - (1 - \varepsilon)^2} = \sqrt{2\varepsilon - \varepsilon^2} \leq \sqrt{2\varepsilon}$ . Thus

$$\begin{aligned} & m\left(\{t \in [0, 2\pi) : b \leq \cos t \leq b + \varepsilon\}\right) \\ & \leq m\left(\{t \in [0, 2\pi) : \cos t \leq -1 + \varepsilon\}\right) = 2t_0 \leq \pi \sin t_0 \leq \pi\sqrt{2\varepsilon}. \end{aligned}$$

□

**Theorem 19.** *Let  $A_j \in B(H)$  ( $j = 1, 2, \dots$ ) be selfadjoint operators. Suppose that  $\sum_{j=1}^\infty c_j^{1/3} < 1$ . Then there exists a unit vector  $u \in H$  such that*

$$|\langle A_j u, u \rangle| \geq \frac{c_j}{4} \|A_j\|$$

for all  $j \in \mathbb{N}$ .

*Proof.* Without loss of generality we may assume that  $\|A_j\| = 1$  for all  $j$ . As mentioned above, there exist  $x, y \in H$ ,  $\|x\| = 1 = \|y\|$  such that

$$|\langle A_j x, y \rangle| \geq c_j^{1/3} \quad (j \in \mathbb{N}).$$

For  $0 \leq t < 2\pi$  set  $v(t) = x + e^{it}y$ . Then for each  $j \in \mathbb{N}$ ,

$$\langle A_j v(t), v(t) \rangle = \langle A_j x, x \rangle + \langle A_j y, y \rangle + 2 \operatorname{Re} e^{-it} \langle A_j x, y \rangle.$$

Let  $M_j = \{t \in [0, 2\pi) : |\langle A_j v(t), v(t) \rangle| < c_j\}$ . We have

$$t \in M_j \iff |\operatorname{Re}(a_j + r_j e^{i(s-t)})| < c_j,$$

where  $a_j = \langle A_j x, x \rangle + \langle A_j y, y \rangle$ ,  $r_j = 2|\langle A_j x, y \rangle|$  and  $2\langle A_j x, y \rangle = r_j e^{is}$ . So  $r_j \geq 2c_j^{1/3}$ .

Thus

$$\begin{aligned} t \in M_j & \iff \left| \operatorname{Re}\left(\frac{a_j}{r_j} + e^{i(s-t)}\right) \right| < \frac{c_j}{r_j} \\ & \iff -\frac{a_j}{r_j} - \frac{c_j}{r_j} \leq \cos(s-t) \leq -\frac{a_j}{r_j} + \frac{c_j}{r_j}. \end{aligned}$$

By Lemma 18, for  $b = -\frac{a_j}{r_j} - \frac{c_j}{r_j}$  and  $\varepsilon = \frac{2c_j}{r_j}$  we have

$$m(M_j) \leq \pi \left(\frac{4c_j}{r_j}\right)^{1/2} \leq 2\pi \left(\frac{c_j}{2c_j^{1/3}}\right)^{1/2} = \pi\sqrt{2}c_j^{1/3}.$$

So

$$m\left(\bigcup_{j=1}^\infty M_j\right) \leq \sum_{j=1}^\infty m(M_j) \leq \pi\sqrt{2} \sum_{j=1}^\infty c_j^{1/3} < 2\pi.$$

Hence there exists  $t \in [0, 2\pi) \setminus \bigcup_{j=1}^\infty M_j$ . For this  $t$  we have  $|\langle A_j v(t), v(t) \rangle| \geq c_j$  for all  $j \in \mathbb{N}$ . Let  $u = \frac{v(t)}{\|v(t)\|}$ . Then  $\|u\| = 1$  and  $|\langle A_j u, u \rangle| \geq \frac{c_j}{4}$  ( $j \in \mathbb{N}$ ). □

**Corollary 20.** *Let  $T_1, \dots, T_n \in B(H)$ ,  $\varepsilon > 0$ . Then there exists a unit vector  $x \in H$  such that*

$$|\langle T_j x, x \rangle| \geq \frac{\|T_j\|}{8n^3} - \varepsilon.$$

*Proof.* If  $T_1, \dots, T_n$  are selfadjoint operators, by Theorem 19 we get the existence of a unit vector  $x \in H$  with  $|\langle T_j x, x \rangle| \geq \frac{\|T_j\|}{4n^3} - \varepsilon$  for  $j = 1, \dots, n$ .

If  $T_1, \dots, T_n$  are general non-selfadjoint operators, then we can consider either the real or imaginary part of each  $T_j$  with greater norm and obtain Corollary 20 in the usual way.  $\square$

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MATHEMATICAL INSTITUTE, CZECH ACADEMY OF SCIENCES, ŽITNA 25, 115 67 PRAHA 1, CZECH REPUBLIC

*E-mail address:* muller@math.cas.cz