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ON THE GORENSTEIN AND COHOMOLOGICAL DIMENSION OF GROUPS

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Abstract. Here we relate the Gorenstein dimension of a group G, $\mathrm{Gcd}_R G$, over $\mathbb{Z}$ and $\mathbb{Q}$ to the cohomological dimension of G, $\mathrm{cd}_R G$, over $\mathbb{Z}$ and $\mathbb{Q}$, and show that if G is in $\mathbf{LH}\mathfrak{F}$, a large class of groups defined by Kropholler, then $\mathrm{cd}_{\mathbb{Q}} G = \mathrm{Gcd}_{\mathbb{Q}} G$ and if G is torsion free, then $\mathrm{Gcd}_{\mathbb{Z}} G = \mathrm{cd}_{\mathbb{Z}} G$. We also show that for any group G, $\mathrm{Gcd}_{\mathbb{Q}} G \leq \mathrm{Gcd}_{\mathbb{Z}} G$.

1. Introduction

The Gorenstein projective dimension of an S-module M, $\mathrm{Gpd}_S\ M$, is a refinement of the projective dimension of M over S and was defined by Enochs and Jenda in [8]. This concept goes back to the concept of a relative homological dimension, called the G-dimension, which was defined by Auslander in [3] for finitely generated modules over commutative Noetherian rings and provided a characterization of the Gorenstein rings.

The Gorenstein dimension of a group G over $\mathbb{Z}$, $\mathrm{Gcd}_{\mathbb{Z}} G$, is the Gorenstein projective dimension of $\mathbb{Z}$, as a trivial $\mathbb{Z}G$-module. It turns out that $\mathrm{Gcd}_{\mathbb{Z}} G$ coincides with the cohomological dimension of G over $\mathbb{Z}$, $\mathrm{cd}_{\mathbb{Z}} G$, when the latter is finite and is related ([1], [2]) to the invariants $\mathrm{spli}\mathbb{Z}G$, $\mathrm{silp}\mathbb{Z}G$ and the generalized cohomological dimension of G, $\underline{\mathrm{cd}}_{\mathbb{Z}} G$, where $\mathrm{spli}\mathbb{Z}G$ is the supremum of the projective dimensions of the injective $\mathbb{Z}G$-modules, $\mathrm{silp}\mathbb{Z}G$ is the supremum of the injective dimensions of the projective $\mathbb{Z}G$-modules, and

$$\underline{\mathrm{cd}}_{\mathbb{Z}} G = \sup\{n \in \mathbb{N} |\ \exists\ M, P : \mathrm{Ext}^n_{\mathbb{Z}G}(M, P) \neq 0,\ M\ \mathbb{Z}\text{-free},\ P\ \mathbb{Z}G\text{-projective}\}.$$

These invariants were considered by Gedrich-Gruenberg [9] and Ikenaga [11] in their study of extending Farrell-Tate cohomology, which is defined for the class of groups of finite virtual cohomological dimension, to a larger class of groups.

In [2] the finiteness of $\mathrm{Gcd}_{\mathbb{Z}}$G was proposed as an algebraic characterization of those groups G which admit a finite dimensional model for $\underline{E}G$, the classifying space for proper actions.

The Gorenstein dimension of a group G over $\mathbb{Q}$, $\mathrm{Gcd}_{\mathbb{Q}} G$, is the Gorenstein projective dimension of $\mathbb{Q}$ as a trivial $\mathbb{Q}G$-module. The invariants $\mathrm{spli}\mathbb{Q}G$, $\mathrm{silp}\mathbb{Q}G$ and the generalized cohomological dimension of G over $\mathbb{Q}$, $\underline{\mathrm{cd}}_{\mathbb{Q}} G$, are defined analogously.

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Although it is not known whether Gorenstein projectivity is a subgroup closed property, the Gorenstein dimension of a group G over $\mathbb{Z}$ and $\mathbb{Q}$ enjoys some nice properties (Theorem 3.1).

Here we study the Gorenstein dimension of a group G over $\mathbb{Z}$ and $\mathbb{Q}$ when G is in $\mathbf{LH}\mathfrak{F}$, a class of groups defined by Kropholler in [12]. The class $\mathbf{H}\mathfrak{F}$ was defined as the smallest class of groups which contains the class of finite groups, and whenever a group G admits a finite dimensional contractible G-CW-complex with stabilizers in $\mathbf{H}\mathfrak{F}$, then G is in $\mathbf{H}\mathfrak{F}$. $\mathbf{LH}\mathfrak{F}$ is the class of groups G such that every finitely generated subgroup of G is in $\mathbf{H}\mathfrak{F}$. The class $\mathbf{LH}\mathfrak{F}$ includes, for example, all soluble by finite groups, or more generally, all elementary amenable groups and all linear groups, and is extension closed and closed under ascending unions, amalgamated free products and HNN extensions.

We show that for G in $\mathbf{LH}\mathfrak{F}$, the Gorenstein dimension of G over $\mathbb{Q}$ coincides with the cohomological dimension of G over $\mathbb{Q}$, $\mathrm{Gcd}_{\mathbb{Q}}G = \mathrm{cd}_{\mathbb{Q}}G$, and when G is torsion free, then the Gorenstein dimension of G over $\mathbb{Z}$ coincides with the cohomological dimension of G over $\mathbb{Z}$, $\mathrm{Gcd}_{\mathbb{Z}}G = \mathrm{cd}_{\mathbb{Z}}G$ (Theorem 3.5).

We also show that for any group G, $\mathrm{Gcd}_{\mathbb{Q}}G \leq \mathrm{Gcd}_{\mathbb{Z}}G$ (Theorem 3.2).

One does not expect $\mathrm{Gcd}_{\mathbb{Q}}G$ to be equal to $\mathrm{Gcd}_{\mathbb{Z}}G$ since there are groups G with $\mathrm{cd}_{\mathbb{Z}}G < \infty$ and $\mathrm{cd}_{\mathbb{Q}}G < \mathrm{cd}_{\mathbb{Z}}G$ [4], in which case $\mathrm{Gcd}_{\mathbb{Q}}G = \mathrm{cd}_{\mathbb{Q}}G < \mathrm{cd}_{\mathbb{Z}}G = \mathrm{Gcd}_{\mathbb{Z}}G$. However, we believe:

Conjecture 1.1. *If G is an elementary amenable group, then* $\mathrm{Gcd}_{\mathbb{Q}}G = \mathrm{Gcd}_{\mathbb{Z}}G$.

Note that if G is an elementary amenable group, then since G is in $\mathbf{LH}\mathfrak{F}$, $\mathrm{Gcd}_{\mathbb{Q}}G = \mathrm{cd}_{\mathbb{Q}}G$.

2. Gorenstein dimension

Let S be an arbitrary ring with unit element $1 \neq 0$. All modules considered are left S-modules.

An S-module M is said to admit a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ if there is an acyclic complex $\mathcal{F} = \{(F_i, \vartheta_i) |\ i \in \mathbb{Z}\}$ of projective modules and a projective resolution $\mathcal{P} = \{(P_i, d_i) |\ i \in \mathbb{Z}, i \geq 0\}$ of M such that $\mathcal{F}$ and $\mathcal{P}$ coincide in dimensions greater than n:

$$\begin{array}{llllllllll}
\mathcal{F}: & \cdots \to F_{n+1} \to & F_n \xrightarrow{\vartheta_n} & F_{n-1} \to & \cdots \to & F_0 \to & F_{-1} \to & F_{-2} \to \cdots \\
 & \qquad \| & \| & & & & & \\
\mathcal{P}: & \cdots \to P_{n+1} \to & P_n \xrightarrow{d_n} & P_{n-1} \to & \cdots \to & P_0 \to & M \to 0 &
\end{array}$$

The number n is called the coincidence index of the complete resolution.

An S-module M is said to admit a complete resolution in the strong sense if there is a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ with $\mathrm{Hom}_S(\mathcal{F}, Q)$ acyclic for every S-projective module Q.

Definition 2.1. An S-module M is called Gorenstein projective if it admits a complete resolution in the strong sense of coincidence index 0.

If A is an S-module and n a non-negative integer, then the Gorenstein projective dimension of A is less than or equal to n, $\mathrm{Gpd}_S A \leq n$, if there is a resolution of A by Gorenstein projective S-modules of length n.

In [10] the following characterization of finite $\mathrm{Gpd}_S M$ is given:

Theorem 2.2 ([10]). *Let M be an S-module with $\mathrm{Gpd}_S M < \infty$ and n an integer. Then the following conditions are equivalent:*

(i) $\mathrm{Gpd}_S M \leq n$.
(ii) $\mathrm{Ext}^i_S(M, L) = 0$ *for all $i > n$ and all S-modules L with finite projective dimension.*
(iii) $\mathrm{Ext}^i_S(M, Q) = 0$ *for all $i > n$ and all projective S-modules Q.*
(iv) *For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, where $G_0, \ldots, G_{n-1}$ are Gorenstein projectives, then K_n is also Gorenstein projective.*

Consequently, the Gorenstein projective dimension of M is determined by the formulas

$$\begin{aligned}\mathrm{Gpd}_S M &= \sup\{i \in \mathbb{N} \mid \exists L \text{ with } \mathrm{pd}_S L < \infty : \mathrm{Ext}^i_S(M, L) \neq 0\} \\ &= \sup\{i \in \mathbb{N} \mid \exists Q \ S\text{-projective} : \mathrm{Ext}^i_S(M, Q) \neq 0\}.\end{aligned}$$

Remark 2.3. If $\mathrm{cd}_R G < \infty$, then it is well known that

$$\mathrm{cd}_R G = \sup\{i \in \mathbb{N} \mid \exists F \ RG\text{-projective} : \mathrm{H}^i(G, F) \neq 0\}.$$

It now follows from the characterization of the finite Gorenstein dimension obtained in Theorem 2.2 that if $\mathrm{Gcd}_R G < \infty$, then $\mathrm{cd}_R G = \mathrm{Gcd}_R G$.

Following Ikenaga [11] we define the generalized cohomological dimension of a group G over $\mathbb{Q}$, $\underline{\mathrm{cd}}_{\mathbb{Q}} G$, as

$$\underline{\mathrm{cd}}_{\mathbb{Q}} G = \sup\{n \in \mathbb{N} \mid \exists\ M, P : \mathrm{Ext}^n_{\mathbb{Q}G}(M, P) \neq 0, P\ \mathbb{Q}G\text{-projective}\}$$

and the generalized homological dimension of a group G over $\mathbb{Q}$ as

$$\underline{\mathrm{hd}}_{\mathbb{Q}} G = \sup\{n \in \mathbb{N} \mid \exists\ M, C : \mathrm{Tor}^{\mathbb{Q}G}_n(M, C) \neq 0, C\ \mathbb{Q}G\text{-injective}\}.$$

It is easy to see that if $\mathrm{cd}_{\mathbb{Q}} G < \infty$, then $\mathrm{cd}_{\mathbb{Q}} G = \underline{\mathrm{cd}}_{\mathbb{Q}} G$, and if $\mathrm{hd}_{\mathbb{Q}} G < \infty$, then $\mathrm{hd}_{\mathbb{Q}} G = \underline{\mathrm{hd}}_{\mathbb{Q}} G$. Moreover, it is easy to see that $\underline{\mathrm{cd}}_{\mathbb{Q}} G = \mathrm{silp}\mathbb{Q}G$.

We obtain the following relations between $\mathrm{spli}RG$, $\mathrm{Gcd}_R G$ and $\underline{\mathrm{cd}}_R G$ when R is the ring of integers $\mathbb{Z}$ or the ring of rationals $\mathbb{Q}$.

Proposition 2.4. *For any group G*

(1) $\mathrm{Gcd}_R G = \underline{\mathrm{cd}}_R G$,
(2) $\underline{\mathrm{hd}}_R G \leq \underline{\mathrm{cd}}_R G$.

Proof. In [2] statement (1) was proved for $R = \mathbb{Z}$, and its proof used the fact that $\mathrm{spli}\mathbb{Z}G = \mathrm{silp}\mathbb{Z}G$ [7, Theorem 4.4]. However $\mathrm{silp}\mathbb{Q}G = \mathrm{spli}\mathbb{Q}G$ [7, Theorem 4.4], and the proof for the case $R = \mathbb{Z}$ is valid also over $\mathbb{Q}$. In [6, Theorem 4.11] the statement (2) was proved for $R = \mathbb{Z}$. It is easy to see that the proof is also valid over $\mathbb{Q}$. Note that the proof of (2) uses a certain characterization of a natural map which relates $\mathrm{Tor}^{RG}_*(\_,\_)$ to $\mathrm{Ext}^*_{RG}(\_,\_)$. □

Proposition 2.5. *For any group G,* $\mathrm{spli}RG < \infty$ *if and only if* $\mathrm{Gcd}_R G < \infty$.

Proof. If $R = \mathbb{Z}$ the result was shown in [1, Remark 2.10]. If $R = \mathbb{Q}$, then by [7, Theorem 4.4] $\mathrm{silp}\mathbb{Q}G = \mathrm{spli}\mathbb{Q}G$; hence by Proposition 2.4(1) $\mathrm{silp}\mathbb{Q}G = \mathrm{spli}\mathbb{Q}G = \underline{\mathrm{cd}}_{\mathbb{Q}} G = \mathrm{Gcd}_{\mathbb{Q}} G$. □

Remark 2.6. It is easy to see that the proof of Theorem 2.7 in [2] is valid for $\mathbb{Q}G$-modules, and thus we obtain that $\mathrm{spli}\mathbb{Q}G < \infty$ if and only if there is a $\mathbb{Q}G$-exact sequence $0 \to \mathbb{Q} \to A$ with $\mathrm{pd}_{\mathbb{Q}G}A < \infty$. Moreover $\mathrm{pd}_{\mathbb{Q}G}A = \mathrm{Gcd}_{\mathbb{Q}}G$.

3. Gorenstein dimension and cohomological dimension

The Gorenstein dimension of a group G over R enjoys the following properties:

Theorem 3.1. (1) *If $H \leq G$, then $\mathrm{Gcd}_R H \leq \mathrm{Gcd}_R G$.*

(2) *If $1 \to N \to G \to K \to 1$ is an extension of groups, then $\mathrm{Gcd}_R G \leq \mathrm{Gcd}_R N + \mathrm{Gcd}_R K$.*

(3) *If F is a finite subgroup of G and $N_G(F)$ its normalizer in G, then $\mathrm{Gcd}_R(N_G(F)/F) \leq \mathrm{Gcd}_R G$.*

(4) *If $1 \to N \to G \to K \to 1$ is an extension of groups with $|N| < \infty$, then $\mathrm{Gcd}_R G = \mathrm{Gcd}_R K$.*

(5) *If $1 \to N \to G \to K \to 1$ is an extension of groups with $|K| < \infty$, then $\mathrm{Gcd}_R G = \mathrm{Gcd}_R N$.*

Proof. The theorem was proved for $R = \mathbb{Z}$ in [2, Theorem 2.8]. The same proof goes through over $\mathbb{Q}$. Note that we are using the characterization of the finiteness of $\mathrm{spli}\mathbb{Q}G$ given in Remark 2.6. □

Theorem 3.2. *For any group G, $\mathrm{Gcd}_{\mathbb{Q}}G \leq \mathrm{Gcd}_{\mathbb{Z}}G$.*

Proof. If $\mathrm{Gcd}_{\mathbb{Z}}G<\infty$, then by Proposition 2.5 $\mathrm{spli}\mathbb{Z}G<\infty$; hence by [9, Lemma 6.4] $\mathrm{spli}\mathbb{Q}G < \infty$, which implies that $\mathrm{Gcd}_{\mathbb{Q}}G < \infty$ by Proposition 2.5. If $\mathrm{Gcd}_{\mathbb{Q}}G = n$ and $\mathrm{Gcd}_{\mathbb{Z}}G = m$, then by Theorem 2.2 we obtain the following characterizations of n and m:

$$\text{(i)} \quad n = \sup\{i \in \mathbb{N} |\ \exists F\ \mathbb{Q}G\text{-projective} : \mathrm{H}^i(G, F) \neq 0\},$$

$$\text{(ii)} \quad m = \sup\{i \in \mathbb{N} |\ \exists L \text{ with } \mathrm{pd}_{\mathbb{Z}G}L < \infty : \mathrm{H}^i(G, L) \neq 0\}.$$

But if F is a projective $\mathbb{Q}G$-module, then clearly $\mathrm{pd}_{\mathbb{Z}G}F \leq 1$; hence it follows from (i) and (ii) that $n \leq m$. □

Definition 3.3 ([14])**.** A group G is said to be of type Φ over R if it has the following property: for any RG-module M, $\mathrm{pd}_{RG}M < \infty$ if and only if $\mathrm{pd}_{RH}M < \infty$ for any finite subgroup H of G.

Proposition 3.4. *If a group G is in $\mathbf{LH}\mathfrak{F}$, then*

(1) $\mathrm{findim}RG = \mathrm{spli}RG$, *where* $\mathrm{findim}RG$ *is the supremum of the projective dimensions of the RG-modules of finite projective dimension.*

(2) $\mathrm{findim}RG < \infty$ *if and only if G is of type Φ over R.*

Proof. (1) For G in $\mathbf{H}\mathfrak{F}$ the result was proved in [5, Theorem C]. For $R = \mathbb{Z}$ and G in $\mathbf{LH}\mathfrak{F}$ the result was proved in [15, Corollary 2], and it is easy to see that the proof of Corollary 2 of [15] is valid over $\mathbb{Q}$.

(2) Theorem 6 of [15] proves statement (2) for $R = \mathbb{Z}$. It is easy to see that its proof is also valid over $R = \mathbb{Q}$, and the result follows. □

Theorem 3.5. *If G is a group in $\mathbf{LH}\mathfrak{F}$, then*

(1) $\mathrm{Gcd}_{\mathbb{Q}}G = \mathrm{cd}_{\mathbb{Q}}G$.
(2) *If G is torsion free, then* $\mathrm{Gcd}_{\mathbb{Z}}G = \mathrm{cd}_{\mathbb{Z}}G$.

Proof. (1) If $\mathrm{Gcd}_{\mathbb{Q}}G < \infty$, then by Proposition 2.5 $\mathrm{spli}\mathbb{Q}G < \infty$; hence by Proposition 3.4 (1) $\mathrm{findim}\mathbb{Q}G < \infty$, and thus by Proposition 3.4 (2), G is of type Φ. Since $\mathrm{cd}_{\mathbb{Q}}H = 0$ for every finite subgroup H of G, it now follows that $\mathrm{cd}_{\mathbb{Q}}G < \infty$.

(2) If $\mathrm{cd}_{\mathbb{Z}}G < \infty$, then by Remark 2.3 $\mathrm{cd}_{\mathbb{Z}}G = \mathrm{Gcd}_{\mathbb{Z}}G$. If $\mathrm{Gcd}_{\mathbb{Z}}G < \infty$, then by Proposition 2.5 $\mathrm{spli}\mathbb{Z}G < \infty$, and since G is a torsion free group in $\mathbf{LH}\mathfrak{F}$ by Corollaries 2 and 4 of [15], it follows that $\mathrm{cd}_{\mathbb{Z}}G < \infty$. The result now follows. □

In [13] the notion of jump cohomology was defined for a group G as follows:

Definition 3.6 ([13]). A group G has jump cohomology over R if there exists an integer $k \geq 0$ such that if H is any subgroup of G with $\mathrm{cd}_R H < \infty$, then $\mathrm{cd}_R H \leq k$. We allow k to be infinite.

The smallest such k is called the jump height of G over R.

Corollary 3.7. *If G is a group in $\mathbf{LH}\mathfrak{F}$, then* $\mathrm{Gcd}_{\mathbb{Q}}G = \mathrm{cd}_{\mathbb{Q}}G =$*the jump height of G over $\mathbb{Q}$.*

Proof. In [13, Theorem 1.4] it was shown that if G is in $\mathbf{H}\mathfrak{F}$, then G has jump cohomology of height k over $\mathbb{Q}$ if and only if $\mathrm{cd}_{\mathbb{Q}}G \leq k$. We obtain the same result for G in $\mathbf{LH}\mathfrak{F}$ through the proof of Theorem 7 in [15]. The result now follows from (1) of Theorem 3.5. □

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