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SENDOV CONJECTURE FOR HIGH DEGREE POLYNOMIALS

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ABSTRACT. Sendov's conjecture says that if all zeros of a complex polynomial P lie in the closed unit disk and a denotes one of them, then the closed disk of center a and radius 1 contains a critical point of P (i.e. a zero of its derivative P'). The main result of this paper is to prove that, for each a, there exists an integer N such that the disk $|\zeta - a| \leq 1$ contains a critical point of P when the degree of P is larger than N. We obtain this by studying the geometry of the zeros and critical points of a polynomial which would eventually contradict Sendov's conjecture.

1. Introduction

Sendov's conjecture can be stated as follows:

Conjecture (Sendov). *Let $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$ be a monic complex polynomial having all its zeros in the closed unit disk. Then, there exists a zero ζ of the derivative P' such that $|\zeta - a| \leq 1$.*

This conjecture appears for the first time in 1967 Hayman's book *Research Problems in Function Theory* [3], where it was improperly attributed to the Bulgarian mathematician Illief. Since 1967, it has been proved for a few special cases, for example: polynomials having at most 8 distinct zeros [1], when $|a| = 1$ [6], if P vanishes at 0, when all the summits of the convex hull of the zeros of P lie on the unit circle [7], but the general case is still open in spite of 80 papers devoted to it. Surveys of the problem have been given by M. Marden [4] and Bl. Sendov [8], and we refer the reader to these for further information and bibliographies.

Set $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$ with a real, $0 < a < 1$ and $|z_k| \leq 1$ for all $k = 1, \ldots, n-1$. We prove in this paper that the derivative P' has a zero in the disk $|\zeta - a| \leq 1$ when n is larger than an integer bound N. Where the bound N depends on a, unfortunately we will not be able to give an explicit formula for N but just describe how it can be computed.

Assuming that P contradicts Sendov's conjecture, we will estimate below and above the positive real number $|P(c)|$ for some real c satisfying $0 < c < a$. This leads to a contradiction when the degree n of the polynomial P is large enough.

2. Exclusion domain

Let us introduce two useful theorems in geometry of polynomials. The first one is due to Q. I. Rahman and G. Schmeisser (see [5, p. 100]).

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Theorem 1. *Let P be a complex polynomial of degree $\leq n$, and δ a complex number satisfying $P'(\delta) \neq 0$. For any $\omega \in \mathbb{C}$ different from $P(\delta)$, the polynomial $P(z) - \omega$ has a zero in the disk whose diameter is the line segment $\left[\delta, \delta - \frac{n(P(\delta)-\omega)}{P'(\delta)}\right]$.*

The second one is due to J. H. Grace (see [2, p. 356]); we will call it the *Perpendicular bisector theorem.*

Theorem 2. *Let P denote a polynomial, and α and β two complex numbers such that $P(\alpha) = P(\beta)$. Then, the perpendicular bisector of the line segment $[\alpha, \beta]$ intersects the convex hull of the zeros of P' (i.e. each half-plane delimited by the bisector contains at least one zero of P').*

The following theorem gives an exclusion domain (i.e. without zero) for a polynomial P that contradicts Sendov's conjecture.

Theorem 3. *Let $P(z) = (z - a)\prod_{k=1}^{n-1}(z - z_k)$, where a is a real, $0 < a < 1$ and $|z_k| \leq 1$ for $k = 1, \dots, n-1$. Suppose that $P'(\zeta) \neq 0$ for $|\zeta - a| \leq 1$. Then, for any $c \in (0, a)$, the polynomial P cannot have any zero in the disk of center c and radius $1 - \sqrt{1 - c(a-c)}$.*

Proof. On the contrary, assume that there exists a zero γ of P such that $|c - \gamma| \leq 1 - \sqrt{1 - c(a-c)}$. Suppose that $\Im(\gamma) \leq 0$ and let ξ denote the complex number, with positive imaginary part, satisfying

$$|\xi - a| = |\xi| = 1,$$

so ξ is at the intersection of circles of radius 1, centered at a and 0. We have

$$|\xi - \gamma| \leq |\xi - c| + |c - \gamma| \leq 1 = |\xi - a|\cdot$$

Therefore, the perpendicular bisector of the line segment $[\gamma, a]$ doesn't intersect the *"lunula region"* $\mathcal{L}$ defined by

$$\mathcal{L} = \{z \in \mathbb{C}\ ;\ |z| \leq 1 \text{ and } |z - a| \geq 1\}\cdot$$

By the Gauss-Lucas theorem and hypothesis, $\mathcal{L}$ contains all the critical points of P. Since $P(\gamma) = P(a)$, Theorem 2 implies that the perpendicular bisector of the line segment $[\gamma, a]$ should intersect the convex hull of the critical points and then the region $\mathcal{L}$. This leads to a contradiction and proves the theorem. ☐

3. Lemmas

We give in this section some technical inequalities. For the convenience of the reader the proofs are deferred to the end of the paper (section 7).

Lemma 1. *Let $\delta \in (0, 1)$ and $z_1,\dots,z_n$ be complex numbers which all lie in the closed unit disk. Writing $m := \frac{1}{n}\Re\left(\sum_{k=1}^{n} z_k\right)$, we have*

$$\prod_{k=1}^{n} |\delta - z_k| \leq \left(\sqrt{1 + \delta^2 - 2\delta m}\right)^n.$$

Lemma 2. *Let δ and a be real numbers such that $0 < \delta < a < 1$ and let $\zeta_1, \dots, \zeta_{n-1}$ denote complex numbers satisfying $|\zeta_k| \leq 1$ and $|\zeta_k - a| > 1$, for $k = 1, \dots, n-1$. Writing $s := \frac{1}{n-1}\Re\left(\sum_{k=1}^{n-1}\zeta_k\right)$ and $q := \frac{a/2 - s}{1 + a/2}$, we have*

$$\prod_{k=1}^{n-1}\left|\frac{\delta - \zeta_k}{a - \zeta_k}\right| \leq \left\{\left(\frac{1+\delta}{1+a}\right)^q \sqrt{1 + \delta^2 - \delta a}^{\,1-q}\right\}^{n-1}.$$

Lemma 3. *Let δ, a and b be real numbers such that $0 < \delta < a < 1 < b$ and let $\xi_1, \dots, \xi_{n-1}$ denote complex numbers satisfying $|\xi_k| \leq 1$ and $|\xi_k - a| > 1$, for $k = 1, ..., n-1$. Set*

$$s := \frac{1}{n-1} \Re \left(\sum_{k=1}^{n-1} \xi_k \right) < \frac{a}{2}$$

and

$$p := \frac{a/2 - s}{1 - a/2}, \quad q := \frac{a/2 - s}{1 + a/2}, \quad B_1 := (1 + b - a)^p \left(\sqrt{1 + b^2 - ba}\right)^{1-p}$$

and $B_2 := (1+b)^q \left(\sqrt{1 + b^2 - ba}\right)^{1-q}$. We have

$$\prod_{k=1}^{n-1} |b - \xi_k| \geq \min(B_1, B_2)^{n-1}.$$

Remark. The important point is that $\min(B_1, B_2) > \sqrt{1 + b^2 - ba}$.

Lemma 4. *Let c and r denote real numbers such that $0 < c < 1$ and $0 < r < 1 - c$. Assume that $z_1, \dots, z_n$ are complex numbers satisfying $0 < |z_k| \leqslant 1$ and $\left| \frac{c - z_k}{1 - c z_k} \right| \geq r$, for $k = 1, \dots, n$. We have*

$$\prod_{k=1}^{n} \left| \frac{c - z_k}{1 - c z_k} \right| \geq r^{\beta}, \quad \textit{where} \quad \beta := \frac{\sum_{k=1}^{n} \log |z_k|}{\log \left(\frac{c + r}{1 + cr} \right)}.$$

4. Upper estimation of $|P(c)|$

Theorem 4. *Let $P(z) = (z - a) \prod_{k=1}^{n-1} (z - z_k)$, where $0 < a < 1$ and $|z_k| \leq 1$ for $k = 1, \dots, n-1$. Assume that $P'(\zeta) \neq 0$ for all $|\zeta - a| \leq 1$. Then, for any real $\delta \in (0, a)$, we have*

$$\left| \frac{P(\delta)}{P'(a)} \right| \geq \frac{1 - \sqrt{1 + \delta^2 - \delta a}}{n}.$$

Proof. Write $R = \frac{nP(\delta)}{P'(a)}$ and denote by ξ the complex number satisfying $|\xi - a| = |\xi| = 1$ with positive imaginary part. Theorem 1 asserts that the disk of center a and radius $|R|$ contains a complex number γ such that $P(\delta) = P(\gamma)$; there is no loss of generality assuming $\Im(\gamma) \geq 0$. By Theorem 2, the perpendicular bisector of the line segment $[\delta, \gamma]$ intersects the convex hull of the zeros of P'; therefore $|\xi - \delta| \geq |\xi - \gamma|$ (see Figure 1 below). We deduce that

$$|R| \geq 1 - |\xi - \gamma|$$

and then

$$\left| \frac{P(\delta)}{P'(a)} \right| \geq \frac{1 - \sqrt{1 + \delta^2 - \delta a}}{n}.$$

□

Corollary 1. *Let P be defined as above, and put $m := \frac{1}{n} \Re \left(a + \sum_{k=1}^{n-1} z_k \right)$. Then*

$$m \leq \inf_{\delta \in (0,a)} \left(\frac{\delta}{2} - \frac{1}{\delta n} \log(1 - \sqrt{1 + \delta^2 - \delta a}) \right).$$

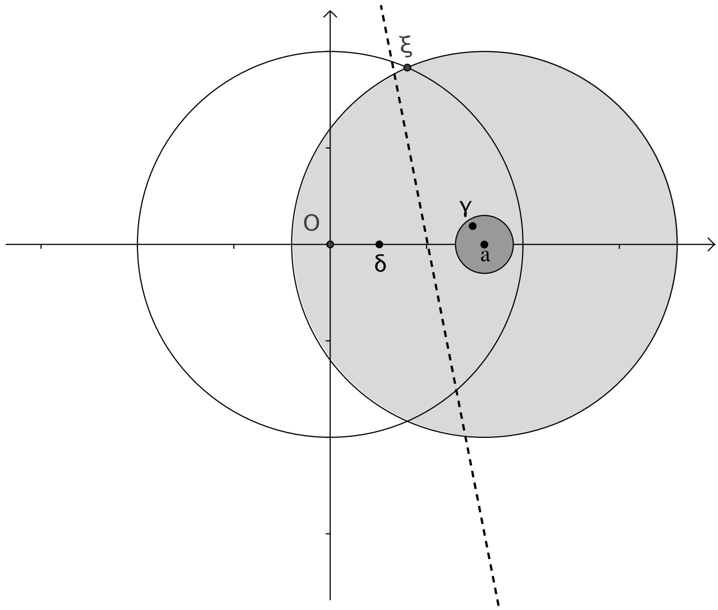

FIGURE 1. Illustration of the proof of Theorem 4

Proof. Let $\delta \in (0, a)$. Theorem 4 gives

$$|P(\delta)| \geq \frac{1 - \sqrt{1 + \delta^2 - \delta a}}{n} |P'(a)|,$$

and then $|P(\delta)| \geq 1 - \sqrt{1 + \delta^2 - \delta a}$. By Lemma 1 we know that

$$|P(\delta)| \leq (\sqrt{1 + \delta^2 - 2\delta m})^n;$$

therefore

$$1 - \sqrt{1 + \delta^2 - \delta a} \leq (\sqrt{1 + \delta^2 - 2\delta m})^n.$$

We deduce that

$$\frac{1}{n} \log(1 - \sqrt{1 + \delta^2 - \delta a}) \leq \frac{\delta^2 - 2\delta m}{2}$$
$$\iff \quad m \leq \frac{\delta}{2} - \frac{1}{\delta n} \log(1 - \sqrt{1 + \delta^2 - \delta a}),$$

which establishes the desired inequality. □

Remark. When $n \to +\infty$, $\inf_{\delta \in (0,a]} \left(\frac{\delta}{2} - \frac{1}{\delta n} \log(1 - \sqrt{1 + \delta^2 - \delta a}) \right) \to 0$. Therefore, if P satisfies the assumptions of Theorem 4, then m should be negative or close to zero for large n. This fact will be of crucial importance in section 5 to see that $K > 1$.

Theorem 5. *Let $P(z) = (z - a) \prod_{k=1}^{n-1} (z - z_k)$, where $0 < a < 1$ and $|z_k| \leq 1$ for all k. Suppose that the critical points $\zeta_1, \ldots, \zeta_{n-1}$ of P all lie in $|z - a| > 1$, which implies that $\Re(\zeta_k) < a/2$ for $k = 1, \ldots, n-1$, and so also*

$$s := \frac{1}{n-1} \Re \left(\sum_{k=1}^{n-1} \zeta_k \right) < a/2.$$

Let $q := \frac{a/2-s}{1+a/2}$ *and* N_1 *be the smallest integer such that*

$$\left(\frac{1+a/2}{1+a}\right)^q \leq \left(\frac{1-\sqrt{1-a^2/4}}{na}\right)^{1/(n-1)} \qquad \text{for all} \quad n \geq N_1.$$

Then, if $n \geq N_1$*, we have*

$$|P'(a)| \leq \frac{16n}{a^2} \qquad \text{and} \qquad |P(0)| \geq \frac{a^2}{16}. \tag{1}$$

Proof. Assume that $n \geq N_1$. Lemma 2 asserts that for all $\delta \in (0, a/2)$,

$$\begin{aligned}\left|\frac{P'(\delta)}{P'(a)}\right| &\leq \left[\left(\frac{1+\delta}{1+a}\right)^q \sqrt{1+\delta^2-\delta a}^{\,1-q}\right]^{n-1} \\ &\leq \left(\frac{1+a/2}{1+a}\right)^{q(n-1)} \\ &\leq \frac{1-\sqrt{1-a^2/4}}{na}.\end{aligned}$$

We deduce that

$$\begin{aligned}\left|\frac{P(0)}{P'(a)} - \frac{P(a/2)}{P'(a)}\right| &\leq \frac{a}{2} \sup_{\delta\in[0,a/2]} \left|\frac{P'(\delta)}{P'(a)}\right| \\ &\leq \frac{1-\sqrt{1-a^2/4}}{2n}.\end{aligned}$$

By Theorem 4, we know that

$$\left|\frac{P(a/2)}{P'(a)}\right| \geq \frac{1-\sqrt{1-a^2/4}}{n};$$

then

$$\left|\frac{P(0)}{P'(a)}\right| \geq \frac{1-\sqrt{1-a^2/4}}{2n} \geq \frac{a^2}{16n}.$$

We deduce the theorem. □

Remark. This theorem expresses that the zeros of P should lie nearby the unit circle and those of P' close to the circle $|z-a| = 1$ when the degree of P is large enough.

We are now in a position to give an upper bound for $|P(c)|$.

Theorem 6. *Let* $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$*, where* $0 < a < 1$ *and* $|z_k| \leq 1$ *for all* k*. Suppose that the derivative* P' *doesn't vanish in the disk* $|\zeta - a| \leq 1$*. Let* q *and* N_1 *be defined as in Theorem* 5*,* $c \in (0, a)$ *and set*

$$D := \max\left\{\left(\frac{1}{1+a}\right)^q \; ; \; \left(\frac{1+c}{1+a}\right)^q \left(\sqrt{1+c^2-ac}\right)^{1-q}\right\} < 1.$$

Define N_2 *to be the smallest integer such that*

$$D^{n-1} \leq \frac{a}{16n} \qquad \text{for all} \qquad n \geq N_2.$$

Then, if $n \geq \max\{N_1, N_2\}$ *we have*

$$|P(c)| \leq 1 + a.$$

Proof. Consider the mapping f defined by

$$\begin{array}{rcccl} f & : & [0,c] & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \log\left[\left(\frac{1+x}{1+a}\right)^q \sqrt{1+x^2-xa}^{\,1-q}\right]; \end{array}$$

the first derivative of f is given by

$$f'(x) = \frac{x^2+(1-a+m)x-m}{(1+x)(1+x^2-xa)}.$$

Since $f'(x)$ shares the sign of $x^2+(1-a+m)x-m$, by Lemma 2 we have

$$\left|\frac{P'(\delta)}{P'(a)}\right| \le D^{n-1}, \qquad \text{and then} \qquad \sup_{\delta\in[0,c]}\left|\frac{P'(\delta)}{P'(a)}\right| \le \frac{a}{16n}.$$

For all $n \ge \max\{N_1, N_2\}$, using Theorem 5, we deduce that

$$|P(0)-P(c)| \le \sup_{\delta\in[0,c]}\left|\frac{P'(\delta)}{P'(a)}\right| c\,|P'(a)| \le \frac{c}{a} \le 1.$$

Therefore

$$|P(c)| \le 1 + |P(0)| \le 1 + a.$$

□

5. Lower estimation of $|P(c)|$

We prove, in this section, that there exist constants $C > 0$ and $K > 1$ such that if n is large enough the inequality $|P(c)| \ge CK^n$ holds. Three new lemmas are needed; their proofs are still deferred to the end (section 7).

Lemma 5. *Let h and c be positive real numbers satisfying $0 < c < 1-h$. For all $z \in \mathbb{C}$, if $|z| \ge 1-h$, we have*

$$|c-z| \ge \frac{c}{1-h}\left|\frac{(1-h)^2}{c} - z\right|.$$

Lemma 6. *Let $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$, where $0 < a < 1$ and $|z_k| \le 1$ for $k = 1,\dots,n-1$. Denote by $\zeta_1,\dots,\zeta_{n-1}$ the critical points of P. Then, for all $b > 1$, we have*

$$(b-a)\prod_{k=1}^{n-1}|b-z_k| \ge (b-1)\prod_{k=1}^{n-1}|b-\zeta_k|.$$

Lemma 7. *Let $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$, where $0 < a < 1$ and $|z_k| \le 1$ for $k = 1,\dots,n-1$. Suppose that $P'(\zeta) \ne 0$ for $|\zeta - a| \le 1$. Let c and h be real numbers such that $0 < h < c < a < 1-h$. Then, the disk $\mathcal{D}$ defined by*

$$\mathcal{D} := \left\{z \in \mathbb{C}\,;\ \left|\frac{(c-z)}{(1-h)^2-cz}\right| \le \frac{c(a-c)}{2((1-h)^2-c^2)}\right\}$$

contains no zero of P.

We can now give a lower estimate of $|P(c)|$.

Theorem 7. *Let* $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$*, where* $0 < a < 1$ *and* $|z_k| \le 1$ *for* $k = 1, \dots, n-1$*. Suppose that* $P'(\zeta) \neq 0$ *for* $|\zeta - a| \le 1$*. Let* $c \in (0, a)$*,* q *and* N_1 *be defined as in Theorem* 5*. Set*

$$p := \frac{a/2 - m}{1 - a/2}, \quad r := \frac{c(a-c)}{2(1-c^2)}, \quad \alpha := \log\left(\frac{a}{16}\right) / \log\left(\frac{c+r}{1+cr}\right)$$

and

$$K := \min\left\{(1+c-ac)^p \sqrt{1+c^2-ac}^{\,1-p} \,;\, (1+c)^q \sqrt{1+c^2-ac}^{\,1-q}\right\}.$$

If $n \ge N_1$*, we have*

$$|P(c)| \ge \frac{(1-c)(a-c)}{1-ac}\, r^\alpha\, K^{n-1}.$$

Remark. Observe that if c is sufficiently close to a, then $K > 1$.

Proof. Fix $0 < h < 1-a$ and assume that the zeros of P are indexed such that for all $k \ge n_0$ we have $|z_k| \ge 1-h$. Let $b_h = \frac{(1-h)^2}{c}$; we have

$$\begin{aligned}
|P(c)| =& |c-a| \prod_{k=1}^{n_0-1} |c-z_k| \prod_{k=n_0}^{n-1} |c-z_k| \\
\ge& (a-c) \prod_{k=1}^{n_0-1} |c-z_k| \left(\frac{c}{1-h}\right)^{n-n_0} \prod_{k=n_0}^{n-1} |b_h - z_k| \quad \text{(using Lemma 5)} \\
=& \left(\frac{a-c}{b_h-a}\right) \prod_{k=1}^{n_0-1} \left|\frac{c-z_k}{b_h-z_k}\right| \left(\frac{c}{1-h}\right)^{n-n_0} |b_h-a| \prod_{k=1}^{n-1} |b_h-z_k|.
\end{aligned}$$

By Lemma 6, we deduce

$$\begin{aligned}
|P(c)| \ge& \left(\frac{a-c}{b_h-a}\right) \prod_{k=1}^{n_0-1} \left|\frac{c-z_k}{b_h-z_k}\right| \left(\frac{c}{1-h}\right)^{n-n_0} (b_h-1) \prod_{k=1}^{n-1} |b_h-\zeta_k| \\
\ge& \left(\frac{(b_h-1)(a-c)}{(b_h-a)}\right) \prod_{k=1}^{n_0-1} \left|\frac{(c-z_k)(1-h)}{(b_h-z_k)c}\right| \left(\frac{c}{1-h}\right)^{n-1} \prod_{k=1}^{n-1} |b_h-\zeta_k|.
\end{aligned}$$

Using Lemma 3, we obtain

$$|P(c)| \ge \left(\frac{(b_h-1)(a-c)}{(b_h-a)}\right) \prod_{k=1}^{n_0-1} \left|\frac{(c-z_k)(1-h)}{(b_h-z_k)c}\right| (K_h)^{n-1}, \tag{2}$$

where K_h is defined by

$$\frac{c}{1-h} \min\left\{(1+b_h-a)^p \sqrt{1+b_h^2-ab_h}^{\,1-p} \,;\, (1+b_h)^q \sqrt{1+b_h^2-ab_h}^{\,1-q}\right\}.$$

Set $c' = \frac{c}{1-h}$, $z_k' = \frac{z_k}{1-h}$ and $r_h = \frac{c(a-c)(1-h)}{2((1-h)^2-c^2)}$; by Lemma 7 we know that

$$\left|\frac{(c-z_k)(1-h)}{(b_h-z_k)c}\right| = \left|\frac{c'-z_k'}{1-c'z_k'}\right| \ge r_h.$$

Lemma 4 gives that

$$\prod_{k=1}^{n_0-1} \left|\frac{c'-z_k'}{1-c'z_k'}\right| \ge r_h^{\beta_h}, \tag{3}$$

where $\beta_h = \log\left(\prod_{k=1}^{n_0-1}|z'_k|\right)/\log\left(\frac{c'+r_h}{1+c'r_h}\right)$. By Theorem 5, we have

$$\prod_{k=1}^{n_0-1}|z'_k| \geq \prod_{k=1}^{n_0-1}|z_k| \geq \prod_{k=1}^{n-1}|z_k| \geq \frac{a}{16};$$

thus in (3) the exponent β_h can be replaced by $\alpha_h = \log\left(\frac{a}{16}\right)/\log\left(\frac{c'+r_h}{1+c'r_h}\right)$. Combining (2) with (3), we deduce that

$$|P(c)| \geq \left(\frac{(b_h-1)(a-c)}{(b_h-a)}\right) r_h^{\alpha_h}(K_h)^{n-1}.$$

The theorem follows by letting $h \to 0$. $\square$

6. Main result and computations

We can now formulate our main result.

Theorem 8. *Let $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$, where $0 < a < 1$ and $|z_k| \leq 1$ for $k = 1, \ldots, n-1$. Let N_1 and N_2 be defined by Theorems 5 and 6, and suppose that $c \in (0, a)$ is chosen sufficiently close to a to have $K > 1$ (K is defined in Theorem 7). If*

$$\deg(P) \geq N := \max\left\{N_1; N_2; \frac{\log\left(\frac{(1+a)(1-ac)}{(1-c)(a-c)}\right) - \alpha\log(r)}{\log(K)} + 1\right\},$$

then the derivative P' has a zero in the disk $|\zeta - a| \leq 1$.

Proof. Assume, on the contrary, that $P'(\zeta) \neq 0$ for $|\zeta - a| \leq 1$. Combining the results of Theorems 6 and 7, we obtain

$$1 + a \geq |P(c)| \geq \frac{(1-c)(a-c)}{1-ac}\, r^{\alpha}\, K^{n-1};$$

thus

$$\log\left(\frac{(1+a)(1-ac)}{(1-c)(a-c)}\right) \geq \alpha\log(r) + (n-1)\log K,$$

i.e.

$$n \leq \left[\log\left(\frac{(1+a)(1-ac)}{(1-c)(a-c)}\right) - \alpha\log(r)\right]/\log(K) + 1.$$

This implies the theorem. $\square$

Before turning to computations, let us recall the main formulae:

$$\begin{aligned}
p &= \tfrac{a/2-m}{1-a/2}\ ; \\
q &= \tfrac{a/2-m}{1+a/2}\ ; \\
K &= \min\left\{(1+c-ac)^p\sqrt{1+c^2-ac}^{\,1-p}\ ;\ (1+c)^q\sqrt{1+c^2-ac}^{\,1-q}\right\}\ ; \\
r &= \tfrac{c(a-c)}{2(1-c^2)}\ ; \\
\alpha &= \log\left(\tfrac{a}{16}\right)/\log\left(\tfrac{c+r}{1+cr}\right)\ ; \\
N &= \max\left\{N_1; N_2; \tfrac{\log\left(\frac{(1+a)(1-ac)}{(1-c)(a-c)}\right)-\alpha\log(r)}{\log(K)} + 1\right\}.
\end{aligned}$$

Therefore, if a and c are given with $0 < c < a$, to compute N one needs to know m. Its exact value is unknown but can be replaced by the upper estimate given by

Corollary 1, which depends only on n. Computations can be done in the following way; assume that $a \in (0,1)$ is given:

(i) Choose arbitrarily $c \in (0,a)$ and $m > 0$.
(ii) Compute p, q, K and check that $K > 1$. If not, go back to the first step, increasing c.
(iii) Compute N and deduce the upper estimate of m given in Corollary 1. If it is larger than m, go back to the first step, increasing m, otherwise decreasing m, and repeat until equality holds.
(iv) Adjust the choice of c to obtain the smallest value of N.

Finally, find below the values of N obtained for many choices of a.

a	c	m	r	α	p	q	K	N
$0,9$	$0,756$	$0,080$	$0,1270$	$13,32$	$0,673$	$0,255$	$1,031$	1006
$0,8$	$0,700$	$0,100$	$0,0686$	$9,66$	$0,500$	$0,214$	$1,049$	616
$0,7$	$0,630$	$0,110$	$0,0366$	$7,3$	$0,369$	$0,178$	$1,051$	560
$0,6$	$0,550$	$0,100$	$0,0197$	$5,73$	$0,286$	$0,154$	$1,048$	563
$0,5$	$0,460$	$0,100$	$0,0117$	$4,58$	$0,200$	$0,120$	$1,035$	718
$0,4$	$0,374$	$0,089$	$0,0057$	$3,8$	$0,139$	$0,093$	$1,024$	1004
$0,3$	$0,284$	$0,073$	$0,0025$	$3,18$	$0,091$	$0,067$	$1,014$	1654
$0,2$	$0,191$	$0,053$	$0,0009$	$2,65$	$0,052$	$0,043$	$1,007$	3587
$0,1$	$0,096$	$0,029$	$0,0002$	$2,17$	$0,022$	$0,020$	$1,002$	15064

7. Proofs of lemmas

Proof of Lemma 1. For $k = 1,...,n$, fix $\Re(z_k)$; then $|\delta - z_k|$ is maximum when $|z_k| = 1$. Therefore, we can assume that $|z_k| = 1$ and write $z_k = e^{i\theta_k}$. The mapping $\Phi(x) = \frac{1}{2}\log(1+\delta^2 - 2\delta x)$ is concave. By Jensen's inequality we get

$$\begin{aligned}\log\left(\left|\prod_{k=1}^{n}(\delta - z_k)\right|^{1/n}\right) &= \frac{1}{n}\sum_{k=1}^{n}\log|\delta - z_k| \\ &= \frac{1}{n}\sum_{k=1}^{n}\Phi(\cos\theta_k) \\ &\le \Phi\left(\frac{1}{n}\sum_{k=1}^{n}\cos\theta_k\right) \\ &= \log\left(\sqrt{1+\delta^2-2\delta s}\right),\end{aligned}$$

which establishes the lemma. □

Proof of Lemma 2. For $k = 1,\dots,n-1$, if $\Re(\zeta_k)$ is given, the modulus $\left|\frac{\delta-\zeta_k}{a-\zeta_k}\right|$ is maximal when $|\zeta_k| = 1$. Therefore, we can assume that for all k, $|\zeta_k| = 1$. The mapping Φ defined on $[-1, a/2]$ by

$$\Phi(x) = \frac{1}{2}\log\left(\frac{1+\delta^2-2\delta x}{1+a^2-2ax}\right)$$

is convex since

$$\Phi''(x) = \frac{2(a-\delta)(1-a\delta)((a+\delta)(1+a\delta)-2a\delta x)}{(1+\delta^2-2\delta x)^2(1+a^2-2ax)^2} \ge 0.$$

We deduce that

$$\begin{aligned}\log\left(\prod_{k=1}^{n-1}\left|\frac{\delta-\zeta_k}{a-\zeta_k}\right|\right) &= \sum_{k=1}^{n-1}\Phi(\mathrm{Re}(\zeta_k))\\ &\le (n-1)\left[q\Phi(-1)+(1-q)\Phi\left(\frac{a}{2}\right)\right]\\ &\le (n-1)\left[q\log\left(\frac{1+\delta}{1+a}\right)+(1-q)\log\sqrt{1+\delta^2-\delta a}\right],\end{aligned}$$

where q satisfies $-q+(1-q)\frac{a}{2}=s$, and so $q=\frac{a/2-s}{1+a/2}$. Applying the exponential function to both sides we obtain the lemma. □

Proof of Lemma 3. Let $k\in\{1,\dots,n-1\}$; for $\Re(\zeta_k)$ given, the modulus $|b-\zeta_k|$ is minimal when $|a-\zeta_k|=1$ or ζ_k is real with $-1\le\zeta_k\le -1+a$. Assume that $|a-\zeta_k|=1$ or $\zeta_k\in[-1,a-1]$ for each k. The mapping Φ, defined on $[-1,a/2]$ by

$$x\longmapsto\begin{cases}\frac{1}{2}\log\left(1+(b-a)(a+b-2x)\right) & \text{if}\quad x\ge a-1,\\ \log(b-x) & \text{if}\quad x\in[-1,a-1],\end{cases}$$

satisfies $\Phi''(x)\le 0$. Then Φ is concave on $[-1,-1+a]$ and on $[-1+a,a/2]$. We deduce that

$$\begin{aligned}\log\left(\prod_{k=1}^{n-1}|b-\zeta_k|\right) &= \sum_{k=1}^{n-1}\Phi(\Re(\zeta_k))\\ &\ge (n-1)\min\{\alpha\Phi(-1)+\beta\Phi(-1+a)+\gamma\Phi(a/2)\},\end{aligned}$$

where the minimum is taken over the set of all $(\alpha,\beta,\gamma)\in\mathbb{R}_+^3$ satisfying

$$\begin{cases}\alpha+\beta+\gamma & = & 1,\\ -\alpha+(a-1)\beta+\frac{a}{2}\gamma & = & s.\end{cases}\tag{4}$$

Let us define the mappings $g,\ f_1,\ f_2:\mathbb{R}_+^3\longrightarrow\mathbb{R}$ by

$$\begin{aligned}g(\alpha,\beta,\gamma)&=\alpha\log(b+1)+\beta\log(b+1-a)+\frac{\gamma}{2}\log(1+b^2-ab)\,,\\ f_1(\alpha,\beta,\gamma)&=\alpha+\beta+\gamma\,,\\ f_2(\alpha,\beta,\gamma)&=-\alpha+(-1+a)\beta+\frac{a}{2}\gamma.\end{aligned}$$

Then, $\frac{1}{n-1}\log\left(\prod_{k=1}^{n-1}|b-\zeta_k|\right)$ is greater than

$$\min_{(\alpha,\beta,\gamma)\in\mathbb{R}_+^3}\left\{g(\alpha,\beta,\gamma)\ ;\ f_1(\alpha,\beta,\gamma)=1 \text{ and } f_2(\alpha,\beta,\gamma)=m\right\}.\tag{5}$$

The Lagrange multipliers theory asserts that if the minimum is reached at $(\alpha_0,\beta_0,\gamma_0)\in(\mathbb{R}_+^*)^3$, then there exist multipliers λ_1 and λ_2 such that

$$\nabla g=\lambda_1\nabla f_1+\lambda_2\nabla f_2,$$

i.e.

$$\begin{pmatrix}\log(b+1)\\ \log(b+1-a)\\ \frac{1}{2}\log(1+b^2-ab)\end{pmatrix}=\lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda_2\begin{pmatrix}-1\\-1+a\\a/2\end{pmatrix}.$$

This is impossible in the generic case. We deduce (if necessary slightly modifying a or b) that if the minimum of (5) is reached at (α,β,γ), then we have $\alpha=0$ or $\beta=0$ or $\gamma=0$. Let us consider the different possibilities.

- If $\alpha = 0$, replacing in (4) we obtain $\beta = \frac{a/2-s}{1-a/2}$; thus

$$\prod_{k=1}^{n-1} |b - b_k| \geq \left((b+1-a)^{\beta}(\sqrt{1+b^2-ab})^{1-\beta}\right)^{n-1} = B_1^{n-1}.$$

- If $\beta = 0$, replacing in (4) we obtain $\alpha = \frac{a/2-s}{1+a/2}$; thus

$$\prod_{k=1}^{n-1} |b - b_k| \geq \left((b+1)^{\alpha}(\sqrt{1+b^2-ab})^{1-\alpha}\right)^{n-1} = B_2^{n-1}.$$

- If $\gamma = 0$, we have

$$\prod_{k=1}^{n-1} |b - b_k| \geq (1+b-a)^{n-1} \geq B_1^{n-1}.$$

This completes the proof of Lemma 3. □

Proof of Lemma 4. Let $k \in \{1, \dots, n\}$. If $\left|\frac{c-z_k}{1-cz_k}\right|$ is given, then $|z_k|$ is maximal when z_k is a real number with $\frac{c+r}{1+cr} \leq z_k \leq 1$. For $k = 1, \dots, n$, assume that $z_k \in [\frac{c+r}{1+cr}, 1]$, write $\alpha_k = \log(z_k)$ and consider the mapping Φ, defined on $\left[\log(\frac{c+r}{1+cr}), 0\right]$, by

$$\Phi(\alpha) = \log\left(\frac{e^{\alpha}-c}{1-ce^{\alpha}}\right).$$

It's easily seen that Φ is concave; therefore,

$$\begin{aligned}\log\left(\prod_{k=1}^{n}\left|\frac{c-z_k}{1-cz_k}\right|\right) &= \sum_{k=1}^{n}\Phi(\alpha_k)\\ &\geq \beta\Phi\left(\log(\frac{c+r}{1+cr})\right) + (n-\beta)\Phi(0),\end{aligned}$$

where $\beta\log(\frac{c+r}{1+cr}) = \log\left(\prod_{k=1}^{n}|z_k|\right)$. Lemma 4 is obtained taking the exponential. □

Proof of Lemma 5. The following inequalities are equivalent:

$$\begin{aligned}&|c-z| \geq \frac{c}{1-h}\left|\frac{(1-h)^2}{c} - z\right|\\ \iff &\left|\frac{z}{1-h} - \frac{c}{1-h}\right|^2 \geq \left|1 - \frac{c}{1-h}\frac{z}{1-h}\right|^2\\ \iff &\left|\frac{z}{1-h}\right|^2 + \left(\frac{c}{1-h}\right)^2 \geq 1 + \left|\frac{cz}{(1-h)^2}\right|^2\\ \iff &\left(\left|\frac{z}{1-h}\right|^2 - 1\right)\left(1 - \left(\frac{c}{1-h}\right)^2\right) \geq 0.\end{aligned}$$

We deduce the lemma. □

Proof of Lemma 6. Let us compute the quotient:

$$\begin{aligned}\left|\frac{P'(b)}{P(b)}\right| &= \left|\frac{1}{b-a}+\sum_{k=1}^{n-1}\frac{1}{b-z_k}\right| \\ &\leq \frac{1}{b-a}+\sum_{k=1}^{n-1}\frac{1}{|b-z_k|} \\ &\leq \frac{n}{b-1}.\end{aligned}$$

Then $|P(b)| \geq \frac{b-1}{n}|P'(b)| = (b-1)\prod_{k=1}^{n-1}|b-\zeta_k|$, and the lemma follows. ☐

Proof of Lemma 7. Set $k = \frac{c(a-c)}{2((1-h)^2-c^2)}$; the disk $\mathcal{D}$ has center ω and radius R given by

$$\omega = c\frac{1-k^2(1-h)^2}{1-(kc)^2} \qquad \text{and} \qquad R = k\frac{(1-h)^2-c^2}{1-(kc)^2}.$$

According to Theorem 3, it suffices to show that

$$R \leq 1-\sqrt{1+\omega^2-\omega a}$$

or equivalently that

$$\omega^2-R^2 \leq \omega a - 2R \qquad \text{and} \qquad R \leq 1.$$

The first inequality holds since

$$\omega^2-R^2 \leq \omega a-2R \quad \Longleftrightarrow \quad k^2(1-h)^2((1-h)^2-ac) \geq 0.$$

On the other hand,

$$R \leq 1 \qquad \Longleftrightarrow \qquad \frac{c(a-c)}{2} \leq 1-k^2c^2.$$

Therefore the second inequality is straightforward since $k \leq \frac{1}{2}$. The lemma is proved. ☐

8. Conclusion

It may be surprising to see that Sendov's conjecture is easily proved in extremal cases, meaning when $a = 0$ or $a = 1$ and in the generic case $0 < a < 1$, only a few partial results are known. In the present paper, we try to fill this gap, but it remains to obtain a definitive proof of the conjecture, that is, to demonstrate that, with our notations, $N = 8$ for all $a \in (0,1)$.

We have shown that if a zero, denoted by a, of P is given, one can compute an integer bound N such that if $\deg P \geq N$, then P' has a zero in the disk $|z-a| \leq 1$. It would be nice if N could be given independently of $|a|$ or, at least, to have an explicit formula for N in terms of a.

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