A TWISTED QUADRATIC MOMENT FOR DIRICHLET L-FUNCTIONS

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Abstract. Given $c$, a positive integer, we give an explicit formula for the quadratic moments

$$\sum_{\chi \in X_f^-} \chi(c)|L(1, \chi)|^2,$$

where $X_f^-$ is the set of the odd Dirichlet characters mod $f$ with $f > 2$.

1. Introduction and statement of results

Let $f > 2$ be an integer. Let $X_f^-$ denote the set of the odd Dirichlet characters mod $f$. Hence, $\#X_f^- = \phi(f)/2$. Set

$$M(f, c) := \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} \chi(c)|L(1, \chi)|^2 \quad (\gcd(f, c) = 1).$$

In 1982, H. Walum gave in [Wal] a formula for $M(p, 1)$, $p$ a prime:

$$M(p, 1) = \frac{\pi^2}{6} \frac{(p - 1)(p - 2)}{p^2}.$$

In 1993, we generalized Walum’s result to arbitrary moduli in [Lou1] by proving that

$$M(f, 1) = \frac{\pi^2}{6} \frac{\phi(f)}{f} \left( \prod_{p|f} (1 + \frac{1}{p}) - \frac{3}{f} c \right),$$

(see [Lou2] for applications). In 2010, Z. Wu and W. Zhang gave in [WZ] a formula for $M(p, 2)$, $M(p, 4)$ and $M(p, 8)$, and conjectured a formula for any $M(p, 2^k)$, $k \geq 1$. In this paper, we state and prove a general formula for $M(f, c)$:

**Theorem 1.** Let $c > 1$ be a given positive integer. For $f > 2$ with $\gcd(f, c) = 1$, we have

$$M(f, c) = \frac{\pi^2}{6c} \frac{\phi(f)}{f} \left( \prod_{p|f} (1 + \frac{1}{p}) - \frac{3c}{f} \right) - \frac{\pi^2}{2cf^2} \sum_{d|f} d\mu(f/d)S(c, d).$$

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where $S(c,d)$, which depends only on $d$ mod $c$, is defined by

$$S(c,d) := \sum_{a=1}^{c-1} \cot \left( \frac{\pi a}{c} \right) \cot \left( \frac{\pi ad}{c} \right) \quad (\gcd(c,d) = 1).$$

Let us rewrite this result for $c = 2, 3$ and $4$. Since $S(2,d) = 0$ for $d$ odd,

$$M(f,2) = \frac{\pi^2 \phi(f)}{12} \left( \prod_{p \mid f} (1 + \frac{1}{p}) - \frac{6}{f} \right) \quad (f > 2 \text{ and odd}).$$

Let $\chi_3$ and $\chi_4$ denote the odd Dirichlet characters mod $3$ and mod $4$, respectively. For $f > 3$ and relatively prime with $3$, we have

$$M(f,3) = \frac{\pi^2 \phi(f)}{18} \left( \prod_{p \mid f} (1 + \frac{1}{p}) - \frac{9}{f} - \frac{2\chi_3(f)}{f} \prod_{p \mid f} \frac{p - \chi_3(p)}{p-1} \right).$$

For $f > 2$ and $f$ odd, we have

$$M(f,4) = \frac{\pi^2 \phi(f)}{24} \left( \prod_{p \mid f} (1 + \frac{1}{p}) - \frac{12}{f} - \frac{6\chi_4(f)}{f} \prod_{p \mid f} \frac{p - \chi_4(p)}{p-1} \right).$$

Moreover, let $p \geq 3$ be an odd prime that does not divide $c$. Theorem 1 and (11) give

$$M(p,c) = \frac{\pi^2(p^2 - 3(c + S(c,p))p + c^2 + 1)}{6cp^2},$$

and $S(c,p)$ depends on $p$ mod $c$ only, and we obtain:

**Corollary 2.** Let $c$ be a given positive integer. Let $p \geq 3$ be an odd prime that does not divide $c$. We have

$$M(p,c) = \frac{\pi^2(p-1)(p-c^2-1)}{6cp^2} \quad \text{if } p \equiv 1 \pmod{c},$$

$$M(p,c) = \frac{\pi^2(p^2 + (c^2 - 6c + 2)p + c^2 + 1)}{6cp^2} \quad \text{if } p \equiv -1 \pmod{c},$$

and

$$M(p,c) = \frac{\pi^2(p^2 - \frac{c^2 + 5}{2}p + c^2 + 1)}{6cp^2} \quad \text{if } p \equiv 2 \pmod{c}.$$

**Proof.** In the case that $p \equiv 1$ (mod $c$), we have $S(c,p) = S(c,1) = (c-1)(c-2)/3$, by (11). In the case that $p \equiv -1$ (mod $c$), we have $S(c,p) = -S(c,1)$. In the case that $p \equiv 2$ (mod $c$), using $(\cot x)(\cot(2x)) = \frac{1}{2} \cot^2 x - \frac{1}{2}$, we have $S(c,p) = S(c,2) = \frac{1}{2}S(c,1) - \frac{c-1}{2} = \frac{(c-1)(c-5)}{6}$. \hfill \Box

From Corollary 2 or from (3), (4) and (5), we get

$$M(p,2) = \frac{\pi^2(p-1)(p-5)}{12p^2},$$

$$M(p,3) = \frac{\pi^2}{18} \times \begin{cases} (p-1)(p-10)/p^2 & \text{if } p \equiv 1 \pmod{3}, \\ (p^2 - 7p + 10)/p^2 & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$
and
\[ M(p, 4) = \frac{\pi^2}{24} \times \begin{cases} (p - 1)(p - 17)/p^2 & \text{if } p \equiv 1 \pmod{4}, \\ (p^2 - 6p + 17)/p^2 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \]

2. A formula for \( M(f, 1) \) and \( M(f, 2) \)

Since
\[ L(1, \chi) = \frac{\pi}{2f} \sum_{a=1}^{f-1} \chi(a) \cot \left( \frac{\pi a}{f} \right) \]
(see [Lou1, Proposition 1]), and since
\[ \frac{2}{\phi(f)} \sum_{\chi \in X_f^{-}} \chi(c)\chi(a)\overline{\chi(b)} = \begin{cases} 1 & \text{if } b \equiv ac \pmod{f} \text{ and } \gcd(abc, f) = 1, \\ -1 & \text{if } b \equiv -ac \pmod{f} \text{ and } \gcd(abc, f) = 1, \\ 0 & \text{otherwise}, \end{cases} \]
we obtain
\[ (7) \quad M(f, c) = \frac{\pi^2}{2f^2} \tilde{S}(f, c), \]
where
\[ \tilde{S}(f, c) := \sum_{\substack{a=1 \\text{gcd}(a, f) = 1}}^{f-1} \cot \left( \frac{\pi a}{f} \right) \cot \left( \frac{\pi ac}{f} \right) \quad (\gcd(f, c) = 1). \]

Moreover, for \( F : (0, 1) \to \mathbb{C} \) we have
\[ (9) \quad \sum_{\substack{a=1 \\text{gcd}(a, f) = 1}}^{f-1} \frac{F(a/f)}{\phi(f)} = \sum_{d \mid f} \mu(f/d) \sum_{\substack{a=1 \\text{gcd}(a, f) = 1}}^{d-1} F(a/d). \]
Hence, we obtain
\[ (10) \quad \tilde{S}(f, c) = \sum_{d \mid f} \mu(f/d) S(d, c), \]
where \( S(d, c) \) is defined in (2).

Lemma 3. We have
\[ \tilde{S}(f, 1) = \frac{f^2}{3} \phi(f) \left( \prod_{p \mid f} \left( 1 + \frac{1}{p} \right) - \frac{3}{f} \right). \]

Proof. We have (see [Lou1, Lemme (a)], or use (12))
\[ (11) \quad S(f, 1) = \sum_{a=1}^{f-1} \cot^2 \left( \frac{\pi a}{f} \right) = \frac{(f - 1)(f - 2)}{3}. \]

Using (9) and (11), the result follows. \( \square \)

Using Lemma 3 and (7), we deduce (11). Notice that using \((\cot x)(\cot(2x)) = \frac{1}{2} \cot^2 x - \frac{1}{2}\), we obtain \( \tilde{S}(f, 2) = \frac{1}{2} \tilde{S}(f, 1) - \frac{1}{2} \phi(f) \), and (3) follows.
3. A formula for $M(f,c)$: Proof of Theorem

**Lemma 4.** Let $c \geq 1$ be a positive integer. It holds that

$$(\cot x)(\cot(cx)) = \frac{1}{c} \cot^2 x - \frac{c^2 - 1}{3c} + \frac{1}{c} \sum_{k=1}^{c-1} \frac{\cos(k\pi/c)}{\sin^3(k\pi/c)} \cot(k\pi/c) - \cot x.$$ 

*Proof.* For $z$ complex, set

$$F(z) := (\cot z)(\cot(cz)) - \frac{1}{c} \cot^2 z + \frac{c^2 - 1}{3c} - \frac{1}{c} \sum_{k=1}^{c-1} \frac{\cos(k\pi/c)}{\sin^3(k\pi/c)} \cot(k\pi/c) - \cot z.$$ 

Clearly, $F(z)$ has no pole at the points $z_k = k\pi/c$, $1 \leq k \leq c - 1$. Now,

$$(\cot z)(\cot(cz)) = \frac{1}{cz^2} - \frac{c^2 + 1}{3c} + O(z^2)$$

yields

$$(\cot z)(\cot(cz)) - \frac{1}{c} \cot^2 z = \frac{1 - c^2}{3c} + O(z^2).$$

Hence $F(z)$ has no pole at $z = 0$ and $F(0) = 0$. Hence, $F(z)$ is $\pi$-periodic and entire. Moreover, $F(z)$ is bounded (use $\cot(\sigma + it) = -i + O(e^{-2t})$ and $\cot(\sigma - it) = \cot(\sigma + it) = i + O(e^{-2t})$ uniformly in $\sigma$ for $t \geq 1$ to see that $F(z)$ is bounded independently of $T \geq 1$ on the border of any rectangle of summits $\alpha + iT$, $\alpha + \pi + iT$, $\alpha + \pi - iT$ and $\alpha - iT$, where $\alpha \not\in \{k\pi/c, 0 \leq k < c\}$ and $T \geq 1$). Hence, $F(z)$ is constant. Since $F(0) = 0$, we have $F(z) = 0$ for any complex number $z$. \(\square\)

**Lemma 5.** Let $d > 1$ be an integer. Let $\alpha$ be a real number. Write $\alpha + i = \rho e^{i\theta}$. Assume that $\alpha \not\in \{\cot(\pi a/d); 1 \leq a \leq d - 1\}$. Then,

$$T(d, \alpha) := \sum_{a=1}^{d-1} \frac{1}{\alpha - \cot(\pi a/d)} = \frac{d \sin((d-1)\theta)}{\rho \sin(d\theta)}.$$

*Proof.* The $\cot(\pi a/d)$, $1 \leq a \leq d - 1$, are the roots of

$$(12)\quad P(X) = \frac{(X + i)^d - (X - i)^d}{2id} = \prod_{a=1}^{d-1} (X - \cot(\pi a/d)).$$

Hence,

$$T(d, \alpha) = \frac{P'}{P}(\alpha) = d \frac{(\alpha + i)^{d-1} - (\alpha - i)^{d-1}}{(\alpha + i)^d - (\alpha - i)^d}.$$ 

The result follows. \(\square\)

**Lemma 6.** For $c > 1$, $d > 1$ and $\gcd(c,d) = 1$, it holds that

$$S(d,c) = \frac{c^2 + d^2 - 3cd + 1}{3c} - \frac{d}{c} S(c,d).$$

*Proof.* By (2) and Lemma 4, we have

$$S(d,c) = \sum_{a=1}^{d-1} \left( \frac{1}{c} \cot^2 \left( \frac{\pi a}{d} \right) - \frac{c^2 - 1}{3c} + \frac{1}{c} \sum_{k=1}^{c-1} \frac{\cos(k\pi/c)}{\sin^3(k\pi/c)} \cot(k\pi/c) - \cot(k\pi/c) \right).$$
Since \( \cot(k\pi/c) + i \frac{1}{\sin(k\pi/c)} e^{ik\pi/c} \), using Lemma 5 we obtain

\[
S(d, c) = \frac{(d-1)(d-2)}{3c} - \frac{(c^2 - 1)(d-1)}{3c} + \frac{d}{c} \sum_{k=1}^{c-1} \frac{\cos(k\pi/c)}{\sin^2(k\pi/c)} \sin((d-1)k\pi/c) \sin(dk\pi/c).
\]

Using \( \sin((d-1)k\pi/c) = \cos(k\pi/c) \sin(dk\pi/c) - \sin(k\pi/c) \cos(k\pi/c) \) and (11), we obtain

\[
S(d, c) = \frac{(d-1)(d-2)}{3c} - \frac{(c^2 - 1)(d-1)}{3c} + \frac{d(c-1)(c-2)}{3c} - \frac{d}{c} S(c, d),
\]

by (2). The desired result follows.

Using (7), (10) and Lemma 6, we obtain

\[
M(f,c) = \frac{\pi^2}{6cf^2} \sum_{d|f \neq 1} \mu(f/d) \left( c^2 + d^2 - 3cd + 1 - 3dS(c,d) \right).
\]

Since \( c^2 + d^2 - 3cd + 1 - 3dS(c,d) = 0 \) for \( d = 1 \), by (11), we can include \( d = 1 \) in this sum and Theorem 1 follows (recall that \( \sum_{d|f} \mu(d) = 0 \) for \( f > 1 \)).

4. Conclusion

One could try to obtain an explicit formula for the mean values

\[
\frac{2}{\phi(f)} \sum_{m, n \text{ mod } f \atop \gcd(m,n)=1} \chi(c)L(m,\chi)L(n,\bar{\chi}).
\]

Such a formula has been obtained in [BR] and [LW] for the case \( c = 1 \).

One could try to deduce from (6) and Lemma 6 a proof of the conjecture given in [WZ] which predicts the value of the \( M(p, q^k) \)'s, \( k \geq 1 \).

Added May 29, 2012. Since \( S(d, c) = 4cs(c,d) \), where \( s(c,d) \) denotes a Dedekind sum (by [Apo, Exercise 11, Chapter 3]), Lemma 6 is nothing but the reciprocity law for Dedekind sums (see [Apo, Theorem 3.7]). Our present proof of Lemma 6 thus provides a proof of this reciprocity law different from the one given in [Apo].

Our previous version of this paper, accepted in February 2012, did not include this Lemma 6 and our statement of Theorem 1 was less satisfactory.

References


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