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A TWISTED QUADRATIC MOMENT FOR DIRICHLET L-FUNCTIONS

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ABSTRACT. Given c, a positive integer, we give an explicit formula for the quadratic moments

$$\sum_{\chi \in X_f^-} \chi(c)|L(1,\chi)|^2,$$

where X_f^- is the set of the odd Dirichlet characters mod f with $f > 2$.

1. Introduction and statement of results

Let $f > 2$ be an integer. Let X_f^- denote the set of the odd Dirichlet characters mod f. Hence, $\# X_f^- = \phi(f)/2$. Set

$$M(f,c) := \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} \chi(c)|L(1,\chi)|^2 \quad (\gcd(f,c) = 1).$$

In 1982, H. Walum gave in [Wal] a formula for $M(p,1)$, p a prime:

$$M(p,1) = \frac{\pi^2}{6} \frac{(p-1)(p-2)}{p^2}.$$

In 1993, we generalized Walum's result to arbitrary moduli in [Lou1] by proving that

$$M(f,1) = \frac{\pi^2}{6} \frac{\phi(f)}{f} \left(\prod_{p|f} (1 + \frac{1}{p}) - \frac{3}{f} \right) \tag{1}$$

(see [Lou2] for applications). In 2010, Z. Wu and W. Zhang gave in [WZ] a formula for $M(p,2)$, $M(p,4)$ and $M(p,8)$, and conjectured a formula for any $M(p,2^k)$, $k \geq 1$. In this paper, we state and prove a general formula for $M(f,c)$:

Theorem 1. *Let $c > 1$ be a given positive integer. For $f > 2$ with* $\gcd(f,c) = 1$, *we have*

$$M(f,c) = \frac{\pi^2}{6c} \frac{\phi(f)}{f} \left(\prod_{p|f} (1 + \frac{1}{p}) - \frac{3c}{f} \right) - \frac{\pi^2}{2cf^2} \sum_{d|f} d\mu(f/d) S(c,d),$$

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where $S(c,d)$, which depends only on d mod c, is defined by

$$S(c,d) := \sum_{a=1}^{c-1} \cot\left(\frac{\pi a}{c}\right)\cot\left(\frac{\pi a d}{c}\right) \quad (\gcd(c,d)=1). \tag{2}$$

Let us rewrite this result for $c = 2$, 3 and 4. Since $S(2,d) = 0$ for d odd,

$$M(f,2) = \frac{\pi^2}{12}\frac{\phi(f)}{f}\left(\prod_{p|f}(1+\frac{1}{p}) - \frac{6}{f}\right) \ (f > 2 \text{ and odd}). \tag{3}$$

Let χ_3 and χ_4 denote the odd Dirichlet characters mod 3 and mod 4, respectively. For $f > 3$ and relatively prime with 3, we have

$$M(f,3) = \frac{\pi^2}{18}\frac{\phi(f)}{f}\left(\prod_{p|f}(1+\frac{1}{p}) - \frac{9}{f} - \frac{2\chi_3(f)}{f}\prod_{p|f}(\frac{p-\chi_3(p)}{p-1})\right). \tag{4}$$

For $f > 2$ and f odd, we have

$$M(f,4) = \frac{\pi^2}{24}\frac{\phi(f)}{f}\left(\prod_{p|f}(1+\frac{1}{p}) - \frac{12}{f} - \frac{6\chi_4(f)}{f}\prod_{p|f}(\frac{p-\chi_4(p)}{p-1})\right). \tag{5}$$

Moreover, let $p \geq 3$ be an odd prime that does not divide c. Theorem 1 and (11) give

$$M(p,c) = \frac{\pi^2(p^2 - 3(c+S(c,p))p + c^2 + 1)}{6cp^2}, \tag{6}$$

and $S(c,p)$ depends on p mod c only, and we obtain:

Corollary 2. *Let c be a given positive integer. Let $p \geq 3$ be an odd prime that does not divide c. We have*

$$M(p,c) = \frac{\pi^2(p-1)(p-c^2-1)}{6cp^2} \text{ if } p \equiv 1 \pmod c,$$

$$M(p,c) = \frac{\pi^2(p^2+(c^2-6c+2)p+c^2+1)}{6cp^2} \text{ if } p \equiv -1 \pmod c,$$

and

$$M(p,c) = \frac{\pi^2(p^2 - \frac{c^2+5}{2}p + c^2 + 1)}{6cp^2} \text{ if } p \equiv 2 \pmod c.$$

Proof. In the case that $p \equiv 1 \pmod c$, we have $S(c,p) = S(c,1) = (c-1)(c-2)/3$, by (11). In the case that $p \equiv -1 \pmod c$, we have $S(c,p) = -S(c,1)$. In the case that $p \equiv 2 \pmod c$, using $(\cot x)(\cot(2x)) = \frac{1}{2}\cot^2 x - \frac{1}{2}$, we have $S(c,p) = S(c,2) = \frac{1}{2}S(c,1) - \frac{c-1}{2} = \frac{(c-1)(c-5)}{6}$. □

From Corollary 2, or from (3), (4) and (5), we get

$$M(p,2) = \frac{\pi^2}{12}\frac{(p-1)(p-5)}{p^2},$$

$$M(p,3) = \frac{\pi^2}{18} \times \begin{cases} (p-1)(p-10)/p^2 & \text{if } p \equiv 1 \pmod 3, \\ (p^2-7p+10)/p^2 & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

and

$$M(p,4)=\frac{\pi^2}{24}\times\begin{cases}(p-1)(p-17)/p^2 & \text{if } p\equiv 1 \pmod 4,\\ (p^2-6p+17)/p^2 & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

2. A formula for $M(f,1)$ and $M(f,2)$

Since

$$L(1,\chi)=\frac{\pi}{2f}\sum_{a=1}^{f-1}\chi(a)\cot\left(\frac{\pi a}{f}\right)$$

(see [Lou1, Proposition 1]), and since

$$\frac{2}{\phi(f)}\sum_{\chi\in X_f^-}\chi(c)\chi(a)\overline{\chi(b)}=\begin{cases}1 & \text{if } b\equiv ac \pmod f \text{ and } \gcd(abc,f)=1,\\ -1 & \text{if } b\equiv -ac \pmod f \text{ and } \gcd(abc,f)=1,\\ 0 & \text{otherwise,}\end{cases}$$

we obtain

$$M(f,c)=\frac{\pi^2}{2f^2}\tilde{S}(f,c), \tag{7}$$

where

$$\tilde{S}(f,c):=\sum_{\substack{a=1\\ \gcd(a,f)=1}}^{f-1}\cot\left(\frac{\pi a}{f}\right)\cot\left(\frac{\pi ac}{f}\right)\quad(\gcd(f,c)=1). \tag{8}$$

Moreover, for $F:(0,1)\to\mathbf{C}$ we have

$$\sum_{\substack{a=1\\ \gcd(a,f)=1}}^{f-1}F(a/f)=\sum_{\substack{d\mid f\\ d\neq 1}}\mu(f/d)\sum_{a=1}^{d-1}F(a/d). \tag{9}$$

Hence, we obtain

$$\tilde{S}(f,c)=\sum_{\substack{d\mid f\\ d\neq 1}}\mu(f/d)S(d,c), \tag{10}$$

where $S(d,c)$ is defined in (2).

Lemma 3. *We have*

$$\tilde{S}(f,1)=\frac{f^2}{3}\frac{\phi(f)}{f}\left(\prod_{p\mid f}(1+\frac{1}{p})-\frac{3}{f}\right).$$

Proof. We have (see [Lou1, Lemme (a)], or use (12))

$$S(f,1)=\sum_{a=1}^{f-1}\cot^2\left(\frac{\pi a}{f}\right)=\frac{(f-1)(f-2)}{3}. \tag{11}$$

Using (9) and (11), the result follows. □

Using Lemma 3 and (7), we deduce (1). Notice that using $(\cot x)(\cot(2x))=\frac{1}{2}\cot^2 x-\frac{1}{2}$, we obtain $\tilde{S}(f,2)=\frac{1}{2}\tilde{S}(f,1)-\frac{1}{2}\phi(f)$, and (3) follows.

3. A formula for $M(f,c)$: Proof of Theorem 1

Lemma 4. *Let $c \geq 1$ be a positive integer. It holds that*

$$(\cot x)(\cot(cx)) = \frac{1}{c}\cot^2 x - \frac{c^2-1}{3c} + \frac{1}{c}\sum_{k=1}^{c-1}\frac{\frac{\cos(k\pi/c)}{\sin^3(k\pi/c)}}{\cot(k\pi/c)-\cot x}.$$

Proof. For z complex, set

$$F(z) := (\cot z)(\cot(cz)) - \frac{1}{c}\cot^2 z + \frac{c^2-1}{3c} - \frac{1}{c}\sum_{k=1}^{c-1}\frac{\frac{\cos(k\pi/c)}{\sin^3(k\pi/c)}}{\cot(k\pi/c)-\cot z}.$$

Clearly, $F(z)$ has no pole at the points $z_k = k\pi/c$, $1 \leq k \leq c-1$. Now,

$$(\cot z)(\cot(cz)) = \frac{1}{cz^2} - \frac{c^2+1}{3c} + O(z^2)$$

yields

$$(\cot z)(\cot(cz)) - \frac{1}{c}\cot^2 z = \frac{1-c^2}{3c} + O(z^2).$$

Hence $F(z)$ has no pole at $z=0$ and $F(0)=0$. Hence, $F(z)$ is π-periodic and entire. Moreover, $F(z)$ is bounded (use $\cot(\sigma+it) = -i+O(e^{-2t})$ and $\cot(\sigma-it) = \overline{\cot(\sigma+it)} = i+O(e^{-2t})$ uniformly in σ for $t \geq 1$ to see that $F(z)$ is bounded independently of $T \geq 1$ on the border of any rectangle of summits $\alpha+iT$, $\alpha+\pi+iT$, $\alpha+\pi-iT$ and $\alpha-iT$, where $\alpha \notin \{k\pi/c,\ 0 \leq k < c\}$ and $T \geq 1$). Hence, $F(z)$ is constant. Since $F(0)=0$, we have $F(z)=0$ for any complex number z. □

Lemma 5. *Let $d > 1$ be an integer. Let α be a real number. Write $\alpha + i = \rho e^{i\theta}$. Assume that $\alpha \notin \{\cot(\pi a/d);\ 1 \leq a \leq d-1\}$. Then,*

$$T(d,\alpha) := \sum_{a=1}^{d-1}\frac{1}{\alpha-\cot(\pi a/d)} = \frac{d}{\rho}\frac{\sin((d-1)\theta)}{\sin(d\theta)}.$$

Proof. The $\cot(\pi a/d)$, $1 \leq a \leq d-1$, are the roots of

$$P(X) = \frac{(X+i)^d-(X-i)^d}{2id} = \prod_{a=1}^{d-1}(X-\cot(\pi a/d)). \tag{12}$$

Hence,

$$T(d,\alpha) = \frac{P'}{P}(\alpha) = d\frac{(\alpha+i)^{d-1}-(\alpha-i)^{d-1}}{(\alpha+i)^d-(\alpha-i)^d}.$$

The result follows. □

Lemma 6. *For $c > 1$, $d > 1$ and* $\gcd(c,d) = 1$, *it holds that*

$$S(d,c) = \frac{c^2+d^2-3cd+1}{3c} - \frac{d}{c}S(c,d).$$

Proof. By (2) and Lemma 4, we have

$$S(d,c) = \sum_{a=1}^{d-1}\left(\frac{1}{c}\cot^2\left(\frac{\pi a}{d}\right) - \frac{c^2-1}{3c} + \frac{1}{c}\sum_{k=1}^{c-1}\frac{\frac{\cos(k\pi/c)}{\sin^3(k\pi/c)}}{\cot(k\pi/c)-\cot(\pi a/d)}\right).$$

Since $\cot(k\pi/c) + i = \frac{1}{\sin(k\pi/c)} e^{ik\pi/c}$, using Lemma 5 we obtain

$$S(d,c) = \frac{(d-1)(d-2)}{3c} - \frac{(c^2-1)(d-1)}{3c} + \frac{d}{c}\sum_{k=1}^{c-1} \frac{\cos(k\pi/c)}{\sin^2(k\pi/c)} \frac{\sin((d-1)k\pi/c)}{\sin(dk\pi/c)}.$$

Using $\sin((d-1)k\pi/c) = \cos(k\pi/c)\sin(dk\pi/c) - \sin(k\pi/c)\cos(k\pi d/c)$ and (11), we obtain

$$S(d,c) = \frac{(d-1)(d-2)}{3c} - \frac{(c^2-1)(d-1)}{3c} + \frac{d(c-1)(c-2)}{3c} - \frac{d}{c} S(c,d),$$

by (2). The desired result follows. □

Using (7), (10) and Lemma 6, we obtain

$$M(f,c) = \frac{\pi^2}{6cf^2} \sum_{\substack{d \mid f \\ d \neq 1}} \mu(f/d) \left(c^2 + d^2 - 3cd + 1 - 3dS(c,d)\right).$$

Since $c^2 + d^2 - 3cd + 1 - 3dS(c,d) = 0$ for $d = 1$, by (11), we can include $d = 1$ in this sum and Theorem 1 follows (recall that $\sum_{d \mid f} \mu(d) = 0$ for $f > 1$).

4. Conclusion

One could try to obtain an explicit formula for the mean values

$$\frac{2}{\phi(f)} \sum_{\substack{\chi \mod f \\ \chi(-1)=(-1)^m}} \chi(c) L(m,\chi) L(n,\bar{\chi}).$$

Such a formula has been obtained in [BR] and [LW] for the case $c = 1$.

One could try to deduce from (6) and Lemma 6 a proof of the conjecture given in [WZ] which predicts the value of the $M(p, 2^k)$'s, $k \geq 1$.

Added May 29, 2012. Since $S(d,c) = 4cs(c,d)$, where $s(c,d)$ denotes a Dedekind sum (by [Apo, Exercise 11, Chapter 3]), Lemma 6 is nothing but the reciprocity law for Dedkind sums (see [Apo, Theorem 3.7]). Our present proof of Lemma 6 thus provides a proof of this reciprocity law different from the one given in [Apo]. Our previous version of this paper, accepted in February 2012, did not include this Lemma 6, and our statement of Theorem 1 was less satisfactory.

References

[Apo] Tom M. Apostol, *Modular functions and Dirichlet series in number theory*, 2nd ed., Graduate Texts in Mathematics, vol. 41, Springer-Verlag, New York, 1990. MR1027834 (90j:11001)

[BR] Abdelmejid Bayad and Abdelaziz Raouj, *Mean values of L-functions and Dedekind sums*, J. Number Theory **132** (2012), no. 8, 1645–1652, DOI 10.1016/j.jnt.2012.01.014. MR2922335

[Lou1] Stéphane Louboutin, *Quelques formules exactes pour des moyennes de fonctions L de Dirichlet* (French, with French summary), Canad. Math. Bull. **36** (1993), no. 2, 190–196, DOI 10.4153/CMB-1993-028-8. MR1222534 (94c:11077)

[Lou2] Stéphane R. Louboutin, *Mean values of L-functions and relative class numbers of cyclotomic fields*, Publ. Math. Debrecen **78** (2011), no. 3-4, 647–658, DOI 10.5486/PMD.2011.4882. MR2867207 (2012h:11154)

[LW] Huaning Liu and Wenpeng Zhang, *On the mean value of $L(m,\chi)L(n,\overline{\chi})$ at positive integers $m,n \geq 1$*, Acta Arith. **122** (2006), no. 1, 51–56, DOI 10.4064/aa122-1-5. MR2217323 (2007a:11122)

[Wal] Herbert Walum, *An exact formula for an average of L-series*, Illinois J. Math. **26** (1982), no. 1, 1–3. MR638548 (82m:10066)

[WZ] Z. Wu and W. Zhang, *On the mean values of $L(1,\chi)$*, Bull. Korean Math. Soc. **49** (2012), no. 6, 1303–1310. MR3002688

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