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REGULARITY AND PROJECTIVE DIMENSION OF THE EDGE IDEAL OF C_5-FREE VERTEX DECOMPOSABLE GRAPHS

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Abstract. In this paper, we explain the regularity, projective dimension and depth of the edge ideal of some classes of graphs in terms of invariants of graphs. We show that for a C_5-free vertex decomposable graph G, $\mathrm{reg}(R/I(G)) = c_G$, where c_G is the maximum number of 3-disjoint edges in G. Moreover, for this class of graphs we characterize $\mathrm{pd}(R/I(G))$ and $\mathrm{depth}(R/I(G))$. As a corollary we describe these invariants in forests and sequentially Cohen-Macaulay bipartite graphs.

Introduction

Let G be a simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. If k is a field, the edge ideal of G in the polynomial ring $R = k[x_1, \ldots, x_n]$ is defined as $I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E(G)\rangle$. The edge ideal of a graph was first considered by Villarreal [18]. Finding connections between algebraic properties of an edge ideal and invariants of a graph, for instance explaining the regularity, projective dimension and depth of the ring $R/I(G)$ by some information from G, is of great interest. For some classes of graphs like trees, chordal graphs and shellable bipartite graphs, some of these invariants are studied in [8], [10], [15] and [20]. Zheng in [20] described regularity and projective dimension for tree graphs. It was proved that if G is a tree, then $\mathrm{reg}\,(R/I(G)) = c_G$, where c_G is the maximum number of pairwise 3-disjoint edges in G. In [8] this description of regularity has been extended to chordal graphs. Also, Kimura in [10] extended the characterization of projective dimension in [20] to chordal graphs. Moreover, Van Tuyl in [15] proved that the equality $\mathrm{reg}\,(R/I(G)) = c_G$ holds when G is a sequentially Cohen-Macaulay bipartite graph.

In this paper, we consider the class of C_5-free vertex decomposable graphs which contains the classes of forests and sequentially Cohen-Macaulay bipartite graphs. For this class of graphs we show that $\mathrm{reg}\,(R/I(G)) = c_G$, which generalizes [15, Theorem 3.3] and [20, Theorem 2.18]. Also, we describe the projective dimension and depth of the ring $R/I(G)$ for this class of graphs and we gain some results that can be compared with [20, Corollary 2.13] and [10, Theorem 5.3 and Corollary 5.6].

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In Section 1, we recall some definitions and theorems that we use in the sequel. In Section 2, first we show that $\operatorname{reg}(R/I(G)) = c_G$ for a C_5-free vertex decomposable graph (Theorem 2.4). Then in Corollary 2.5 we deduce that this equality holds for forests and sequentially Cohen-Macaulay bipartite graphs. The notion d'_G was introduced in [10], and it was shown that for a chordal graph G, one has $\operatorname{pd}(R/I(G)) = \operatorname{bight}(I(G)) = d'_G$, where $\operatorname{bight}(I(G))$ is the maximum height among the minimal prime divisors of $I(G)$. In Theorem 2.9, we show that for a C_5-free vertex decomposable graph G we also have these equalities and by some corollaries we show this is true for chordal graphs, forests and sequentially Cohen-Macaulay bipartite graphs (see Corollaries 2.10 and 2.13). Moreover, in Corollary 2.12 we give a description for $\operatorname{depth}(R/I(G))$ for a C_5-free vertex decomposable graph G.

1. Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G the set of all neighborhoods of v is denoted by $N_G(v)$, or briefly $N(v)$, and we denote the set $N_G(v) \cup \{v\}$ by $N_G[v]$, or briefly $N[v]$. An independent set of G is a subset $F \subseteq V(G)$ such that $e \nsubseteq F$, for any $e \in E(G)$.

Vertex decomposability was introduced by Provan and Billera in [13] in the pure case and extended to the non-pure case by Björner and Wachs in [1] and [2]. We need and use the following definition of a vertex decomposable graph, which is an interpretation of the definition of vertex decomposability for the independence complex of a graph studied in [3] and [19].

Definition 1.1. A graph G is recursively defined to be **vertex decomposable** if G is totally disconnected (with no edges) or if

(i) there is a vertex x in G such that $G \setminus \{x\}$ and $G \setminus N[x]$ are both vertex decomposable, and
(ii) no independent set in $G \setminus N[x]$ is a maximal independent set in $G \setminus \{x\}$.

A vertex x which satisfies the second condition is called a **shedding vertex** of G.

The **Castelnuovo-Mumford regularity** (or simply regularity) of a $\mathbb{Z}$-graded R-module M is defined as

$$\operatorname{reg}(M) := \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Also, the projective dimension of M is defined as

$$\operatorname{pd}(M) := \max\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\},$$

where $\beta_{i,j}(M)$ is the (i, j)-th graded Betti number of M.

Let G be a graph. A subset $C \subseteq V(G)$ is called a **vertex cover** of G if it intersects all edges of G. A vertex cover of G is called *minimal* if it has no proper subset which is also a vertex cover of G. When G is a graph with $V(G) = \{x_1, \ldots, x_n\}$ and $C = \{x_{i_1}, \ldots, x_{i_t}\}$ is a vertex cover of G, by x^C we mean the monomial $x_{i_1} \ldots x_{i_t}$ in the ring of polynomials $R = k[x_1, \ldots, x_n]$. For a monomial ideal $I = \langle x_{11} \cdots x_{1n_1}, \ldots, x_{t1} \cdots x_{tn_t}\rangle$ of the polynomial ring R, the **Alexander dual ideal** of I, denoted by $I^\vee$, is defined as

$$I^\vee := \langle x_{11}, \ldots, x_{1n_1}\rangle \cap \cdots \cap \langle x_{t1}, \ldots, x_{tn_t}\rangle.$$

One can see that, for a graph G,

$$I(G)^\vee = \langle x^C \mid C \text{ is a minimal vertex cover of G}\rangle$$

and

$$I(G) = \bigcap (P_C \mid C \text{ is a minimal vertex cover of } G),$$

where $P_C = \langle x_i \mid x_i \in C\rangle$. The **big height** of $I(G)$, denoted by bight$(I(G))$, is defined as the maximum height among the minimal prime divisors of $I(G)$, that is, the maximal size of a minimal vertex cover of G. In fact, for a graph G we have

$$\text{bight}\,(I(G)) = \max\{|C| \mid C \text{ is a minimal vertex cover of G}\}.$$

The following theorem, which was proved in [14], is one of our main tools in the study of the regularity of the ring $R/I(G)$.

Theorem 1.2 (See [14, Theorem 2.1])**.** *Let I be a square-free monomial ideal. Then* $\text{pd}\,(I^\vee) = \text{reg}\,(R/I)$.

Two edges $\{x, y\}$ and $\{w, z\}$ of G are called **3-disjoint** if the induced subgraph of G on $\{x, y, w, z\}$ consists of exactly two disjoint edges or, equivalently, in the complement of G, the induced graph on $\{x, y, w, z\}$ is a four-cycle. A subset $A \subseteq E(G)$ is called a **pairwise 3-disjoint set of edges in** G if every pair of distinct edges in A are 3-disjoint in G. Set

$$c_G = \max\{|A| \mid A \text{ is a pairwise 3-disjoint set of edges in } G\}.$$

The graph B with vertex set $V(B) = \{z, w_1, \ldots, w_d\}$ and edge set $E(B) = \{\{z, w_i\} \mid 1 \le i \le d\}$ is called a **bouquet**. The vertex z is called the **root** of B, the vertices w_i **flowers** of B and the edges $\{z, w_i\}$ the **stems** of B. A subgraph of G which is a bouquet is called a bouquet of G. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a set of bouquets of G. We use the following notation:

$$F(\mathcal{B}) = \{w \in V(G) \mid w \text{ is a flower of some bouquet in } \mathcal{B}\},$$

$$R(\mathcal{B}) = \{z \in V(G) \mid z \text{ is a root of some bouquet in } \mathcal{B}\},$$

$$S(\mathcal{B}) = \{e \in E(G) \mid e \text{ is a stem of some bouquet in } \mathcal{B}\}.$$

Kimura in [10] introduced two notions of disjointness of a set of bouquets.

Definition 1.3 (See [10, Definition 2.1])**.** A set of bouquets $\mathcal{B} = \{B_1, \ldots, B_n\}$ is called **strongly disjoint** in G if

(i) $V(B_i) \cap V(B_j) = \emptyset$ for all $i \neq j$, and
(ii) we can choose a stem e_i from each bouquet $B_i \in \mathcal{B}$ such that $\{e_1, \ldots, e_n\}$ is pairwise 3-disjoint in G.

Definition 1.4 (See [10, Definition 5.1])**.** A set of bouquets $\mathcal{B} = \{B_1, \ldots, B_n\}$ is called **semi-strongly disjoint** in G if

(i) $V(B_i) \cap V(B_j) = \emptyset$ for all $i \neq j$, and
(ii) $R(\mathcal{B})$ is an independent set of G.

Set

$$d_G := \max\{|F(\mathcal{B})| \mid \mathcal{B} \text{ is a strongly disjoint set of bouquets of } G\}$$

and

$$d'_G := \max\{|F(\mathcal{B})| \mid \mathcal{B} \text{ is a semi-strongly disjoint set of bouquets of } G\}.$$

It is easy to see that any pairwise 3-disjoint set of edges in G is a strongly disjoint set of bouquets in G and that any strongly disjoint set of bouquets is semi-strongly disjoint. In this regard, we have the inequalities

$$c_G \le d_G \le d'_G. \tag{1.1}$$

As some auxiliary tools, we need some results of [8] and [10], which present lower bounds for the regularity and projective dimension of the ring $R/I(G)$. We end this section by recalling these results.

Theorem 1.5. *For any graph G, the following hold:*

(i) (*See* [8, Theorem 6.5].) $\operatorname{reg}(R/I(G)) \ge c_G$.
(ii) (*See* [10, Theorem 3.1].) $\operatorname{pd}(R/I(G)) \ge d_G$.

2. Main results

In this section, among other things, we give some descriptions for the regularity, projective dimension and depth of the ring $R/I(G)$, when G is a C_5-free vertex decomposable graph. This class of graphs contains some nice classes such as forests and sequentially Cohen-Macaulay bipartite graphs (see [6, Theorem 3.2] and [15, Theorem 2.10]). For this purpose, we use the duality concept in Theorem 1.2 and induction. In this way, we need the following lemma.

Lemma 2.1. *Suppose that G is a graph, $x \in V(G)$ and $N_G(x) = \{y_1, \dots, y_t\}$. Let $G' = G \setminus \{x\}$ and $G'' = G \setminus N_G[x]$. Then:*

(i) $I(G)^\vee = xI(G')^\vee + y_1 \cdots y_t I(G'')^\vee$;
(ii) $xy_1 \cdots y_t I(G'')^\vee = xI(G')^\vee \cap y_1 \cdots y_t I(G'')^\vee$;
(iii) *there exists a short exact sequence of R-modules*

$$0 \to xy_1 \cdots y_t I(G'')^\vee \to xI(G')^\vee \oplus y_1 \cdots y_t I(G'')^\vee \to I(G)^\vee \to 0.$$

Proof. (i) For any minimal vertex cover C of G, if $x \in C$, then clearly $C' = C \setminus \{x\}$ is a vertex cover of G', so $x^C = xx^{C'} \in xI(G')^\vee$. Now let $x \notin C$. Then $\{y_1, \dots, y_t\} \subseteq C$ and $C'' = C \setminus \{y_1, \dots, y_t\}$ is a vertex cover of G''. Therefore, $x^C = y_1 \cdots y_t x^{C''} \in y_1 \cdots y_t I(G'')^\vee$. Thus $I(G)^\vee \subseteq xI(G')^\vee + y_1 \cdots y_t I(G'')^\vee$.

Conversely, for any minimal vertex cover C' of G', $C' \cup \{x\}$ is a vertex cover of G. So $xI(G')^\vee \subseteq I(G)^\vee$. Also, if C'' is a minimal vertex cover of G'', then $C'' \cup \{y_1, \dots, y_t\}$ is a vertex cover of G. Thus $y_1 \cdots y_t I(G'')^\vee \subseteq I(G)^\vee$. Therefore, $xI(G')^\vee + y_1 \cdots y_t I(G'')^\vee \subseteq I(G)^\vee$. Hence, $I(G)^\vee = xI(G')^\vee + y_1 \cdots y_t I(G'')^\vee$ as desired.

(ii) For any minimal vertex cover C'' of G'', the set $C' = C'' \cup \{y_1, \dots, y_t\}$ is a vertex cover of G'. Thus, $xy_1 \cdots y_t x^{C''} = xx^{C'} \in xI(G')^\vee \cap y_1 \cdots y_t I(G'')^\vee$. Conversely, let $f \in xI(G')^\vee \cap y_1 \cdots y_t I(G'')^\vee$ be a monomial. Then $f = xf_1 = y_1 \cdots y_t f_2$ for some monomials $f_1 \in I(G')^\vee$ and $f_2 \in I(G'')^\vee$. So $f_2 = xf_3$ for some $f_3 \in I(G'')^\vee$. Thus $f = xy_1 \cdots y_t f_3 \in xy_1 \cdots y_t I(G'')^\vee$. The proof is complete.

(iii) By using (i) and (ii) in the short exact sequence

$$\begin{aligned} 0 &\to xI(G')^\vee \cap y_1 \cdots y_t I(G'')^\vee \\ &\to xI(G')^\vee \oplus y_1 \cdots y_t I(G'')^\vee \to xI(G')^\vee + y_1 \cdots y_t I(G'')^\vee \to 0, \end{aligned}$$

the result holds. $\square$

Thus we can deduce:

Corollary 2.2. *Suppose that G is a graph, $x \in V(G)$ and $|N_G(x)| = t$. Let $G' = G \setminus \{x\}$ and $G'' = G \setminus N_G[x]$. Then*

(i) $\mathrm{pd}\,(I(G)^\vee) \leq \max\{\mathrm{pd}\,(I(G')^\vee), \mathrm{pd}\,(I(G'')^\vee) + 1\}$;

(ii) $\mathrm{reg}\,(I(G)^\vee) \leq \max\{\mathrm{reg}\,(I(G')^\vee) + 1, \mathrm{reg}\,(I(G'')^\vee) + t\}$.

Proof. Considering the short exact sequence

$$0 \to xy_1 \cdots y_t I(G'')^\vee \to xI(G')^\vee \oplus y_1 \cdots y_t I(G'')^\vee \to I(G)^\vee \to 0$$

insures that

$$\mathrm{pd}\,(I(G)^\vee) \leq \max\{\mathrm{pd}\,(xI(G')^\vee), \mathrm{pd}\,(y_1 \cdots y_t I(G'')^\vee), \mathrm{pd}\,(xy_1 \cdots y_t I(G'')^\vee) + 1\}$$

and

$$\mathrm{reg}\,(I(G)^\vee) \leq \max\{\mathrm{reg}\,(xy_1 \cdots y_t I(G'')^\vee) - 1, \mathrm{reg}\,(xI(G')^\vee), \mathrm{reg}\,(y_1 \cdots y_t I(G'')^\vee)\}$$

(see [5, Corollary 20.19]).

On the other hand, we know that for any monomial ideal I and monomial f with the property that the support of f is disjoint from the support of any generators of I, we have $\mathrm{pd}\,(fI) = \mathrm{pd}\,(I)$ and $\mathrm{reg}\,(fI) = \mathrm{reg}\,(I) + \deg(f)$. This fact completes the proof. □

The following lemma is frequently needed in the sequel.

Lemma 2.3. *Assume that G is a C_5-free graph and x is a shedding vertex of G. Then there is a vertex y of $N(x)$ such that $N[y] \subseteq N[x]$.*

Proof. Let x be a shedding vertex of a C_5-free graph G and $N(x) = \{y_1, \ldots, y_t\}$. Suppose, on the contrary, that for each $1 \leq i \leq t$, there exists a vertex w_i in $N(y_i) \cap (V(G) \setminus N[x])$. Now, if w_i is adjacent to w_j for some $1 \leq i, j \leq t$ with $i \neq j$, then $x - y_i - w_i - w_j - y_j - x$ is a C_5-subgraph of G, and so it is a contradiction. Hence, $\{w_1, \ldots, w_t\}$ is an independent set in $G \setminus N[x]$. Now, if F is a maximal independent set in $G \setminus N[x]$ containing $\{w_1, \ldots, w_t\}$, it is also a maximal independent set in $G \setminus \{x\}$. This contradicts our assumption that x is a shedding vertex and so completes our proof. □

Now we are ready to describe the regularity of $R/I(G)$ for a C_5-free vertex decomposable graph G.

Theorem 2.4. *Let G be a C_5-free vertex decomposable graph. Then*

$$\mathrm{reg}\,(R/I(G)) = c_G.$$

Proof. In view of Theorems 1.5(i) and 1.2 it is enough to show that $\mathrm{pd}\,(I(G)^\vee) \leq c_G$. We proceed by induction on $|V(G)|$. For $|V(G)| = 2$, G is totally disconnected or a single edge. Hence, $I(G)^\vee = 0$ or $I(G)^\vee = \langle x, y\rangle$. Therefore, $\mathrm{pd}\,(I(G)^\vee) = 0 \leq 0 = c_G$ or $\mathrm{pd}\,(I(G)^\vee) = 1 \leq 1 = c_G$. Suppose that G is a C_5-free vertex decomposable graph with $|V(G)| > 2$ and the result holds for each C_5-free vertex decomposable graph H with smaller values of $|V(H)|$. Since G is vertex decomposable, there exists a shedding vertex $x \in V(G)$ such that $G' = G \setminus \{x\}$ and $G'' = G \setminus N_G[x]$ are vertex decomposable. Let $N_G(x) = \{y_1, \ldots, y_t\}$. By Corollary 2.2(i), we have

$$\mathrm{pd}\,(I(G)^\vee) \leq \max\{\mathrm{pd}\,(I(G')^\vee), \mathrm{pd}\,(I(G'')^\vee) + 1\}.$$

Clearly G' and G'' are C_5-free and vertex decomposable. So, by the induction hypothesis we have $\mathrm{pd}\,(I(G')^\vee) \leq c_{G'}$ and $\mathrm{pd}\,(I(G'')^\vee) \leq c_{G''}$. Hence,

$$\mathrm{pd}\,(I(G)^\vee) \leq \max\{c_{G'}, c_{G''}+1\}.$$

Now, since $c_{G'} \leq c_G$, it is enough to show that $c_{G''}+1 \leq c_G$.

By Lemma 2.3 there is a vertex y such that $N[y] \subseteq N[x]$. Thus we can add the edge $\{x, y\}$ to any set of pairwise 3-disjoint edges in G'' and get a pairwise 3-disjoint set of edges in G, which proves the inequality $c_{G''}+1 \leq c_G$. □

As a corollary we can recover results of Zheng [20] and Van Tuyl [15] as follows.

Corollary 2.5. *The following hold:*

(i) [15, Theorem 3.3] *Let G be a sequentially Cohen-Macaulay bipartite graph. Then* $\mathrm{reg}\,(R/I(G)) = c_G$.
(ii) [20, Theorem 2.18] *Let G be a forest. Then* $\mathrm{reg}\,(R/I(G)) = c_G$.

Proof. (i) By [15, Theorem 2.10], G is vertex decomposable. Since a bipartite graph is C_5-free, Theorem 2.4 yields the result.

(ii) Since any forest is a chordal graph, it is vertex decomposable by [19, Corollary 7]. Clearly G is also C_5-free. So we can apply Theorem 2.4 to get the result. □

The following result presents an upper bound for the projective dimension of the ring $R/I(G)$. As we shall see later, this is a technical tool for characterizing the projective dimension of the ring $R/I(G)$.

Proposition 2.6. *Let G be a C_5-free vertex decomposable graph. Then*

$$\mathrm{pd}\,(R/I(G)) \leq d'_G.$$

Proof. From the equalities $I(G) = (I(G)^\vee)^\vee$, $\mathrm{pd}\,(R/I(G)) = \mathrm{pd}\,(I(G)) + 1$ and $\mathrm{reg}\,(I(G)) = \mathrm{reg}\,(R/I(G)) + 1$ and Theorem 1.2, one can see that $\mathrm{pd}\,(R/I(G)) = \mathrm{reg}\,(I(G)^\vee)$. So, it is enough to show that $\mathrm{reg}\,(I(G)^\vee) \leq d'_G$. We prove the assertion by induction on $|V(G)|$. For $|V(G)| = 2$, G is totally disconnected or a single edge. Hence, $\mathrm{reg}\,(I(G)^\vee) = 0 \leq 0 = d'_G$ or $\mathrm{reg}\,(I(G)^\vee) = 1 \leq 1 = d'_G$. Now, suppose inductively that G is a C_5-free vertex decomposable graph with $|V(G)| > 2$ and the result holds for smaller values of $|V(G)|$. Assume that $x \in V(G)$ is a shedding vertex of G such that $G' = G \setminus \{x\}$ and $G'' = G \setminus N_G[x]$ are vertex decomposable and $N_G(x) = \{y_1, \ldots, y_t\}$. In view of Corollary 2.2(ii), we have

$$\mathrm{reg}\,(I(G)^\vee) \leq \max\{\mathrm{reg}\,(I(G')^\vee) + 1, \mathrm{reg}\,(I(G'')^\vee) + t\}.$$

By induction hypothesis $\mathrm{reg}\,(I(G')^\vee) \leq d'_{G'}$ and $\mathrm{reg}\,(I(G'')^\vee) \leq d'_{G''}$. Thus

$$\mathrm{reg}\,(I(G)^\vee) \leq \max\{d'_{G'} + 1, d'_{G''} + t\}. \tag{2.1}$$

Now, let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a semi-strongly disjoint set of bouquets in G'' with $d'_{G''} = |F(\mathcal{B})|$. Then by adding the bouquet with root x and flowers $\{y_1, \ldots, y_t\}$ to $\mathcal{B}$, we have a semi-strongly disjoint set of bouquets $\mathcal{B}'$ in G with $|F(\mathcal{B}')| = d'_{G''} + t$. Therefore

$$d'_{G''} + t \leq d'_G. \tag{2.2}$$

Now let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a semi-strongly disjoint set of bouquets in G' with $d'_{G'} = |F(\mathcal{B})|$. We consider the following two cases.

Case I. If $y_i \in R(\mathcal{B})$ for some $1 \leq i \leq t$, then by adding the stem $\{x, y_i\}$ to the bouquet with root y_i, G has a semi-strongly disjoint set of bouquets $\mathcal{B}'$ with $|F(\mathcal{B}')| = d'_{G'} + 1$.

Case II. If $R(\mathcal{B}) \cap \{y_1, \ldots, y_t\} = \emptyset$, then by Lemma 2.3, there exists $1 \leq i \leq t$, such that $N[y_i] \subseteq N[x]$. So $y_i \notin R(\mathcal{B}) \cup F(\mathcal{B})$. Thus, adding the bouquet with a single stem $\{x, y_i\}$ to $\mathcal{B}$ induces a semi-strongly disjoint set of bouquets $\mathcal{B}'$ of G with $|F(\mathcal{B}')| = d'_{G'} + 1$.

Therefore, in each case we have

$$d'_{G'} + 1 \leq d'_G. \tag{2.3}$$

Now (2.1), (2.2) and (2.3) imply the result. □

Dao and Schweig in [4] introduce a new graph domination parameter called edgewise domination. Let $F \subseteq E(G)$. We say that F is edgewise dominant if any $v \in V(G)$ is adjacent to an endpoint of some edge $e \in F$. They also define

$$\epsilon(G) := \min\{|F| \mid F \subseteq E(G) \text{ is edgewise dominant}\}.$$

Moreover, recall that $S \subseteq V(G)$ is called a dominating set of G if each vertex in $V(G) \setminus S$ is adjacent to some vertex in S. Also,

$$\gamma(G) = \min\{|A| \mid A \text{ is a dominating set of } G\}$$

is another graph domination parameter.

The following proposition declares when a semi-strongly disjoint set of bouquets of a graph corresponds to a minimal vertex cover. The argument is the same as in [10, Corollary 5.6].

Proposition 2.7. *Let G be a graph and $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a semi-strongly disjoint set of bouquets in G with $|F(\mathcal{B})| = d'_G$. Then:*

(i) *$F(\mathcal{B})$ is a minimal vertex cover of G.*

(ii) *If, moreover, G has no isolated vertex, we have that $F(\mathcal{B})$ is a dominating set of G and $S(\mathcal{B})$ is edgewise dominant in G.*

Proof. (i) First we show that the set $F(\mathcal{B})$ is a vertex cover of G. Assume, on the contrary, that $\{x, y\}$ is an edge which is not covered by $F(\mathcal{B})$. Then $x, y \notin F(\mathcal{B})$. Moreover x and y are not adjacent to any vertex in $R(\mathcal{B})$; otherwise, if $\{x, z\} \in E(G)$ for some $z \in R(\mathcal{B})$, then by adding the stem $\{x, z\}$ to the bouquet with root z, we have a semi-strongly disjoint set of bouquets $\mathcal{B}'$ with $|F(\mathcal{B}')| = d'_G + 1$, which is a contradiction. Moreover $x, y \notin R(\mathcal{B})$; otherwise, if x (respectively y) is a root, then y (respectively x) is adjacent to a vertex in $R(\mathcal{B})$, which is not possible by the above argument. Therefore, if we add the bouquet with a single stem $\{x, y\}$ to the set $\mathcal{B}$, we have a semi-strongly disjoint set of bouquets $\mathcal{B}'$ with $|F(\mathcal{B}')| = d'_G + 1$, which is again a contradiction. So $F(\mathcal{B})$ is a vertex cover of G as desired. Moreover, it is a minimal one, since removing any flower makes the corresponding stem uncovered.

(ii) Let $v \in V(G)$. Since G has no isolated vertex, there exists an edge e containing v. Since $F(\mathcal{B})$ is a vertex cover of G, $F(\mathcal{B}) \cap e \neq \emptyset$. This means that there exists a stem $e' \in S(\mathcal{B})$ such that v is adjacent to one endpoint of e'. Therefore, $S(\mathcal{B})$ is edgewise dominant. Since G has no isolated vertex, any vertex cover is a dominating set. Hence, (i) insures that $F(\mathcal{B})$ is a dominating set of G. □

The following corollary provides a chain of inequalities between some algebraic invariants of the edge ideal $I(G)$ and some invariants of G.

Corollary 2.8. *For a graph G we have*

$$c_G \leq d_G \leq d'_G \leq \text{bight}\,(I(G)) \leq \text{pd}\,(R/I(G)) \leq |V(G)| - \epsilon(G).$$

If, moreover, G has no isolated vertex, we have

$$\max\{\epsilon(G), \gamma(G)\} \leq d'_G.$$

Proof. The first assertion can be gained from the inequality (1.1), Proposition 2.7(i), [12, Theorem 3.31] and [4, Theorem 4.3]. The second one is an immediate consequence of Proposition 2.7(ii). ☐

Now, we are ready to prove another main result of this note. This shows that the upper bound determined in Proposition 2.6 is tight and also that it is equal to big height of the edge ideal.

Theorem 2.9. *Let G be a C_5-free vertex decomposable graph. Then*

$$\text{pd}\,(R/I(G)) = \text{bight}\,(I(G)) = d'_G.$$

Proof. By Proposition 2.6 and Corollary 2.8 the result holds. ☐

In [10], it was proved that for a chordal graph G, $\text{pd}\,(R/I(G)) = d_G$, and later with another argument, it was shown that for a chordal graph G, we moreover have $\text{pd}\,(R/I(G)) = d_G = d'_G$ (see [10, Theorems 4.1 and 5.3]). Now, the next corollary shows that Theorem 5.3 and Corollary 5.6 in [10] can be directly gained by Corollary 2.8 and [10, Theorem 4.1].

Corollary 2.10 (See [10, Theorem 5.3 and Corollary 5.6])**.** *Let G be a chordal graph. Then*

$$\text{pd}\,(R/I(G)) = \text{bight}\,(I(G)) = d_G = d'_G.$$

Looking at many examples of C_5-free vertex decomposable graphs (which were not necessarily chordal), we observed that $d_G = d'_G$. So the following question comes to mind:

Question 2.11. Does the equality $d_G = d'_G$ hold for a C_5-free vertex decomposable graph?

In the next corollary we are interested in characterizing the depth of the ring $R/I(G)$ when G is a C_5-free vertex decomposable graph. Recall that a graph G is called unmixed if all maximal independent sets in G have the same cardinality.

Corollary 2.12. *Let G be a C_5-free vertex decomposable graph. Then*

$$\text{depth}\,(R/I(G)) = \min\{|F| \mid F \subseteq V(G) \text{ is a maximal independent set in } G\}.$$

Moreover, $R/I(G)$ is Cohen-Macaulay if and only if G is unmixed.

Proof. By applying the Auslander-Buchsbaum formula for $R/I(G)$, we have

$$\text{pd}\,(R/I(G)) + \text{depth}\,(R/I(G)) = n,$$

where $n = |V(G)|$. So, by Theorem 2.9,

$$\text{depth}\,(R/I(G)) = n - \text{bight}\,(I(G)).$$

In view of the definition of the big height of $I(G)$, there exists a minimal vertex cover C of G with $\text{bight}\,(I(G)) = |C|$. Since every minimal vertex cover C' of G

corresponds to the maximal independent set $F = V(G)\setminus C'$ and C has the maximal cardinality among minimal vertex covers, the first assertion holds. Now, since

$$\operatorname{depth}(R/I(G)) = \min\{|F| \mid F \text{ is a maximal independent set in } G\}$$

and by [17, Corollary 5.3.11], we have

$$\dim(R/I(G)) = \max\{|F| \mid F \text{ is an independent set in } G\}.$$

Cohen-Macaulayness is equivalent to saying that all maximal independent sets in G have the same cardinality. This yields the result. □

By considering the fact that forest graphs and sequentially Cohen-Macaulay bipartite graphs are C_5-free vertex decomposable and by applying Theorem 2.9 and Corollary 2.12, one has:

Corollary 2.13. *Let G be a forest or a sequentially Cohen-Macaulay bipartite graph. Then:*

(i) $\operatorname{pd}(R/I(G)) = d'(G) = \operatorname{bight}(I(G))$.
(ii) $\operatorname{depth}(R/I(G)) = \min\{|F| \mid F \subseteq V(G)$ *is a maximal independent set in* $G\}$.
(iii) *The ring $R/I(G)$ is Cohen-Macaulay if and only if G is unmixed.*

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