PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 5, May 2014, Pages 1773–1776
S 0002-9939(2014)11912-5
Article electronically published on February 19, 2014

A COUNTEREXAMPLE TO LJAMIN'S THEOREM

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(Communicated by Richard Rochberg)

Abstract. One of the well-known results ensuring that a nonautonomous superposition operator maps the set of functions of one variable of bounded variation in the sense of Jordan into itself is the theorem by A. G. Ljamin. According to that theorem it suffices to consider the class of functions which are uniformly Lipschitz w.r.t. the second variable and of uniformly bounded variation w.r.t. the first variable. Unfortunately, Ljamin's result is false. Here we deliver an example contradicting sufficiency of those conditions.

1. Introduction

Nonlinear superposition operators have a particular importance among nonlinear operators; one can find many papers concerning this topic (see e.g. [2], [1] and the references therein). It is connected with the fact that these operators have found many applications, especially in the theory of nonlinear differential equations as well as nonlinear integral equations. One of the basic problems in the theory of nonlinear superposition operators is finding necessary and/or sufficient conditions which guarantee that a nonlinear superposition operator maps a function space under scrutiny into itself.

Among various function spaces a significant amount of interest has been obtained on the space of functions of bounded variation in the sense of Jordan. It stems from the fact that solutions to many integral equations which describe concrete physical phenomena are often functions of bounded variation in the sense of Jordan (cf. [5], [4], [6] and others).

An interesting result concerning an autonomous superposition operator in the space of functions of bounded variation in the sense of Jordan was proved by Josephy [8]. To be more specific: an autonomous superposition operator maps such a space into itself if and only if a function that generates that operator is locally Lipschitz. Let us recall that Josephy's theorem was extended to the space of functions of bounded φ-variation by Ciemnoczołowski and Orlicz in [7].

On the other hand, the situation becomes more complex in the case of so-called nonautonomous superposition operators. In [9] Ljamin stated (without a proof) the following result:

> Suppose that the function $h(\cdot, u) : [0, 1] \to \mathbb{R}$ has bounded variation uniformly w.r.t. $u \in \mathbb{R}$, and the function $h(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, uniformly w.r.t. $t \in [0, 1]$. Then the operator

Received by the editors May 2, 2012 and, in revised form, June 26, 2012.
2010 *Mathematics Subject Classification.* Primary 47H30, 26A45; Secondary 45G10.
Key words and phrases. Nonautonomous superposition operator, functions of bounded variation in the sense of Jordan.

> $H(f)(t) := h(t, f(t))$, $t \in [0,1]$, maps the space $BV([0,1])$ into itself. (...)[1]

In [2] (Theorem 6.12, p. 174) or in [1] (Theorem 4.3, p. 174), one can find Ljamin's theorem along with information that its proof is 'straightforward'. In the paper [3] one can find other sufficient conditions (stronger than Ljamin's) which guarantee that a nonautonomous superposition operator maps the space under consideration into itself. Moreover, in the introduction of this article the author writes:

> The motivation for writing this note is the main result from [14] (see also [11], Th. 6.12, p. 174), whose statement is not clear (let us mention that there is no proof of this result in [14]).[2]

Actually, the hidden conjecture from [3] that Ljamin's result may not be correct, is true. The aim of this note is to indicate the example which shows that Ljamin's theorem is not correct.

2. The example

For any nonnegative integer a let $I_a := \{a, a+1, a+2, \ldots\}$. Let us note that the series $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent.

The following example presents a function which is uniformly Lipschitz w.r.t. the second variable and of uniformly bounded variation w.r.t. the first variable. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 0, & \text{if } \forall n \in I_2 : x \neq 1 - \frac{1}{n} \text{ or } y \notin (c_n - w_n,\, c_n + w_n), \\ \frac{1}{n}(1 - \frac{|y-c_n|}{w_n}), & \text{if } \exists n \in I_2 : x = 1 - \frac{1}{n},\, y \in [c_n - w_n,\, c_n + w_n], \end{cases}$$

where $c_n = 1 - \frac{1}{n}$, $w_n = \frac{1}{2n}$, $n \in I_2$. It is obvious that $c_n \pm w_n \in (0,1)$, $n \in I_2$. For any given $x \in [0,\,1]$, $f(x,\cdot)$ is either constant (vanishing) or a bounded piecewise linear function whose greatest absolute value of the slope is $\frac{1}{nw_n} = \frac{1}{n\frac{1}{2n}} = 2$, $n \in I_2$. So we deduce that for any $x \in [0,\,1]$ the function $f(x,\cdot)$ satisfies the Lipschitz condition with a Lipschitz constant equal to 2.

We now turn to the 'uniformly bounded variation' part. Let us fix any $n_0 \in I_2$. We claim that:

(a) $c_n - w_n > c_{n_0} + w_{n_0}$ for $n \in I_{4n_0}$.

It holds that $c_n - w_n > c_{n_0} + w_{n_0} \Leftrightarrow 1 - \frac{1}{n} - \frac{1}{2n} > 1 - \frac{1}{n_0} + \frac{1}{2n_0} \Leftrightarrow -\frac{1}{n} - \frac{1}{2n} > -\frac{1}{n_0} + \frac{1}{2n_0}$. Substituting $n = kn_0$ for some $k \in \mathbb{N}$ into the last inequality we get $-\frac{1}{kn_0} - \frac{1}{2kn_0} > -\frac{1}{n_0} + \frac{1}{2n_0}$, which can be equivalently written as $\frac{1}{2} > \frac{3}{2k}$, and it is evident that $k = 4$ satisfies this inequality. Suppose that $n > 4n_0$. Let $g(s) = 1 - \frac{1}{s} - \frac{1}{2s} = 1 - \frac{3}{2s}$, $s \in [1, +\infty)$. It is clear that g is a continuous and strictly increasing function in $[1,\,+\infty)$. Because $g(n) = c_n - w_n$ for $n \in I_2$ and $4n_0 > 1$, it is a fact that $g(n) > g(4n_0) > 1 - \frac{1}{n_0} + \frac{1}{2n_0} = c_{n_0} + w_{n_0}$, $n \in I_{4n_0+1}$, which implies that claim (a) holds true.

(b) $\sum_{n=n_0}^{4n_0} \frac{1}{n} < 4$.

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[1] The formulation is taken from [1], Theorem 4.3. $BV([0,1])$ denotes the Banach space of functions of bounded variation in the sense of Jordan from $[0,1]$ to $\mathbb{R}$.

[2] In the above quote, [11] denotes [2], while [14] denotes [9] in our References.

Since $n_0 \geq 2$, we have

$$\sum_{n=n_0}^{4n_0} \frac{1}{n} = \frac{1}{n_0} + \ldots + \frac{1}{4n_0} \leq \underbrace{\frac{1}{n_0} + \frac{1}{n_0} + \ldots + \frac{1}{n_0}}_{3n_0+1 \text{ terms}} = (3n_0+1)\frac{1}{n_0} < \frac{4n_0}{n_0} = 4,$$

which shows validity of claim (b).

Note that claim (a) implies that $(c_{n_0} - w_{n_0}, c_{n_0} + w_{n_0}) \cap (c_n - w_n, c_n + w_n) = \emptyset$ if $n \geq 4n_0$ or $n \leq \frac{n_0}{4}$. Now suppose that $y \in [0,1]$. If $y \notin (c_n - w_n,\, c_n + w_n), n \in I_2$, then $f(\cdot, y) \equiv 0$. If $y \in (c_{n_0} - w_{n_0},\, c_{n_0} + w_{n_0})$ for some $n_0 \in I_2$, then, by claim (a), $y \notin (c_n - w_n,\, c_n + w_n)$ for $n \geq 4n_0$ or $n \leq \frac{n_0}{4}$. Claim (b) and the definition of f imply that[3]

$$\begin{aligned} V_0^1(f(\cdot,y)) &\leq 2 \sum_{n=q(n_0)}^{4n_0} \frac{1}{n} \\ &\leq 2\left(\sum_{n=q(n_0)}^{4q(n_0)} \frac{1}{n} + \sum_{n=4q(n_0)+1}^{n_0} \frac{1}{n} + \sum_{n=n_0}^{4n_0} \frac{1}{n}\right) \leq 2(4+3+4) = 22, \end{aligned}$$

where $V_0^1(g(\cdot))$ denotes the variation of g over $[0,1]$, $q(n_0) = \max\{\lfloor n_0/4 \rfloor, 2\}$, and $\lfloor a \rfloor$ is the greatest integer not greater than a, $a \in \mathbb{R}$. The upper bound on the variation does not depend on y, so the functions $f(\cdot, y)$, $y \in \mathbb{R}$, satisfy $V_0^1(f(\cdot,y)) \leq 22$.

We conclude that f meets the desired properties. Now let $u(x) = x$ and $g(x) := f(x, u(x))$ for $x \in [0,\, 1]$. Obviously, $V_0^1(u(\cdot)) = 1$ and

$$g(x) = \begin{cases} 0, & x \neq 1 - \frac{1}{n}, \\ \frac{1}{n}, & x = 1 - \frac{1}{n}, \end{cases}$$

where $n \in I_2$. Since series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, it follows that $V_0^1(g(\cdot)) = +\infty$.

References

[1] J. Appell, N. Guanda, N. Merentes, and J. L. Sanchez, *Boundedness and continuity properties of nonlinear composition operators: a survey*, Commun. Appl. Anal. **15** (2011), no. 2-4, 153–182. MR2867343

[2] Jürgen Appell and Petr P. Zabrejko, *Nonlinear superposition operators*, Cambridge Tracts in Mathematics, vol. 95, Cambridge University Press, Cambridge, 1990. MR1066204 (91k:47168)

[3] Daria Bugajewska, *On the superposition operator in the space of functions of bounded variation, revisited*, Math. Comput. Modelling **52** (2010), no. 5-6, 791–796, DOI 10.1016/j.mcm.2010.05.008. MR2661764 (2011j:47180)

[4] Daria Bugajewska and Dariusz Bugajewski, *On nonlinear integral equations and nonabsolute convergent integrals*, Dynam. Systems Appl. **14** (2005), no. 1, 135–147. MR2128317 (2005k:45012)

[5] Dariusz Bugajewski, *On BV-solutions of some nonlinear integral equations*, Integral Equations Operator Theory **46** (2003), no. 4, 387–398, DOI 10.1007/s00020-001-1146-8. MR1997978 (2004f:45006)

[3] We take the following convention: $\sum_{i=k}^{l} a_i := \sum_{i \in \mathbb{Z}:\, k \leq i \leq l} a_i$, where $a_i \in \mathbb{R}$, $\mathbb{Z}$ denotes the set of integers, and $\sum_{i \in \emptyset} a_i := 0$.

[6] Dariusz Bugajewski and Donal O'Regan, *Existence results for BV-solutions of nonlinear integral equations*, J. Integral Equations Appl. **15** (2003), no. 4, 343–357, DOI 10.1216/jiea/1181074981. MR2058808 (2005g:45006)
[7] J. Ciemnoczołowski and W. Orlicz, *Composing functions of bounded φ-variation*, Proc. Amer. Math. Soc. **96** (1986), no. 3, 431–436, DOI 10.2307/2046589. MR822434 (87k:26012)
[8] Michael Josephy, *Composing functions of bounded variation*, Proc. Amer. Math. Soc. **83** (1981), no. 2, 354–356, DOI 10.2307/2043527. MR624930 (83c:26009)
[9] A. G. Ljamin, *On the acting problem for the Nemytskij operator in the space of functions of bounded variation*, 11th School Theory Operators Function Spaces, Chelyabinsk, 1986, 63–64 (in Russian).

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