PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 5, May 2014, Pages 1591–1606
S 0002-9939(2014)12077-6
Article electronically published on February 10, 2014

ON THE PATH PROPERTIES OF A LACUNARY POWER SERIES

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(Communicated by Richard Rochberg)

Abstract. A power series $f(z)$ which converges in $\mathbb{D} = \{|z| < 1\}$ maps the radii $[0, \zeta)$ onto paths $\Gamma(\zeta)$, $\zeta \in \mathbb{T} = \partial\mathbb{D}$. These are studied under several aspects in the case of the special lacunary series $f(z) = z + z^2 + z^4 + z^8 + \dots$. First, the $\Gamma(\zeta)$ are considered as random functions on the probability space $(\mathbb{T}, \mathscr{B}, \mathrm{mes}/2\pi)$, where $\mathscr{B}$ is the σ-algebra of Borel sets and mes the Lebesgue measure. Then analytical properties of the $\Gamma(\zeta)$ are discussed which hold on subsets A of $\mathbb{T}$ with Hausdorff dimension 1 in spite of $\mathrm{mes} A = 0$. Furthermore, estimates of the derivative of f and of the arc length of sections of the $\Gamma(\zeta)$ are given. Finally, these results are used to derive connections between the distribution of critical points of f and the overall behaviour of the paths.

1. Introduction

Let $\mathbb{D}$ be the unit disk and $\mathbb{T} = \partial\mathbb{D}$ the unit circle. We consider the lacunary power series

$$f(z) = \sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + z^8 + \dots \qquad (z \in \mathbb{D}). \tag{1.1}$$

Our main interest is to study the paths

$$\Gamma(\zeta)\colon f((1 - 2^{-t})\zeta), \quad 0 \le t < \infty, \tag{1.2}$$

for $\zeta \in \mathbb{T}$, that is, the images of the radii $[0, \zeta)$ under f.

The aim of this paper is to present an overview of the different kinds of features of f and its paths $\Gamma(\zeta)$. For clarity and ease of presentation we consider only this particular function f, which is typical for a large class of lacunary power series. Generality would mean having different technical assumptions for different theorems.

In Section 2 we study the most interesting question, namely the relation between the lacunary power series and Brownian motion. Here $\mathbb{T}$ is considered as a probability space with the normalized Lebesgue measure as probability measure. The tails of almost all $\Gamma(\zeta)$ look like Brownian paths. But it turns out to be difficult to give a precise meaning to this statement. In particular, it is doubtful that $\mathbb{T}$ suffices as probability space to describe this behaviour.

The approach of P. Billingsley [Bil99] shows that the real and imaginary parts of $\sqrt{2/n} f((1 - 2^{-n\tau})\zeta)$ $(0 \le \tau \le 1)$, as random variables of ζ on $\mathbb{T}$, converge

Received by the editors May 22, 2012 and, in revised form, June 1, 2012.
2010 *Mathematics Subject Classification.* Primary 30B10, 60G17, 60J65.
Key words and phrases. Lacunary series, Brownian motion, path properties, critical points, Bloch functions.

in distribution to linear Brownian motion on the probability space of continuous functions on $[0,1]$ with the σ-algebra of Borel sets induced by the uniform norm. The result of W. Philipp and W. Stout [PS75] deals with almost sure asymptotic behaviour of the $\Gamma(\zeta)$, even in a quantitative sense, but on an unspecified probability space.

Then we turn to analytical problems. In Section 3 we sketch some known results and discuss the standing of our function f within the class of lacunary power series. Then we state three theorems of N. Makarov [Mak89a], [Mak89b] and J. Hawkes [Haw80], who have shown that on certain subsets of $\mathbb{T}$ with measure 0 but Hausdorff dimension 1, the tail behaviour of the paths $\Gamma(\zeta)$ is much more regular than for almost all $\zeta \in \mathbb{T}$. Also we prove the tightness (in the sense of Billingsley) of a complex sequence associated with f.

In Section 4 we prove some estimates for the derivative f' which we need later. The derivative has no counterpart for Brownian motion. In Section 5 we show (Theorem 5.1) that the parametrization of $\Gamma(\zeta)$ by t is essentially the same as the parametrization by arc length.

Then we try to find out whether there are any function-theoretic reasons for $\Gamma(\zeta)$ to be so wild. Now all sheets of the Riemann image surface R of $\mathbb{D}$ under f cover $\mathbb{C}$ completely; there are no finite boundary points. Hence only the branch points can shape R. Their projections onto $\mathbb{C}$ are the critical values $w = f(z)$ where z is a critical point; that is, $f'(z) = 0$ holds. We study (Theorem 5.2) the distribution of the critical points in $\mathbb{D}$ and the critical values in $\mathbb{C}$. Finally we show (Theorem 5.3) how the critical values influence the form of $\Gamma(\zeta)$ for all $\zeta \in \mathbb{T}$.

2. The lacunary function and Brownian motion

2.1. Let $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$ and let $\operatorname{mes} E$ denote the Lebesgue measure on $\mathbb{T}$, with $\operatorname{mes} \mathbb{T} = 2\pi$. In this section, it will be convenient to represent $z \in \mathbb{D}$ in the form $z = r\zeta = (1 - 2^{-t})\zeta$, $0 \le t < \infty$, $\zeta \in \mathbb{T}$. The key to the connection between our function

$$f(z) = \sum_{k=0}^{\infty} z^{2^k} \qquad (z \in \mathbb{D}) \tag{2.1}$$

and the Brownian motion is the following well-known elementary lemma [Pom92, Prop. 8.13].

Lemma 2.1. *There is a constant c such that, for $0 \le t < \infty$,*

$$\Big| f((1 - 2^{-t})\zeta) - \sum_{k=0}^{[t]} \zeta^{2^k} \Big| < c \qquad (\zeta \in \mathbb{T}). \tag{2.2}$$

This lemma often allows us to go freely from the function to the partial sums on $\mathbb{T}$ and vice versa provided that $n = \left[\log \frac{1}{1-r} / \log 2\right]$ is observed. Now we mention two classical theorems formulated for the function f.

The *central limit theorem* for lacunary series [SZ47], [Zyg68b, p. 264] implies

Theorem 2.2 (Salem–Zygmund)**.** *Let $E \subset \mathbb{T}$ with $\operatorname{mes} E > 0$ and let $x \in \mathbb{R}$. Then, as $r \to 1$,*

$$\operatorname{mes}\{\zeta \in E\colon \operatorname{Re} f((1 - 2^{-t})\zeta)\} \le x\sqrt{\frac{t}{2}} \to \frac{\operatorname{mes} E}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} \mathrm{d}y. \tag{2.3}$$

The same holds for $\operatorname{Im} f((1-2^{-t})\zeta)$. Contrary to what is claimed in [Pom92, p. 193], we do not know of a complex form.

The *law of the iterated logarithm* for lacunary series [EG55], [Wei59] implies

Theorem 2.3 (Erdös–Gal and Weiss)**.** *For almost all $\zeta \in \mathbb{T}$ we have*

$$\limsup_{r\to 1} |f((1-2^{-t})\zeta)| \Big/ \sqrt{t \log\log t} = 1. \tag{2.4}$$

The upper estimate was proved by Erdös and Gal, whereas the lower estimate is due to M. Weiss. The corresponding result for $\operatorname{Re} f(r\zeta)$ follows from the powerful Hartmann–Wintner theorem [HW41] for Brownian motion; see [MP10, p. 127].

2.2. Now we turn to Brownian motion; see e.g. [Bil99], [MP10] and the survey paper [Kah97]. We consider the classical probability space

$$S_0 := \big(\mathbb{T}, \text{Borel}, \frac{\text{mes}}{2\pi}\big) \tag{2.5}$$

and define a stochastic process

$$F(t) = F(t,\zeta) := f((1-2^{-t})\zeta) \qquad (\zeta \in \mathbb{T}, t \geq 0). \tag{2.6}$$

It follows from (2.1) that $\mathbb{E}(F(t)) = 0$ and, with $r = 1-2^{-t}$,

(2.7)

$$\mathbb{V}(F(t)) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} r^{2^{k+1}} = f(r^2) \sim \frac{\log \frac{1}{1-r}}{\log 2} = t \qquad (t \to \infty).$$

There are two different approaches to deal with real-valued lacunary series. The first method is that of W. Philipp and W. Stout; compare also [Ber75]. From [PS75, Th. 3.1] and Lemma 2.1 we obtain

Theorem 2.4 (Philipp–Stout)**.** *Without changing the distribution of* $\{\operatorname{Re} F(t),\, t \geq 0\}$*, we can redefine this process on a richer probability space S together with standard Brownian motion* $\{B(t),\, t \geq 0\}$ *such that, for each $\gamma > 0$,*

$$\operatorname{Re} F(t) = B(t) + O(t^{\frac{5}{12}+\gamma}) \qquad (t \to \infty) \tag{2.8}$$

almost surely on S.

It is of course important that the exponent is $< \frac{1}{2}$ for small γ. The redefinition of the stochastic process on the richer probability space S was necessary in order to apply the martingale version [Str65, Th. 4.3] of the Skorohod representation theorem. For our purposes this raises the important

Question 1. Does almost surely on S imply almost surely on S_0?

If the answer to this question is positive, then (2.8) would say very much about the tail behaviour of the paths $\Gamma(\zeta)$ for almost all ζ.

2.3. P. Billingsley follows a different route [Bil99]. Put in our special context, he considers the sequence of stochastic processes

$$X_n(\tau,\omega) := \frac{\sqrt{2}}{\sqrt{n}}\left(\sum_{j=1}^{[n\tau]} \cos 2^j\omega + (n\tau - [n\tau]) \cos 2^{[n\tau]+1}\omega\right) \qquad (0 \leq \tau \leq 1) \tag{2.9}$$

which are constructed by linear interpolation between the first n partial sums of $\sum \cos 2^j\omega$. We have $\mathbb{E}(X_n(\tau,\cdot)) = 0$ and $1 \leq \mathbb{V}(X_n(\tau,\cdot)) \leq 1 + 1/n$. The paths

$X_n(\cdot,\omega)$ are regarded as random elements on $\mathbb{T}$ in the space S_1 of all real-valued continuous functions on $[0,1]$ with the Borel σ-field induced by the uniform topology. From [Bil99, Theorem 11.1] we obtain

Theorem 2.5 (Billingsley)**.** *The random elements X_n converge in distribution to a standard linear Brownian motion, i.e. $X_n \Rightarrow W$ as $n \to \infty$.*

The proof is clearly written and shorter than the proof of Theorem 2.4. The theorem implies Theorem 2.2 of Salem and Zygmund but only for the case that $E = \mathbb{T}$. We need the following extension to mixed Fourier series:

Theorem 2.6 (Billingsley)**.** *For every $a \in \mathbb{C}$, $|a| = 1$, the random elements*

(2.10)
$$X_n^{(a)}(\tau,\omega) := \frac{\sqrt{2}}{\sqrt{n}} \operatorname{Re}\left[a\left(\sum_{j=1}^{[n\tau]} \mathrm{e}^{\mathrm{i}2^j\omega} + (n\tau - [n\tau])\mathrm{e}^{\mathrm{i}2^{[n\tau]+1}\omega}\right)\right] \quad (0 \le \tau \le 1)$$

converge in distribution to a standard linear Brownian motion, i.e. $X_n^{(a)} \Rightarrow W$ as $n \to \infty$.

The proof requires merely some modifications and additions to Billingsley's proof for pure cosine series. It is, however, important to realize that it rests upon [Bil99, Theorem 7.5] and essentially verifies the hypotheses (7.13) and (7.14) of the latter.

Now let S_2 be the probability space of all complex-valued continuous functions on $[0,1]$ with the Borel σ-field induced by the uniform topology. We consider the random elements $Z_n(\cdot,\omega)$ on $\mathbb{T}$ in the space S_2 defined by

$$Z_n(\tau,\omega) := \frac{1}{\sqrt{n}}\left(\sum_{j=1}^{[n\tau]} \mathrm{e}^{\mathrm{i}2^j\omega} + (n\tau - [n\tau])\mathrm{e}^{\mathrm{i}2^{[n\tau]+1}\omega}\right) \quad (0 \le \tau \le 1). \tag{2.11}$$

By Theorem 2.6 we have

$$\operatorname{Re} Z_n \to B_1, \quad \operatorname{Im} Z_n \to B_2 \quad \text{as } n \to \infty$$

in distribution on S_1. Furthermore it is easy to show that $\operatorname{Re} Z_n$ and $\operatorname{Im} Z_n$ are uncorrelated for each n. This suggests the following generalization of Theorem 2.5:

Theorem 2.7. *The random elements Z_n converge in distribution to a standard planar Brownian motion, i.e. $Z_n \Rightarrow W$ as $n \to \infty$.*

A planar Brownian has the form $B_1 + \mathrm{i}B_2$ where B_j are independent linear Brownian motions. In view of the topology in S_2 the convergence applies to the joint distributions of the real and imaginary parts. Note that $X_n = \sqrt{2}\operatorname{Re} Z_n$ and $\mathbb{E}(Z_n(\tau,\cdot)) = 0$ and $1 \le \mathbb{V}(Z_n(\tau,\cdot)) \le 1 + 1/n$.

Proof of Theorem 2.7. Besides (2.9) let

$$Y_n(\tau,\omega) := \frac{\sqrt{2}}{\sqrt{n}}\left(\sum_{j=1}^{[n\tau]} \sin 2^j\omega + (n\tau - [n\tau])\sin 2^{[n\tau]+1}\omega\right). \tag{2.12}$$

Then

(2.13)

$$\lambda X_n(\tau,\omega)+\mu Y_n(\tau,\omega)=\frac{\sqrt{2}}{\sqrt{n}}\operatorname{Re}\left[(\lambda-\mathrm{i}\mu)\left(\sum_{j=1}^{[n\tau]}\mathrm{e}^{\mathrm{i}2^j\omega}+(n\tau-[n\tau])\mathrm{e}^{\mathrm{i}2^{[n\tau]+1}\omega}\right)\right].$$

Now we apply Theorem 2.6 for $a=1$, $a=-\mathrm{i}$ and $a=\dfrac{\lambda-\mathrm{i}\mu}{\sqrt{\lambda^2+\mu^2}}$ and obtain

$$X_n\Rightarrow W^X,\ Y_n\Rightarrow W^Y,\ \lambda X_n+\mu Y_n\Rightarrow\sqrt{\lambda^2+\mu^2}W^{(\lambda,\mu)}, \tag{2.14}$$

where W^X, W^Y and $W^{(\lambda,\mu)}$ are standard Brownian motions on some probability spaces. We may assume that W^X and W^Y are defined on the same space and are independent.

For every fixed τ we have $X_n(\tau,\cdot)\Rightarrow W^X(\tau,\cdot)$, $Y_n(\tau,\cdot)\Rightarrow W^Y(\tau,\cdot)$ and $\lambda X_n(\tau,\cdot)+\mu Y_n(\tau,\cdot)\Rightarrow\sqrt{\lambda^2+\mu^2}\,W^{(\lambda,\mu)}(\tau,\cdot)$. Since the sum of independent normally distributed variables is normally distributed, it follows that

$$\lambda X_n^n(\tau,\cdot)+\mu Y_n^n(\tau,\cdot)\Rightarrow\lambda W^X(\tau,\cdot)+\mu W^Y(\tau,\cdot). \tag{2.15}$$

In the following we make repeated use of the Cramér-Wold Theorem (see [Bil95, p. 383]), which states that for a sequence of $\mathbb{R}^m$-valued random variables, weak convergence is equivalent to weak convergence of the sequences of linear combinations of their components.

Let $0\le\tau_1\le\ldots\le\tau_k\le 1$. Since (2.15) holds for every τ_j, it follows by the Cramér-Wold Theorem that for arbitrary real u_j,

$$\begin{aligned}\lambda\sum_j u_jX_n(\tau_j,\cdot)+\mu\sum_j u_jY_n(\tau_j,\cdot)&=\sum_j u_j(\lambda X_n(\tau_j,\cdot)+\mu Y_n(\tau_j,\cdot))\\&\Rightarrow\sum_j u_j(\lambda W^X(\tau_j,\cdot)+\mu W^Y(\tau_j,\cdot))\\&=\lambda\sum_j u_jW^X(\tau_j,\cdot)+\mu\sum_j u_jW^Y(\tau_j,\cdot).\end{aligned}$$

This holds for all λ and μ, so by the reverse direction of the Cramér-Wold Theorem,

$$(\sum_j u_jX_n(\tau_j,\cdot),\sum_j u_jY_n(\tau_j,\cdot))\Rightarrow(\sum_j u_jW^X(\tau_j,\cdot),\sum_j u_jW^Y(\tau_j,\cdot)),$$

which can be written in the form

$$\sum_j u_j(X_n(\tau_j,\cdot),Y_n(\tau_j,\cdot))\Rightarrow\sum_j u_j(W^X(\tau_j,\cdot),W^Y(\tau_j,\cdot)).$$

This holds for all $u_1,\ldots,u_k$, so applying the Cramér-Wold Theorem once more we obtain

$$\begin{aligned}&((X_n(\tau_1,\cdot),Y_n(\tau_1,\cdot)),\ldots,(X_n(\tau_k,\cdot),Y_n(\tau_k,\cdot)))\\&\Rightarrow((W^X(\tau_1,\cdot),W^Y(\tau_1,\cdot)),\ldots,(W^X(\tau_k,\cdot),W^Y(\tau_k,\cdot))),\end{aligned}$$

which is just (7.13) for (X_n,Y_n) and (W^X,W^Y). This shows that condition (7.13) in [Bil99, Theorem 7.5] is satisfied for (X_n,Y_n) in the role of X^n; this theorem is clearly valid, not only for real- but also for $\mathbb{R}^m$-valued, in particular $\mathbb{C}$-valued,

functions. The other hypothesis (7.14) of the theorem is obviously true since it holds for the components X_n and Y_n. So $(X_n, Y_n) \Rightarrow (W^X, W^Y)$ or

$$\frac{1}{\sqrt{2}}(X_n + \mathrm{i}Y_n) \Rightarrow W, \tag{2.16}$$

where now W is a standard planar Brownian motion. □

By Lemma 2.1 we have for $0 \le \tau \le 1$

$$\left| f(r\zeta) - \sum_{j=0}^{k} \zeta^{2^j} \right| < c \quad \text{for} \quad 1 - 2^{-[n\tau]} \le r \le 1 - 2^{[n\tau]+1},$$

so for $r = 1 - 2^{-n\tau}$

$$\left| f((1 - 2^{-n\tau})\zeta) - \sum_{j=0}^{[n\tau]} \zeta^{2^j} \right| < c.$$

From

$$\frac{\sqrt{n}}{\sqrt{2}}(X_n(\tau,\omega) + \mathrm{i}Y_n(\tau,\omega)) = \sum_{j=1}^{[n\tau]} \mathrm{e}^{\mathrm{i}2^j\omega} - (n\tau - [n\tau])\mathrm{e}^{\mathrm{i}2^{[n\tau]+1}\omega}$$

it follows that

$$\left| \frac{1}{\sqrt{n}} f((1 - 2^{-n\tau})\mathrm{e}^{\mathrm{i}\omega}) - \frac{1}{\sqrt{2}}(X_n(\tau,\omega) + \mathrm{i}Y_n(\tau,\omega)) \right| < \frac{c+1}{\sqrt{n}}.$$

Using [Bil99, Theorem 3.1] we now obtain from (2.16):

Theorem 2.8. *The stochastic processes*

$$\Phi_n(\tau,\zeta) := \frac{1}{\sqrt{n}} f((1 - 2^{-n\tau})\zeta) \quad (\zeta \in \mathbb{T},\ \ 0 \le \tau \le 1) \tag{2.17}$$

converge in distribution to a standard planar Brownian motion, i.e. $\Phi_n \Rightarrow W$ *as* $n \to \infty$.

Now we discuss the possibility of ordinary convergence on S_0 instead of convergence in distribution on S_2.

Proposition 2.9. *Let* $\Phi_n(\tau,\zeta)$ *be defined by* (2.17). *If* h *is continuous in* $[0,1]$ *and if*

$$\operatorname{Re}\Phi_n(\tau,\zeta) \to h(\tau) \ \ (n \to \infty) \quad \textit{for } \tau \in \mathbb{Q} \cap [0,1], \tag{2.18}$$

then $h(\tau) = \sqrt{\tau}h(1)$ *for* $\tau \in [0,1]$.

Proof. Given $\tau \in [0,1]$ we choose $p_k, q_k \in \mathbb{N}$ such that $p_k/q_k \to \tau$ as $k \to \infty$. It follows from (2.17) that

$$\Phi_{nq_k}\Big(\frac{p_k}{q_k}, \zeta\Big) = \frac{1}{\sqrt{q_k n}} f((1 - 2^{-np_k})\zeta) = \sqrt{\frac{p_k}{q_k}}\Phi_{np_k}(1,\zeta).$$

Hence (2.18) implies $h(p_k/q_k) = \sqrt{p_k/q_k}\, h(1)$. Since h is continuous and $p_k/q_k \to \tau$, it follows that $h(\tau) = \sqrt{\tau}\, h(1)$. □

This proposition destroys all hope that $\Phi_n(\tau,\zeta)$ converges pointwise on a subset of $\mathbb{T}$ of positive measure as $n \to \infty$.

3. Some results from analysis

3.1. The general lacunary power series with Hadamard gaps has the form

$$g(z) = \sum_{k=0}^{\infty} b_k z^{n_k} \quad \text{with } b_k \in \mathbb{C},\ \frac{n_{k+1}}{n_k} \geq q > 1. \tag{3.1}$$

We list some important conditions in increasing order of strength.

(a) The condition $\sup |b_k| < \infty$ holds if and only if g is a *Bloch function*, that is, if

$$\|g\|_{\mathcal{B}} := \sup_{|z|<1} (1-|z|^2)|g'(z)| < \infty. \tag{3.2}$$

The space of Bloch functions is denoted by $\mathcal{B}$. See [ACP74] for basic properties.

(b) The condition $b_k \to 0$ $(k \to \infty)$ holds if and only if there exist $w \in \mathbb{C}$ and a curve $C \subset \mathbb{D}$ such that $\overline{C} \setminus C \subset \mathbb{T}$ and $g(z) \to w$ as $|z| \to 1$, $z \in C$. Indeed there exists $\zeta \in \mathbb{T}$ such that $\lim_{r\to 1} g(r\zeta)$ exists and is finite [Bin69].

(c) The condition $\sum |b_k|^2 < \infty$ holds if and only if g belongs to the Hardy space $\mathcal{H}^p$ for some $p \in (1,\infty)$ or all $p \in (1,\infty)$ [Zyg68a, p. 203].

(d) The condition $\sum |b_k| < \infty$ holds if and only if $g(\mathbb{D}) \neq \mathbb{C}$ [Mur81], [GP83].

Condition (c) is important in connection with Brownian motion; see e.g. [Bil99, Th. 11.1]. Now let f be our lacunary function. Then conditions (a), (b) and (d) hold, and therefore we obtain

Proposition 3.1. *If f is defined by* (1.1), *then*

(i) f is a Bloch function with $f(\mathbb{D}) = \mathbb{C}$,

(ii) $f(z)$ does not have a finite limit as $|z| \to 1$ along any curve $C \subset \mathbb{D}$ with $\overline{C} \setminus C \subset \mathbb{T}$.

3.2. The four theorems of Section 2 are about the behaviour almost everywhere or almost surely. The connection with Brownian motion indicates that the paths $\Gamma(\zeta)$ defined in (1.2) behave wildly as $t \to \infty$ for almost all $\zeta \in \mathbb{T}$. Now we shall see that the behaviour is much more regular on certain large subsets of $\mathbb{T}$. Let $\dim A$ denote the Hausdorff dimension of the Borel set $A \subset \mathbb{T}$. For $A \subset \mathbb{T}$ we have $0 \leq \dim A \leq 1$, and $\operatorname{mes} A > 0$ implies $\dim A = 1$ but not vice versa. See e.g. [Fal85]. The next two theorems remain true for Bloch functions that have radial limits only on a set of measure 0.

Theorem 3.2 (Makarov)**.** *There exists a set $A_1 \subset \mathbb{T}$ with $\dim A_1 = 1$ such that*

$$\operatorname{Re} f(r\zeta) \to +\infty \quad (r \to 1) \quad \textit{for } \zeta \in A_1.$$

N. Makarov describes the size of A_1 very precisely in terms of a specific Hausdorff measure; see [Mak89b, Th. 4.1], [Pom92, Th. 10.14].

Theorem 3.3 (Makarov)**.** *There exists a set $A_2 \subset \mathbb{T}$ with $\dim A_2 = 1$ such that*

$$\sup_{0<r<1} |f(r\zeta)| < \infty \quad \textit{for } \zeta \in A_2.$$

See [Mak89a, Th. 5.1] and also [Roh89] for a generalization. Thus the path $\Gamma(\zeta)$ defined in (1.2) remains bounded for $\zeta \in A_2$, whereas almost everywhere it is strongly unbounded by the law of the iterated logarithm (Theorem 2.3). The next theorem [Haw80] also stands in marked contrast to this law.

Theorem 3.4 (Hawkes)**.** *There exists a set* $A_3 \subset \mathbb{T}$ *with* $\dim A_3 = 1$ *such that*

$$\lim_{r\to 1} f(r\zeta)\Big/ \log\frac{1}{1-r} \quad \textit{exists} \neq 0 \textit{ for } \zeta \in A_3.$$

3.3. We return to the sequence

$$\Phi_n = \Phi_n(\tau,\zeta) = \frac{1}{\sqrt{n}} f((1-2^{-n\tau})\zeta) \tag{3.3}$$

defined in (2.17). We want to show that (Φ_n) is tight in the sense of Billingsley [Bil99, Th. 7.3] but now generalized to complex sequences. Let $c_1, c_2, \dots$ denote suitable positive constants.

Lemma 3.5. *For* $0 < r < 1$ *we have*

$$\int_{\mathbb{T}} \exp\Big[c_1 \max_{0\le\rho\le r} |f(r\zeta)|^2 \Big/ \log\frac{1}{1-r}\Big] |\mathrm{d}\zeta| < c_2. \tag{3.4}$$

Our function f is a Bloch function by Proposition 3.1. Hence we can apply [Pom92, Th. 8.9]. Using the Hardy–Littlewood maximal theorem [Dur70, Th. 1.9] and the power series expansion of e^x we obtain (3.4).

Theorem 3.6. *Let* $\epsilon > 0$ *and* $0 < \delta < 1$. *For* $n \in \mathbb{N}$ *we set*

$$E_n = \{\zeta \in \mathbb{T}\colon \max_{0\le t-s\le\delta} |\Phi_n(t,\zeta) - \Phi_n(s,\zeta)| \ge \epsilon\}. \tag{3.5}$$

Then there is n_0 *such that*

$$\operatorname{mes} E_n < c_3 \exp\Big(-c_4 \frac{\epsilon^2}{\delta}\Big) \quad \textit{for } n > n_0. \tag{3.6}$$

For every $\eta > 0$ we can therefore find $\delta > 0$ such that $\operatorname{mes} E_n < \eta$ for large n. Hence condition (7.7) in [Bil99, p. 82] is satisfied so that (Φ_n) is tight in our complex sense.

Proof. Let $0 < s < t$ and $\mu = \lfloor ns \rfloor$, $\nu = \lfloor nt \rfloor$. We have

$$\sum_{k=0}^{\nu} \zeta^{2^k} - \sum_{k=0}^{\mu-1} \zeta^{2^k} = \sum_{k=\mu}^{\nu} \zeta^{2^k} = \sum_{j=0}^{\nu-\mu} (\zeta^{2^\mu})^{2^j}.$$

Now we apply Lemma 2.1 three times with $r_1 = 1 - 2^{-ns}$, $r_2 = 1 - 2^{-nt}$ and $r_3 = 1 - 2^{-n(t-s)}$. Using (3.3) we obtain

$$|\Phi_n(t,\zeta) - \Phi_n(s,\zeta)| \le |\Phi_n(t-s, \zeta^{2^\mu})| + \frac{c_5}{\sqrt{n}}. \tag{3.7}$$

With $\rho = 1 - 2^{-n(t-s)}$ and $r = 1 - 2^{-n\delta}$ we have, by (3.3),

$$\max_{0\le t-s\le\delta} |\Phi_n(t-s,\zeta')| = \frac{1}{\sqrt{n}} \max_{\rho\le r} |f(\rho\,\zeta')|.$$

Since $\log\dfrac{1}{1-r} = (\log 2) n\delta$, we thus obtain from (3.7) that, with the constant c_1 of (3.4),

$$\int_{\mathbb{T}} \exp\Big(\frac{c_1}{2\delta} \max_{t-s\le\delta} |\Phi_n(t,\zeta) - \Phi_n(s,\zeta)|^2\Big) |\mathrm{d}\zeta| \le c_6 \int_{\mathbb{T}} \exp\big(c_1 \max_{\rho\le r} |f(\rho\,\zeta^{2^k})|^2\big) |\mathrm{d}\zeta|$$

for large n. The value of the last integral is not changed if we substitute $\zeta' = \zeta^{2^k}$. Hence the left-hand side is $\le c_7$ by (3.4). Hence it follows from definition (3.5) of E_n that

$$(\operatorname{mes} E_n) \exp\left(\frac{c_1}{2\delta}\epsilon^2\right) \le c_7,$$

which implies (3.6). □

4. Estimates of the derivative

As always in this paper, let f be the lacunary function defined by (1.1). First we shall prove two estimates of its derivative,

$$f'(z) = \sum_{k=0}^{\infty} 2^k z^{2^k-1} = 1 + 2z + 4z^3 + \dots . \tag{4.1}$$

Let $c_1, c_2, \dots$ denote suitable constants > 0.

Proposition 4.1. *There exist c_1 and c_2 such that*

$$0 < c_1 < \int_{1-2^{-k}}^{1-2^{-k-1}} |f'(r\zeta)| \mathrm{d}r < c_2 \tag{4.2}$$

for $k \in \mathbb{N}$ and $\zeta \in \mathbb{T}$.

Proof of the upper estimate (4.2). Since $f \in \mathcal{B}$ we obtain from (3.2) that

$$|f'(z)| \le \|f\|_{\mathcal{B}} / (1 - |z|^2),$$

and the upper estimate (4.2) follows by integration. □

For the lower estimate we need a lemma which is established by the method of K. Binmore; see [GHP87, Th. 2]. We formulate this lemma for the function (4.1), which is also lacunary.

Lemma 4.2. *There exist $c_3, c_4, c_5 > 0$ and $m \in \mathbb{N}$ such that if $0 < \gamma < c_3$ and $z_\mu \in \mathbb{D}$ ($\mu = 0, \dots, m$) satisfy*

$$1 - \frac{1}{2^k} \le |z_\mu| \le 1 - \frac{1}{2^k} + \frac{c_3}{2^k}, \tag{4.3}$$

$$\frac{\gamma}{2^k} \le |z_\mu - z_\nu| \le \frac{1}{2^{k+1}} \qquad (\mu \ne \nu) \tag{4.4}$$

for some $k \in \mathbb{N}$ with $k > c_4$, then

$$\max_{\mu=0,\dots,m} |f'(z_\mu)| \ge c_5 \gamma^m 2^k. \tag{4.5}$$

Proof of the lower estimate (4.2). We apply Lemma 4.2 with $\gamma = c_3/(2m+1)$; we may assume that $c_3 \le \frac{1}{2}$. Now we consider the disjoint intervals

$$I_\nu = \left[1 - \frac{1}{2^k} + \frac{2\nu\gamma}{2^{k+1}}, 1 - \frac{1}{2^k} + \frac{(2\nu+1)\gamma}{2^k}\right] \qquad (\nu = 0, \dots, m). \tag{4.6}$$

We claim that there exists μ depending on ζ and k such that

$$|f'(r\zeta)| \ge c_5 \gamma^m 2^k \quad \text{for all } r \in I_\mu. \tag{4.7}$$

Suppose that this claim is false. Then, for every ν, there exists $z_\nu = r_\nu \zeta$ with $r_\nu \in I_\nu$ such that

$$|f'(r_\nu \zeta)| < c_5 \gamma^m 2^k \qquad (\nu = 0, \dots, m). \tag{4.8}$$

The intervals I_ν have mutual distances $\geq \gamma 2^{-k}$ by (4.6). Since $r_\nu \in I_\nu$ it follows that $|z_\mu - z_\nu| \geq \gamma 2^{-k}$ for $\mu \neq \nu$. Since $(2m+1)\gamma = c_3 \leq \frac{1}{2}$ the upper inequality (4.4) is also satisfied, and (4.3) follows from (4.6). Hence (4.8) contradicts (4.5).

We conclude from (4.6) and (4.7) that

$$\int_{1-2^{-k}}^{1-2^{-k-1}} |f'(r\zeta)|\mathrm{d}r \geq \int_{I_\mu} |f'(r\zeta)|\mathrm{d}r \geq \frac{\gamma}{2^k}\, c_5 \gamma^m 2^k =: c_1$$

because $\gamma = c_3/(2m+1)$ and m is a constant. □

Proposition 4.3. *There exist constants $c_1, c_2 > 0$ such that every component H of*

$$H_0 = \{z \in \mathbb{D} \colon (1-|z|^2)|f'(z)| < c_1\} \tag{4.9}$$

has a hyperbolic diameter $< c_2$ and contains at least one zero of f'.

We need the following lemma [GP84, Th. 2], again formulated for f'. The maximum term of the power series (4.1) is defined by

$$\mu(r) := \max_{k \in \mathbb{N}_0} 2^k r^{2^k - 1} = 2^{\nu(r)} r^{2^{\nu(r)}-1}. \tag{4.10}$$

Lemma 4.4. *There are constants $c_3, c_4, c_5 > 0$ with $c_4 < 1$ such that if H is a domain in $\mathbb{D}$ with*

$$|f'(z)| \leq c_3 \mu(|z|) \qquad \text{for } z \in H, \tag{4.11}$$

then $\rho := \sup\{z \colon z \in H\} < 1$. If $\rho > c_4$, then (see (4.10))

$$\operatorname{diam} H < c_5 2^{-\nu(\rho)}. \tag{4.12}$$

Proof of Proposition 4.3. (a) With $\nu = \nu(r)$ we deduce from (4.10) that

$$2^{\nu-1} r^{2^{\nu-1}} \leq 2^\nu r^{2^\nu} \quad \text{and} \quad 2^{\nu+1} r^{2^{\nu+1}} \leq 2^\nu r^{2^\nu}. \tag{4.13}$$

The second inequality implies $2r^{2^\nu} \leq 1$ and thus

$$2^\nu \geq \log 2 \big/ \log \frac{1}{r}. \tag{4.14}$$

The first inequality (4.13) implies $1 \leq 2r^{2^{\nu-1}}$ and thus $r^{2^\nu} \geq \frac{1}{4}$. Hence we obtain from (4.10) and (4.14) that

$$(1-r^2)\mu(r) \geq \frac{\log 2}{4} \frac{1-r^2}{\log(1/r)} > \frac{1}{8} \tag{4.15}$$

if $\frac{1}{2} \leq r < 1$. If $0 < r < \frac{1}{2}$, then $\mu(r) = 1$ and (4.15) holds trivially.

Now we apply Lemma 4.4 assuming first that $\rho > c_5$. Let $c_1 = \frac{1}{8} c_3$; see (4.11). If $z \in H$ and $r = |z|$, then

$$|f'(z)| < \frac{c_1}{1-r^2} \leq 8 c_1 \mu(r) = c_3 \mu(r)$$

by (4.15). Hence (4.11) is satisfied, and it follows from (4.12) and (4.14) that

$$\operatorname{diam} H \leq \frac{c_5}{2^{\nu(\rho)}} \leq \frac{c_5}{\log 2} \log \frac{1}{\rho} \leq c_6 (1-\rho^2) \tag{4.16}$$

because $\rho > c_5$. If however $\rho \leq c_5$, then $\operatorname{diam} H \leq 2c_5 \leq c_7(1-\rho^2)$.

Let $z_1, z_2 \in H$ and let S be the hyperbolic line segment from z_1 to z_2. Then $|z| \le \rho$ for $z \in S$ and therefore

$$\int_S \frac{|\mathrm{d}z|}{1-|z|^2} \le \frac{1}{1-\rho^2} \int_S |\mathrm{d}z| \le \frac{\pi|z_1 - z_2|}{1-\rho^2} < c_8$$

by (4.16). Hence H has a hyperbolic diameter $< c_2 := c_8$.

(b) Suppose that f' has no zero in H. Then

$$v(z) := \log[(1-|z|^2)|f'(z)|]^{-1} \qquad (z \in H) \tag{4.17}$$

is finite and continuous and thus subharmonic because

$$\frac{\partial^2}{\partial z \partial \bar{z}} v(z) = \frac{1}{(1-|z|^2)^2} > 0.$$

Now H is a component of H_0 with $\overline{H} \subset \mathbb{D}$. Hence $v(z) = \log(1/c_1)$ for $z \in \partial H$ by (4.17) and (4.9). Since v is subharmonic in H it follows that $v(z) \le \log(1/c_1)$ for $z \in H$ and therefore

$$(1-|z|^2)|f'(z)| = \mathrm{e}^{-v(z)} \ge c_1 \qquad \text{for } z \in H.$$

This is false by (4.9). Hence f' has a zero in H. □

5. Paths and critical points

5.1. Now we turn to the paths of our function f defined in (1.1), namely

$$\Gamma(\zeta)\colon \quad f((1-2^{-t})\zeta), \quad 0 \le t < \infty, \tag{5.1}$$

for $\zeta \in \mathbb{T}$, defined in (1.2). For almost all $\zeta \in \mathbb{T}$, the path $\Gamma(\zeta)$ is dense in $\mathbb{C}$; see [AP87, Prop. 2.2]. We also consider the sections

$$\Gamma_{s,t}(\zeta)\colon \quad f((1-2^{-u})\zeta), \quad s \le u \le t, \tag{5.2}$$

for $\zeta \in \mathbb{T}$ and $0 \le s < t < \infty$. Let len denote the arc length.

Theorem 5.1. *There exist constants $c_1, c_2 > 0$ such that, for $t - s \ge 2$ and every $\zeta \in \mathbb{T}$,*

$$c_1(t-s) < \operatorname{len} \Gamma_{s,t}(\zeta) < c_2(t-s). \tag{5.3}$$

Therefore the geometric parameter arc length is, within multiplicative constants, the same as the dynamical parameter t except if the sections are small.

Proof. Substituting $r = 1 - 2^{-u}$ we obtain from (5.2) that

$$\operatorname{len} \Gamma_{k,k+1} = \int_{1-2^{-k}}^{1-2^{-k-1}} |f'(r\zeta)| \mathrm{d}r \qquad \text{for } k \in \mathbb{N}_0.$$

Hence (5.3) follows from Proposition 4.1 by taking sums. □

5.2. Let R be the Riemann image surface of $\mathbb{D}$ under f. This is a branched covering surface of $\mathbb{C}$, but it is difficult to give a definition that is both rigorous and intuitive. Therefore we use R only to illustrate other well-defined concepts.

We need some facts about Bloch functions. Our function f satisfies

$$\|f\|_{\mathcal{B}} = \sup_{|z|<1} (1-|z|^2)|f'(z)| < 2.9, \tag{5.4}$$

since it can be proved that the number B_2 in the table in [Pom92, p. 190] is an upper bound for the Bloch norm. Let $z \in \mathbb{D}$. If $f'(z) = 0$ we set $d_f(z) = 0$. Otherwise $d_f(z)$ is, by definition, the largest radius for which f maps some domain $U(z) \subset \mathbb{D}$ one-to-one onto the disk

$$V(z) = \{w \in \mathbb{C} \colon |w - f(z)| < d_f(z)\}. \tag{5.5}$$

Thus $d_f(z)$ is the radius of the largest unbranched disk on R around $f(z) \in R$. Using the lower estimate $\frac{1}{4}\sqrt{3}$ for the Bloch constant we obtain [ACP74, (1.2)]

$$d_f(z) \le (1-|z|^2)|f'(z)| \le 5.9\sqrt{d_f(z)} \qquad \text{for } z \in \mathbb{D}. \tag{5.6}$$

A *critical point* of f is a point $z_0 \in \mathbb{D}$ such that $f'(z_0) = 0$. A *critical value* w_0 is a point $w_0 \in \mathbb{C}$ such that $w_0 = f(z_0)$ for some $z_0 \in \mathbb{D}$ with $f'(z_0) = 0$. The critical values are the projections onto $\mathbb{C}$ of the branch points of R.

Theorem 5.2. *(i) Every euclidean disk in $\mathbb{C}$ of radius 3 contains a critical value.*

(ii) There is a constant $c_0 > 0$ such that every hyperbolic disk in $\mathbb{D}$ of hyperbolic radius c_0 contains a critical point.

Proof. (i) Let D_1 be a disk of center w_1 and radius 3. Since $f(\mathbb{D}) = \mathbb{C}$ by Proposition 3.1(i), there exists $z_1 \in \mathbb{D}$ with $f(z_1) = w_1$. Let $V(z_1)$ be as in (5.5) and let $w_2 \in \partial V(z_1)$. Then $C = f^{-1}([w_1, w_2))$ is a half-open Jordan arc in $\mathbb{D}$. Now suppose that $\overline{C} \setminus C \subset \mathbb{T}$. Then $f(z)$ would have the finite limit w_2 as $|z| \to 1$, $z \in C$, which is impossible by Proposition 3.1(ii). It follows that $\overline{C} = C \cup \{z_2\}$ for some $z_2 \in \mathbb{D}$.

Now suppose that $f'(z_2) \ne 0$ for all such $z_2 \in \partial U(z_1)$. Then f would map a larger domain U^* onto a disk $\{|w - w_1| < d^*\}$ with $d^* > d_f(z_1)$. This would contradict the maximality of $d_f(z_1)$. Hence we have $f'(z_2) = 0$ for some $w_2 = f(z_2) \in \partial V(z_1)$. Since $d_f(z_1) < 3$ by (5.4) and (5.6), we see that the critical value w_2 lies in D_1.

(ii) Let c_1 and c_2 be the constants of Proposition 4.3. Let $z_1 \in \mathbb{D}$ be given. We proceed as in (i) using the same notation. We obtain a critical point z_2, but we do not yet know its hyperbolic distance from z_1. If $(1-|z_1|^2)|f'(z_1)| < c_1$, then let $z^* = z_1$. Otherwise let z^* be the first point on C with $(1-|z^*|^2)|f'(z^*)| = c_1$; if there is no such point we set $z^* = z_2$. Let C^* be the arc of C from z_1 to z^*.

Then we have $(1-|z|^2)|f'(z)| \ge c_1$ for $z \in C^*$. Since $f(C^*)$ is a segment, we obtain

$$\int_{C^*} \frac{|\mathrm{d}z|}{1-|z|^2} \le \frac{1}{c_1} \int_{C^*} |f'(z)||\mathrm{d}z| = \frac{1}{c_1}|f(z_1) - f(z^*)| \le \frac{d_f(z_1)}{c_1} < \frac{3}{c_1}\,.$$

Hence z_1 and z^* have a hyperbolic distance $< 3/c_1$. If $z^* = z_2$ we are finished. Otherwise, by Proposition 4.3, there is a critical point z_3 of hyperbolic distance $< c_2$ from z^*, and z_3 has a hyperbolic distance $< c_0 := c_2 + 3/c_1$ from z_1. $\square$

The proof of Theorem 5.2(ii) uses Proposition 4.3, which is based on technical results about lacunary power series. On the other hand, Theorem 5.2(i) is rather obvious when looking at the Riemann image surface R as we show now.

It follows from Proposition 3.1(ii) that R has no finite boundary points. Hence every maximal unbranched disk on R has a branch point on its periphery. Since f is Bloch function of norm < 3, every disk in $\mathbb{C}$ of radius 3 contains a critical value, the projection of a branch point on R.

Question 2. Are the critical values of f dense in the plane?

5.3. Finally we study the connection between the critical values of f and the loops of the paths $\Gamma(\zeta)$ for $\zeta \in \mathbb{T}$. See (5.2) for the definition of $\Gamma_{s,t}(\zeta)$.

First we indicate how to obtain, as is well known, for arbitrary holomorphic functions f and an essential class of critical values $f(z_0)$, closed Jordan curves of the form $\Gamma_{s,t}(\zeta)$ which contain $f(z_0)$ in its interior. Let $z_0 = r_0\zeta$ with $r_0 = 1-2^{-t_0}$. We assume that

$$f'(z_0) = 0, \quad f''(z_0) \neq 0, \quad \operatorname{Im}\Big[\zeta\,\frac{f'''(z_0)}{f''(z_0)} > 0\Big], \tag{5.7}$$

say. With $a_2 = \frac{1}{2}f''(z_0)$ and $a_3 = \frac{1}{6}f'''(z_0)$ we have

$$f(r\zeta) = f(z_0) + \zeta^2 a_2 (r-r_0)^2\big[1 + \zeta\,\frac{a_3}{a_2}(r-r_0) + O((r-r_0)^2)\big]$$

as $r \to r_0$. With $\alpha = 2\arg\zeta + \arg a_2$ and $\beta = \zeta\, a_3/a_2$ we obtain

$$\begin{aligned} \arg[f(r\zeta) - f(r_0\zeta)] &= \alpha + (r-r_0)\operatorname{Im}\beta + O((r-r_0)^2) \\ &= \alpha \pm |r-r_0|\operatorname{Im}\beta + O((r-r_0)^2) \end{aligned} \tag{5.8}$$

with the sign $-$ if $r < r_0$ and $+$ if $r > r_0$. Let $\delta > 0$ be small. Then

$$\Gamma_{t_0-\delta,t_0+\delta}(\zeta) = A^- \cup A^+, \quad A^- = \Gamma_{t_0-\delta,t_0}(\zeta), \quad A^+ = \Gamma_{t_0,t_0+\delta}(\zeta)$$

where A^+ and A^- meet at $f(z_0)$ in a cusp but with different curvatures because $\operatorname{Im}\beta > 0$. It follows from (5.8) that A^+ and A^- have the cusped region to their right.

Let $\vartheta > 0$ be small. Then, near z_0, the path $\Gamma(\zeta e^{i\vartheta})$ lies to the right of both A^+ and A^-. It follows that $\Gamma(\zeta e^{i\vartheta})$ must intersect itself near $f(z_0)$, as shown in Figure 1. Note that $\Gamma(\zeta e^{-i\vartheta})$ does not intersect itself near $f(z_0)$.

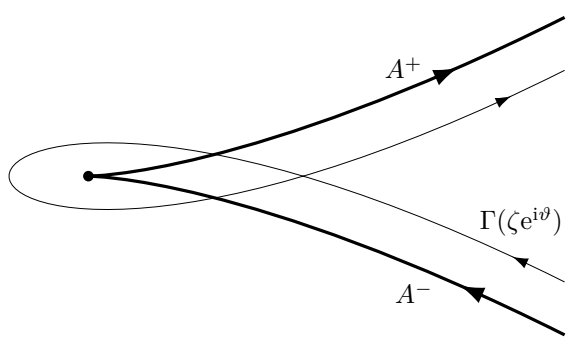

FIGURE 1. Closed Jordan curve enclosing a critical value

Now we prove for our function (1.1) a theorem which goes in the opposite direction. We shall use only $f(\mathbb{D}) = \mathbb{C}$ and Proposition 3.1(ii).

Theorem 5.3. *Let $\zeta \in \mathbb{T}$. If $\Gamma_{s,t}(\zeta)$ is a closed Jordan curve, then the inner domain of $\Gamma_{s,t}(\zeta)$ contains a critical value.*

Proof. Suppose that the inner domain G of $\Gamma_{s,t}(\zeta)$ contains no critical value. Let $B := [w_0, w_1] \subset G$, where $w_0 = f(z_0)$. Now let C be the arc of $f^{-1}(B)$ that begins at z_0. It follows from Proposition 3.1(ii) that $f(C) = B$. Since G contains no critical value we have $f'(z) \neq 0$ for $f(z) \in G$ and thus for $z \in C$.

Let φ be the inverse function of f near w_0 that satisfies $\varphi(w_0) = z_0$. By what we have just shown, we can continue φ analytically along B. Repeating this process we can continue φ along any polygonal curve in G. Since G is simply connected we conclude from the monodromy theorem that φ is well-defined and holomorphic in G. Furthermore φ is continuous in $\overline{G}$, $\overline{\varphi(G)} \subset \varphi(\overline{G}) \subset \mathbb{D}$ and $\partial(\varphi(G)) \subset \varphi(\partial G)$. But $\varphi(\partial G) = \varphi(\Gamma_{s,t}(\zeta))$ is a line segment in $\mathbb{D}$ which cannot contain the boundary of a set with a non-void interior; see Figure 2. $\square$

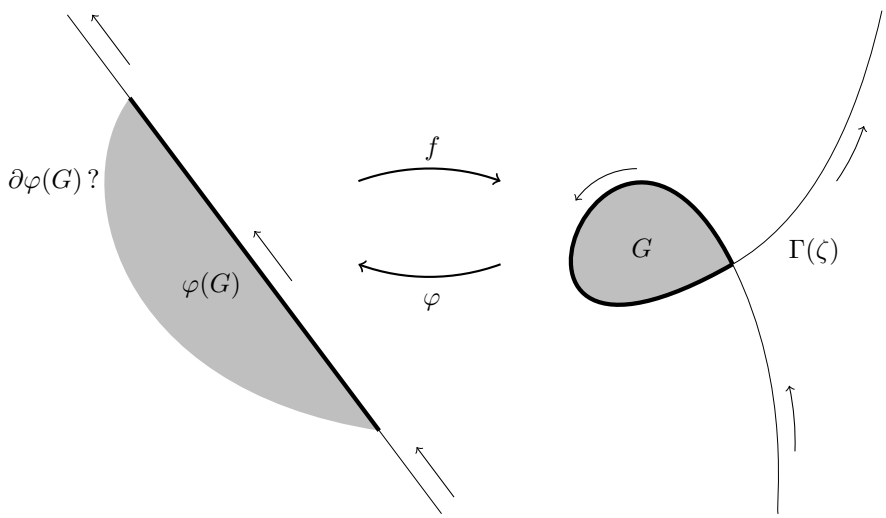

FIGURE 2. How to obtain a contradiction in Theorem 5.3

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