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A CONJECTURE OF GRAY AND THE p-TH POWER MAP ON $\Omega^2 S^{2np+1}$

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ABSTRACT. For $p \geq 2$, the p-th power map $[p]$ on $\Omega^2 S^{2np+1}$ is homotopic to a composite $\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$, where the fiber of ϕ_n is BW_n.

1. Introduction

Localizing spaces and maps at any prime $p \geq 2$, we prove a conjecture of Gray.

Theorem 1.1. *For $p \geq 2$, the p-th power map $\Omega^2 S^{2np+1} \xrightarrow{[p]} \Omega^2 S^{2np+1}$ is homotopic to a composite $\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$, where the fiber of ϕ_n is BW_n.*

Gray found a serious gap in Theriault's [14, 17] proof of Gray's conjecture for $p \geq 5$, which we fill in [11]. Our proof combines [12] with Gray's [7] spectacular construction of BW_n. Using ideas of Barratt, Boardman and Steer, we show (Theorem 2.6) that p times the unstable p^{th} James-Hopf invariant is a cup product. Our factorization Theorem 3.3 applies such cup product information to a map ρ similar to a map of Gray's, but defined in a more combinatorial way. This proves Theorem 4.1, the factorization $E^2\phi_n = [p]$. A result of Gray's (Theorem 4.5), which uses only the Serre spectral sequence, shows that the fiber of ϕ_n is BW_n, as ϕ_n kills the Hopf invariant $\Omega^2 S^{2n+1} \to \Omega^2 S^{2np+1}$ and is degree p on the bottom cell.

Theorem 1.1 relates to [4], where Cohen, Moore and Neisendorfer showed that $\pi_*(S^{2n+1})$ has exponent p^n for p odd, by showing that $[p]$ on $\Omega^2 S^{2n+1}$ factors, for some π_n, as the composite $\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$. Anick [1] solved a conjecture of [4], in 270 pages, constructing a fibration sequence $\Omega^2 S^{2n+1} \xrightarrow{\alpha_n} S^{2n-1} \to T_n \to \Omega S^{2n+1}$, for $p \geq 5$. Gray and Theriault [9] gave a shorter construction of Anick's fibrations, for $p \geq 3$, and showed that α_n was essentially π_n of [4]. Gray noted that Theorem 1.1 gives evidence that $\phi_n = \pi_{np}$, as $E^2\phi_n = [p] = E^2\pi_{np}$, and that this would imply that BW_n is the loop space ΩT_{np}. See [15] for 2-primary Anick fibration analogues. Theriault [16] constructed the odd-primary Anick fibrations not using [4] (thus re-proving the [4] exponent theorem), but using Gray's conjecture. As Theriault observes [17], with our proof of Theorem 1.1, [16] is now correct.

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2. Combinatorial James-Hopf invariants and cup products

We follow [19] and work in the category TOP_* of pointed compactly generated weak Hausdorff spaces. Whitehead largely follows Strøm [13], who shows that TOP_* satisfies all the axioms of a proper model category (cf. [5]), except the limit and colimit axioms, using the model structure of homotopy equivalences, Hurewicz fibrations, and NDR pairs. So cofibration ($\rightarrowtail$) will mean an NDR pair, and equivalence ($\sim$) will mean homotopy equivalence. Assume all spaces are well-pointed and have the homotopy type of a CW-complex.

We use the conventions of [12] and [2]. Suspension is given by smashing on the right with $S^1 = I/\{0,1\}$, so $\Sigma X = X \wedge S^1$. There are adjoint functors given by the evaluation map $\sigma\colon \Sigma\Omega B \to B$ and the suspension map $E\colon B \to \Omega\Sigma B$. Given maps $f\colon \Sigma A \to X$ and $g\colon A \to \Omega X$, we call their adjoints $f^\vee\colon A \to \Omega X$ and $g^\wedge\colon \Sigma A \to X$. A composite $A \xrightarrow{f} \Omega Y \xrightarrow{\Omega E} \Omega^2\Sigma Y$ is adjoint to the suspension of $f^\wedge\colon \Sigma A \to Y$. Recall the *shuffle*

$$\text{shuffle}\colon \Sigma^{n+m}(A\wedge B) = A\wedge B\wedge S^n\wedge S^m \xrightarrow{1_A\wedge T\wedge 1_{S^m}} A\wedge S^n\wedge B\wedge S^m = \Sigma^n A\wedge\Sigma^m B$$

and the permutation group Σ_r action on $X^{[r]}$. Given $f\colon A \to X$ and $g\colon A \to Y$, define the *cup product* $f\cdot g\colon A \to X\wedge Y$ to be the composite $A \xrightarrow{\Delta} A\wedge A \xrightarrow{f\wedge g} X\wedge Y$. The cup product is compatible with permutations: given $\rho\in\Sigma_r$ and maps $f_i\colon A\to X$, we have

$$\rho(f_1\cdot\dots\cdot f_r) = f_{\rho^{-1}(1)}\cdot\dots\cdot f_{\rho^{-1}(r)}\colon A \to X^{[r]}. \tag{2.1}$$

Let $\bar k = \{1,\dots,k\}$, and call $\binom{\bar k}{r}$ the set of subsets $S\subset \bar k$ of size r. For each S, let $(s_1,\dots,s_r)$ be its ordered sequence of elements. Order $\binom{\bar k}{r}$ using the left-lexicographical (left-lex) order of the $(s_1,\dots,s_r)$. Given $S\in\binom{\bar k}{r}$, define the map $\pi_S\colon X^k \to X^{[r]}$ by $\pi_S(x_1,\dots,x_k) = x_{s_1}\wedge\dots\wedge x_{s_r}$, so $\pi_S = \pi_{s_1}\cdot\dots\cdot\pi_{s_r}$ is an iterated cup product. Then

Lemma 2.1. *Let X be a co-H space. Then for any $x\in\bar k$ and any subset $T\in\binom{\bar k}{r-1}$, the cup product $\pi_x\cdot\pi_T\colon X^k\to X^{[r]}$ is nullhomotopic if $x\in T$.*

Proof. Write $\pi_T = \pi_{t_1}\cdot\pi_{T'}$, where T' is the complement of the smallest element $t_1\in T$. By (2.1), it suffices to consider the case $x = t_1$. Then $\pi_x\cdot\pi_T = (\pi_x\cdot\pi_x)\cdot\pi_{T'}$. But $\pi_x\cdot\pi_x$, the composite $X^k \xrightarrow{\pi_x} X \xrightarrow{\Delta} X\wedge X$, is nullhomotopic, since X is a co-H space. □

The James construction $J(X)$ (cf. [19, §VII(2)], [2, §3]) has subspaces $J_k(X)$. The identification map $\iota_k\colon X^k \to J_k(X)$ induces a cofibration $\bar\iota_k\colon X^{[k]} \rightarrowtail J(X)/J_{k-1}(X)$ and a homeomorphism $\bar\iota_k\colon X^{[k]} \xrightarrow{\cong} J_k(X)/J_{k-1}(X)$ defined by $\bar\iota_k(x_1\wedge\dots\wedge x_k) = [x_1,\dots,x_k]$, which gives a cofibration sequence $J_{k-1}(X)\rightarrowtail J_k(X) \xrightarrow{\partial} X^{[k]}$. For X connected, there is an equivalence $J(X)\xrightarrow{\sim}\Omega\Sigma X$. We define the r^{th} combinatorial James-Hopf invariant $j_r\colon J(X)\to J(X^{[r]})$ using the left-lex order, and call H_r the composite $J(X)\xrightarrow{j_r} J(X^{[r]}) \xrightarrow{\sim} \Omega\Sigma X^{[r]}$. $H_1\colon J(X)\xrightarrow{\sim}\Omega\Sigma X$ is the equivalence, as $j_1\colon J(X)\to J(X)$ is the identity. $H_k\colon J(X)\to\Omega\Sigma X^{[k]}$ factors through a map $\overline{H_k}\colon J(X)/J_{k-1}(X)\to\Omega\Sigma X^{[k]}$, by construction, for X connected.

The composite $\Sigma X^k \xrightarrow{\Sigma\iota_k} \Sigma J(X) \xrightarrow{H_r^\wedge} \Sigma X^{[r]}$ satisfies the crucial property ($\prec$ means the left-lex order)

$$H_r^\wedge \Sigma\iota_k = \sum\nolimits^{\prec}_{S\in\binom{\bar{k}}{r}} \Sigma\pi_S \in [\Sigma X^k, \Sigma X^{[r]}]. \tag{2.2}$$

Note that $H_1\Sigma\iota_k = \sum_{x\in\bar{k}} \pi_x \in [\Sigma X^k, \Sigma X]$. By the proof of [2, Lem. 3.7], we have

Lemma 2.2. *Given two maps $f, g\colon J(X) \to \Omega Y$, suppose for each k that the composites $f\cdot\iota_k, g\cdot\iota_k\colon X^k \to \Omega Y$ are homotopic. Then f and g are homotopic.*

Define $\Omega\Sigma X \wedge \Omega\Sigma Y \xrightarrow{\otimes} \Omega\Sigma(X\wedge Y)$ to be the adjoint of the composite

$$\Sigma\Omega\Sigma X \wedge \Omega\Sigma Y \xrightarrow{\sigma\wedge 1} \Sigma X \wedge \Omega\Sigma Y \xrightarrow{1\wedge\sigma} \Sigma X \wedge Y.$$

Given $f\colon \Sigma A \to \Sigma X$ and $g\colon \Sigma A \to \Sigma Y$, define $f\sharp g\colon \Sigma A \to \Sigma X \wedge Y$ as the adjoint of the composite $A \xrightarrow{\Delta} A\wedge A \xrightarrow{f^\vee\wedge g^\vee} \Omega\Sigma X\wedge\Omega\Sigma Y \xrightarrow{\otimes} \Omega\Sigma X\wedge Y$. Then $f\sharp g$ is the composite

$$f\sharp g\colon \Sigma A \xrightarrow{\Delta} \Sigma A\wedge A \xrightarrow{f\wedge 1} \Sigma X\wedge A \xrightarrow{1\wedge g} \Sigma X \wedge Y. \tag{2.3}$$

Note that $f\sharp g$ is a desuspension of what Boardman and Steer [2] call the cup product $f\cdot g$.

Lemma 2.3. *Given maps $f_i\colon A\to X$ and $g_j\colon A\to Y$, let $f = \sum_i \Sigma f_i\colon \Sigma A\to\Sigma X$ and $g = \sum_j \Sigma g_j\colon \Sigma A \to \Sigma Y$. Then $f\sharp g = \sum\nolimits^{\prec}_{i,j} \Sigma f_i\cdot g_j \in [\Sigma A, \Sigma X\wedge Y]$.*

Proof. Composition is left-distributive, and composition is right-distributive if the right map is a suspension. So $(1\wedge g)(f\wedge 1) = \sum_i\big(\sum_j 1\wedge\Sigma g_j\big)\Sigma f_i\wedge 1 = \sum\nolimits^{\prec}_{i,j} \Sigma f_i \wedge g_j$. □

Let $\theta_i = (12\ldots i)$ be the cyclic permutation of length i. Then

Proposition 2.4. *For any co-H space X and any r, the diagram homotopy commutes:*

$$\begin{tikzcd}
J(X) \arrow[r, "H_r"] \arrow[d, "\Delta"'] & \Omega\Sigma X^{[r]} \arrow[rr, "1+\Omega\theta_2+\cdots+\Omega\theta_r"] & & \Omega\Sigma X^{[r]} \arrow[d, "\Omega E"] \\
J(X)^{[2]} \arrow[r, "H_1\wedge H_{r-1}"] & \Omega\Sigma X\wedge\Omega\Sigma X^{[r-1]} \arrow[r, "\otimes"] & \Omega\Sigma X^{[r]} \arrow[r, "\Omega E"] & \Omega^2\Sigma^2 X^{[r]}
\end{tikzcd}$$

Proof. The lower and upper composites of the diagram composed with $\iota_k\colon X^k\to J(X)$ are adjoint to maps which we will call $L, U \in [\Sigma^2 X^k, \Sigma^2 X^{[r]}]$. Note this group is abelian. By Lemma 2.2, it suffices to show that $L = U$. L is the suspension of the composite

$$\Sigma X^k \xrightarrow{\Sigma\iota_k} \Sigma J(X) \xrightarrow{H_1^\wedge\sharp H_{r-1}^\wedge} \Sigma X^{[r]}.$$

By naturality of the $\sharp$ product, L is the suspension of $(H_1^\wedge\Sigma\iota_k)\sharp(H_{r-1}^\wedge\Sigma\iota_k)$. Then

$$L = \sum\nolimits_{x\in\bar{k},\, T\in\binom{\bar{k}}{r-1}} \Sigma^2\pi_x\cdot\pi_T \in [\Sigma^2 X^k, \Sigma^2 X^{[r]}], \tag{2.4}$$

by (2.2) and Lemma 2.3. U is the suspension of the sum $\sum_{i=1}^r \theta_i H_r^\wedge\Sigma\iota_k$. By (2.2), this sum equals $\sum_{i=1}^r\sum_{S\in\binom{\bar{k}}{r}} \Sigma\theta_i\pi_S$. But $\theta_i\pi_S = \pi_x\cdot\pi_T$, where x is the i^{th} element of S and $T = S - \{x\}$, by (2.1). We have a bijection $\bar{r} \times \binom{\bar{k}}{r} \cong \{(x,T) \in \bar{k} \times \binom{\bar{k}}{r-1} : x \notin T\}$ given by $(i,S) \mapsto (s_i, S - \{s_i\})$. Hence, after suspending,

$$U = \sum\nolimits_{x\in\bar{k},\, T\in\binom{\bar{k}}{r-1}, x\notin T} \Sigma^2\pi_x \cdot \pi_T \in [\Sigma^2 X^k, \Sigma^2 X^{[r]}].$$

By Lemma 2.1, the condition $x \notin T$ is unnecessary, as the terms with $x \in T$ are nullhomotopic. By comparing with (2.4), we have $L = U \in [\Sigma^2 X^k, \Sigma^2 X^{[r]}]$. ☐

We specialize soon to $X = S^{2n}$. Let $\Phi\colon \Omega P \wedge Q \to \Omega(P \wedge Q)$ be the adjoint of $\Sigma\Omega P \wedge Q \xrightarrow{\sigma\wedge 1} P \wedge Q$, so $\Phi(\alpha \wedge q)(t) = \alpha(t) \wedge q$, for spaces P and Q. Then

Lemma 2.5. *For e even, the map $\Phi\colon \Omega U \wedge S^e \to \Omega(U \wedge S^e)$ is homotopic to the composite*

$$\Omega U \wedge S^e \xrightarrow{\Sigma^{e-1}\sigma} U \wedge S^{e-1} \xrightarrow{E} \Omega(U \wedge S^e), \tag{2.5}$$

for any space U. Take also a space V and a map $f\colon U \to \Omega\Sigma V$. Then the composite $z\colon \Sigma^e U = U \wedge S^e \xrightarrow{f\wedge E} \Omega\Sigma V \wedge \Omega S^{e+1} \xrightarrow{\otimes} \Omega(V \wedge S^{e+1}) = \Omega(\Sigma^{e+1}V)$ is homotopic to the composite $\Sigma^e U \xrightarrow{\Sigma^{e-1} f^\wedge} \Sigma^e V \xrightarrow{E} \Omega(\Sigma^{e+1}V)$.

Proof. The shuffle $\tau\colon S^e \wedge S^1 \to S^1 \wedge S^e$ is homotopic to the identity on S^{e+1}, as it is defined by a permutation with even sign, since e is even. The adjoint of Φ is the composite

$$\Omega U \wedge S^{e+1} = \Omega U \wedge S^e \wedge S^1 \xrightarrow{1_{\Omega U}\wedge\tau} \Omega U \wedge S^1 \wedge S^e \xrightarrow{\sigma\wedge 1_{S^e}} U \wedge S^e.$$

Thus $\Phi^\wedge\colon \Omega U \wedge S^{e+1} \to U \wedge S^e$ is homotopic to $\sigma \wedge 1_{S^e} = \Sigma^e\sigma$, since τ is homotopic to the identity. As the adjoint of (2.5) is $\Sigma(\Sigma^{e-1}\sigma) = \Sigma^e\sigma$ as well, this proves the first part.

The second part is similar. By the smash product version of the cup product formula (2.3), the adjoint of the map $z\colon \Sigma^e U \to \Omega(\Sigma^{e+1}V)$ is the composite

$$z^\wedge\colon U \wedge S^e \wedge S^1 \xrightarrow{1\wedge\tau} U \wedge S^1 \wedge S^e \xrightarrow{f^\wedge\wedge 1_{S^e}} V \wedge S^1 \wedge S^e \xrightarrow{1\wedge\tau^{-1}} V \wedge S^e \wedge S^1,$$

since the adjoint of $E\colon S^e \to \Omega S^{e+1}$ is the identity map on $S^e \wedge S^1 = S^{e+1}$. Since τ is homotopic to the identity, $z^\wedge\colon \Sigma^{e+1}U \to \Sigma^{e+1}V$ is homotopic to $f^\wedge \wedge 1_{S^e} = \Sigma^e f^\wedge$. But $\Sigma^e f^\wedge$ is also the adjoint of $\Sigma^e U \xrightarrow{\Sigma^{e-1} f^\wedge} \Sigma^e V \xrightarrow{E} \Omega(\Sigma^{e+1}V)$. ☐

The composite $J(X) \xrightarrow{\Delta} J(X) \wedge J(X)/J_{r-2}(X) \xrightarrow{H_1\wedge\overline{H_{r-1}}} \Omega\Sigma X \wedge \Omega\Sigma X^{[r-1]}$ is homotopic to the composite $(H_1 \wedge H_{r-1})\cdot\Delta$ of Proposition 2.4, and the suspension map E is homotopic to the composite $X^{[k]} \xrightarrow{\bar{\iota}_k} J(X)/J_{k-1}(X) \xrightarrow{\overline{H_k}} \Omega\Sigma X^{[k]}$.

Now we specialize Proposition 2.4 to $X = S^{2n}$. Let $m = 2n(p-1)$. Then

Theorem 2.6. *Localize at prime $p \geq 2$. The diagram is homotopy commutative:*

$$\begin{tikzcd}
J(S^{2n}) \arrow[rr, "H_p"] \arrow[d, "\Delta"'] & & \Omega S^{2np+1} \arrow[d, "{[p]}"] \\
J(S^{2n}) \wedge J(S^{2n})/J_{p-2}(S^{2n}) \arrow[r, "H_1\wedge\overline{H_{p-1}}"] & \Omega S^{2n+1} \wedge \Omega S^{m+1} \arrow[r, "\otimes"] & \Omega S^{2np+1} \\
J(S^{2n}) \wedge S^m \arrow[u, "\mathrm{id}\wedge\bar{\iota}_{p-1}"] \arrow[rr, "\Sigma^{m-1}H_1^\wedge"] & & S^{2np} \arrow[u, "E"']
\end{tikzcd}$$

Proof. The bottom rectangle homotopy commutes by Lemma 2.5 and the above remarks. For $p = 2$, the top rectangle homotopy commutes [12, Thm. 2.3]. For $p > 2$, we can desuspend Proposition 2.4, because ΩS^{2np+1} is a retract of $\Omega^2 S^{2np+2}$, since S^{2np+1} is an H-space. Then use the above factorization through $(H_1 \wedge \overline{H_{r-1}}) \cdot \Delta$. $\square$

3. Cup products and a factorization result for a map ρ

For a co-H-space X, we construct a map $\rho\colon \Omega(J(X), J_{p-1}(X)) \to \Omega J(X)^+ \wedge X^{[p-1]}$ analogous to Gray's clutching construction collapse map [7] and prove a factorization result involving ρ (Theorem 3.3) that we combine in §4 with Theorem 2.6.

Given a cofibration $K \rightarrowtail L$, call $\Omega(L, K) = \{\lambda \in PL : \lambda(1) \in K\}$ its homotopy fiber. The natural map $v\colon \Omega(L, K) \to \Omega(L/K)$, adjoint to the natural map from the homotopy fiber to the cofiber, sends a path λ to the loop $[\lambda]$.

Let $p_1, p_2\colon X \to X$ be the factors of the comultiplication map $X \to X \vee X$ of the co-H-space X. The self-maps $J(p_1)$ and $J(p_2)$ of $J(X)$ are homotopic to the identity. Then

Lemma 3.1. *Let $k = a + b + 1$ for $a, b \geq 0$. The map $J(p_1) \times J(p_2)$ defines a factorization*

$$J_k(X) \to J_a(X) \times J_k(X) \cup J_k(X) \times J_b(X)$$

up to homotopy of the diagonal map on $J_k(X)$. Furthermore $J(p_1) \times J(p_2)$ induces a map

$$\rho\colon \Omega(J(X), J_k(X)) \to \Omega(J(X)/J_a(X))^+ \wedge J_k(X)/J_b(X),$$

defined explicitly by the formula $\rho(\lambda) = J(p_1)\lambda^+ \wedge J(p_2)\lambda(1)$, and also the map $\tilde{\Delta}$ below which makes the following diagram homotopy commutative:

$$\begin{tikzcd}
J(X) \arrow[r, "\bar{\Delta}"] \arrow[d] & J(X) \wedge J(X) \arrow[d] \\
J(X)/J_k(X) \arrow[r, "\tilde{\Delta}"] & J(X)/J_a(X) \wedge J(X)/J_b(X)
\end{tikzcd}$$

Proof. The map $J(p_1) \times J(p_2)$ is homotopic to the diagonal map, as is its restriction $J_k(X) \to J_k(X) \times J_k(X)$. Any $\alpha \in J_k(X)$ is represented by some $(y_1, \ldots, y_k) \in X^k$. Let N_1 be the number of indices i for which $p_1(y_i) = *$, and N_2 be the number of indices i for which $p_2(y_i) = *$. For any index i, either $p_1(y_i) = *$ or $p_2(y_i) = *$, so $N_1 + N_2 \geq k$. Thus either $N_1 > b$ or $N_2 > a$, since $k > a + b$. Use $k = a + b + 1$ to rewrite this as either $N_1 \geq k - a$ or $N_2 \geq k - b$. Thus either $J(p_1)(\alpha) \in J_a(X)$ or $J(p_2)(\alpha) \in J_b(X)$.

This proves the first claim, which then implies the second and third claims. $\square$

Let $E = J(X)$, $M = J_{p-1}(X)$, and $Y = J_{p-2}(X)$. Lemma 3.1 in the case $k = p - 1$ and $a = 0$ gives the maps $\tilde{\Delta}\colon E/M \to E \wedge E/Y$ and $\rho\colon \Omega(E, M) \to \Omega E^+ \wedge X^{[p-1]}$, defined explicitly by $\rho(\lambda) = J(p_1)\lambda^+ \wedge \partial J(p_2)\lambda(1)$. Then:

Proposition 3.2. *The composite* $\Omega(E) \to \Omega(E, M) \xrightarrow{\rho} \Omega E^+ \wedge X^{[p-1]}$ *is trivial, and the following diagram is homotopy commutative:*

$$\begin{tikzcd}
\Omega(E,M) \arrow[rrr, "v"] \arrow[d, "\rho"'] & & & \Omega(E/M) \arrow[d, "\Omega\tilde{\Delta}"] \\
\Omega E^+ \wedge X^{[p-1]} \arrow[r] & \Omega E \wedge X^{[p-1]} \arrow[r, "\mathrm{id} \wedge \bar{\iota}_{p-1}"] & \Omega E \wedge E/Y \arrow[r, "\Phi"] & \Omega(E \wedge E/Y)
\end{tikzcd}$$

Proof. The first assertion is immediate from the definition of ρ, as $\lambda(1) = *$ if $\lambda \in \Omega E$.

The composite $M \rightarrowtail E \to E/Y$ factors as $M \xrightarrow{\partial} X^{[p-1]} \overset{\bar{\iota}_{p-1}}{\rightarrowtail} E/Y$. The straight line homotopy of $[\lambda(s)] \in E/Y$ to $[\lambda(1)] = \bar{\iota}_{p-1}\partial\lambda(1) \in M/Y \cong X^{[p-1]}$ gives a homotopy of the diagram, as $(\Omega\tilde{\Delta} \cdot v)(\lambda)(s) = J(p_1)\lambda(s) \wedge [J(p_2)\lambda(s)]$, for any $\lambda \in \Omega(E, M)$, and $(\Phi \cdot (\mathrm{id} \wedge \bar{\iota}_{p-1}) \cdot \rho)(\lambda)(s) = J(p_1)\lambda(s) \wedge \bar{\iota}_{p-1}\partial J(p_2)\lambda(1)$. $\square$

Let $p\colon E \to B$ be a map with $p(M) = *$. Let F be the homotopy fiber of p, and suppose that the natural lift of $M \rightarrowtail E$ is an equivalence $M \xrightarrow{\sim} F$. Call $\bar{p}\colon E/M \to B$ the map induced by p. Then the following composite is an equivalence:

$$\Omega(E, M) \xrightarrow{v} \Omega(E/M) \xrightarrow{\Omega\bar{p}} \Omega B. \tag{3.1}$$

Call $\rho'\colon \Omega B \to \Omega E^+ \wedge X^{[k]}$ the composite of ρ with the inverse of equivalence (3.1). We prove an analogue of [12, Thm. 4.1].

Theorem 3.3. *Let $Z = \Omega Z_0$ be a loop space, and assume that $\Sigma E \to \Sigma E/M$ has a right homotopy inverse. Take maps $f\colon B \to Z$ and $\alpha\colon E \wedge E/Y \to Z$ making the diagram*

$$\begin{tikzcd}
E \arrow[r, "p"] \arrow[d, "\Delta"'] & B \arrow[d, "f"] \\
E \wedge E/Y \arrow[r, "\alpha"] & Z
\end{tikzcd} \tag{3.2}$$

homotopy commute. Then $\Omega f\colon \Omega B \to \Omega Z$ is homotopic to the composite

$$\Omega B \xrightarrow{\rho'} \Omega E^+ \wedge X^{[p-1]} \xrightarrow{\Phi} \Omega(E \wedge X^{[p-1]}) \xrightarrow{\Omega(\mathrm{id} \wedge \bar{\iota}_{p-1})} \Omega(E \wedge E/Y) \xrightarrow{\Omega\alpha} \Omega Z.$$

If $X = S^{2n}$, let $m = 2n(p-1)$. Then $\Omega f\colon \Omega B \to \Omega Z$ is homotopic to the composite

$$\Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+ \xrightarrow{\Sigma^{m-1}\sigma} \Sigma^{m-1} E \xrightarrow{\zeta^\vee} \Omega Z, \tag{3.3}$$

where $\zeta^\vee$ is the adjoint of the composite $\zeta\colon \Sigma^m E \xrightarrow{\mathrm{id} \wedge \bar{\iota}_{p-1}} E \wedge E/Y \xrightarrow{\alpha} Z$.

Proof. The triangle of the following diagram homotopy commutes by Lemma 3.1:

$$\begin{tikzcd}
E \arrow[r] \arrow[dr, "\Delta"'] & E/M \arrow[r, "\bar{p}"] \arrow[d, "\tilde{\Delta}"] & B \arrow[d, "f"] \\
& E \wedge E/Y \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

By (3.2), the outer polygon homotopy commutes. We see that the square homotopy commutes by adjointing with $Z = \Omega Z_0$ and using the right homotopy inverse.

By looping the right square and using Proposition 3.2, we see that the diagram

$$\begin{tikzcd}
\Omega(E,M) \arrow[rr, "v"] \arrow[d, "\rho"] & & \Omega(E/M) \arrow[r, "\Omega\bar{p}"] \arrow[d, "\Omega\tilde{\Delta}"] & \Omega B \arrow[d, "\Omega f"] \\
\Omega E^+ \wedge X^{[p-1]} \arrow[r, "\mathrm{id}\wedge\bar{\iota}_{p-1}"] & \Omega E \wedge E/Y \arrow[r, "\Phi"] & \Omega(E\wedge E/Y) \arrow[r, "\Omega\alpha"] & \Omega Z
\end{tikzcd}$$

homotopy commutes. Now invert the top horizontal composite using the equivalence (3.1).

In the case $X = S^{2n}$, we showed that $\Omega f\colon \Omega B \to \Omega Z$ is homotopic to the composite

$$\Omega B \xrightarrow{\rho'} \Omega E^+ \wedge S^m \to \Omega E \wedge S^m \xrightarrow{\Phi} \Omega(E\wedge S^m) \xrightarrow{\Omega(\mathrm{id}\wedge\bar{\iota}_{p-1})} \Omega(E\wedge E/Y) \xrightarrow{\Omega a} \Omega Z.$$

Lemma 2.5 and the fact that $\Omega(\zeta)\cdot E = \zeta^\vee\colon \Sigma^{m-1}E \to \Omega Z$ establishes (3.3). $\square$

An obvious result follows from the first part of Proposition 3.2, as the composite of equivalence (3.1) with $\Omega E \to \Omega(E,M)$ is $\Omega p\colon \Omega E \to \Omega B$.

Lemma 3.4. *The composite $\Omega E \xrightarrow{\Omega p} \Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+$ is nullhomotopic.*

4. Proof of Gray's conjecture

Specialize to $X = S^{2n}$, so $E = J(S^{2n})$. Let $B = \Omega S^{2np+1}$ and let $p\colon E \to B$ be the p^{th} James-Hopf invariant H_p. The EHP sequences of James [10] for $p = 2$ and Toda [18] for odd primes show that the hypothesis of §3 is satisfied, so we have the equivalence $\Omega(E,M) \xrightarrow{\sim} \Omega B$ of (3.1), where $M = J_{p-1}(S^{2n})$ and $Y = J_{p-2}(S^{2n})$. Thus we have the map $\rho'\colon \Omega B \to \Omega E^+ \wedge X^{[k]} = \Sigma^m \Omega E^+$. Let $Z = B$ and $f = [p]\colon B \to B$, the p-th power map on B. The loop of $[p]$ on B is $[p]$ on ΩB. Define $\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$ as the composite $\Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+ \xrightarrow{\mathfrak{s}} S^{2np-1}$, where $\mathfrak{s}$ is the composite

$$\mathfrak{s}\colon \Sigma^m \Omega E^+ \to \Sigma^m \Omega E \xrightarrow{\Sigma^{m-1}\sigma} \Sigma^{m-1} E \xrightarrow{\Sigma^{m-2}H_1^\wedge} \Sigma^{m-1} X = S^{2np-1}, \tag{4.1}$$

where as usual $m = 2n(p-1)$. Now we state our main result.

Theorem 4.1. $E^2\phi_n = [p] \in [\Omega^2 S^{2np+1}, \Omega^2 S^{2np+1}]$.

Proof. $\Sigma E \to \Sigma E/M$ has a right homotopy inverse by the James splitting, so we can apply Theorem 3.3 and Theorem 2.6. Thus $[p]\colon \Omega B \to \Omega B$ is the homotopy class of the composite $\Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+ \xrightarrow{\Sigma^{m-1}\sigma} \Sigma^{m-1} E \xrightarrow{\zeta^\vee} \Omega B$, using the adjoint of the composite $\zeta\colon \Sigma^m E \xrightarrow{\mathrm{id}\wedge\bar{\iota}_{p-1}} E\wedge E/Y \xrightarrow{\alpha} B$. By Theorem 2.6, α is the composite

$$\alpha\colon J(S^{2n}) \wedge J(S^{2n})/J_{p-2}(S^{2n}) \xrightarrow{H_1\wedge\overline{H_{p-1}}} \Omega S^{2n+1} \wedge \Omega S^{m+1} \xrightarrow{\otimes} \Omega S^{2np+1}$$

and ζ is the composite $\Sigma^m J(X) \xrightarrow{\Sigma^{m-1}H_1^\wedge} \Sigma^m X \xrightarrow{E} \Omega(\Sigma^{m+1}X)$. So $\zeta^\vee$ is the composite $\Sigma^{m-1}J(X) \xrightarrow{\Sigma^{m-2}H_1^\wedge} \Sigma^{m-1}X \xrightarrow{E^2} \Omega^2(\Sigma^{m+1}X)$. Thus $[p]$ is the composite

$$\Omega B \xrightarrow{\rho'} \Sigma^m \Omega E^+ \xrightarrow{\mathfrak{s}} \Sigma^{m-1} X \xrightarrow{E^2} \Omega^2(\Sigma^{m+1}X)$$

or $\Omega B \xrightarrow{\phi_n} S^{2np-1}X \xrightarrow{E^2} \Omega^2 S^{2np+1}$. Thus $E^2\phi_n = [p] \in [\Omega^2 S^{2np+1}, \Omega^2 S^{2np+1}]$. $\square$

We show the homotopy fiber of ϕ_n is BW_n using an argument essentially due to Gray and largely contained in [8], an earlier version of [7]. Lemma 3.4 immediately implies

Lemma 4.2. *The composite* $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$ *is nullhomotopic.*

The next result is due to James and Toda [10, 18].

Lemma 4.3. *Let* $h\colon \Omega J_{p-1}(S^{2n}) \to \Omega S^{2np-1}$ *be a map which gives an isomorphism in* $(2np-2)$*-dimensional integral cohomology. Then the homotopy fiber of* h *is* S^{2n-1}*.*

Remark 4.4. Gray's proof of Lemma 4.3 for odd primes [6, Thm. 1(a)] using the $\mathbb{Z}$ cohomology Serre ss also works for $p = 2$. Letting F be the homotopy fiber of h, the cohomology of the total space $\Omega J_{p-1}(S^{2n})$ is the tensor product of the cohomologies of the base and fiber. The other 2-primary EHP sequence $S^{2n} \to \Omega S^{2n+1} \to \Omega S^{4n+1}$ has a harder proof [3, Thm. 3.1], using the binomial coefficient identity $\binom{2n}{2k} \equiv \binom{n}{k} \pmod{\mathbb{Z}/2}$.

Note that $\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$ is degree p on the bottom cell by Theorem 4.1. We prove

Theorem 4.5. *A map* $\Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$ *has homotopy fiber* BW_n *if* f *is degree* p *on the bottom cell and the composite* $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$ *is nullhomotopic.*

Proof. Looping the EHP fibration [10, 18] gives the homotopy fibration sequence

$$\Omega J_{p-1}(S^{2n}) \xrightarrow{\Omega E} \Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1}. \tag{4.2}$$

Take the fibration sequence $\Omega S^{2np-1} \xrightarrow{\partial} L \overset{\pi}{\twoheadrightarrow} \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$. A null-homotopy of the composite $f \circ \Omega H$ gives a lift $\nu\colon \Omega^2 S^{2n+1} \to L$ of H and a map of homotopy fibers $h\colon \Omega J_{p-1}(S^{2n}) \to \Omega S^{2np-1}$, yielding the homotopy commutative diagram

$$\begin{tikzcd}
\Omega S^{2np-1} \arrow[r, "\partial"] & L \arrow[r, two heads, "\pi"] & \Omega^2 S^{2np+1} \arrow[r, "f"] & S^{2np-1}, \\
\Omega J_{p-1}(S^{2n}) \arrow[u, "h"] \arrow[r, "\Omega E"] & \Omega^2 S^{2n+1} \arrow[u, "\nu"] \arrow[ur, "\Omega H"'] & & \\
& \mathcal{F} \arrow[u] \arrow[ul] & &
\end{tikzcd}$$

where $\mathcal{F}$ is the homotopy fiber of both h and ν. The $\mathbb{Z}$ cohomology Serre ss of the path fibration proves $H^{2np-1}(\Omega^2 S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}/p$. The $\mathbb{Z}/p$ cohomology Serre ss proves $H^*\Omega^2 S^{2n+1}$ is $\mathbb{Z}/p$ in dimensions $2np-2$ and $2np-1$, connected by a Bockstein by the integral result. Since $H^{2np-1}(\Omega J_{p-1}(S^{2n}, \mathbb{Z}/p) = 0$, the $\mathbb{Z}/p$ cohomology Serre exact sequence of (4.2) shows that $(\Omega H)^*$ is an isomorphism in dimension $2np-1$ and $(\Omega E)^*$ is an isomorphism in dimension $2np-2$. The $\mathbb{Z}/p$ cohomology of L is also $\mathbb{Z}/p$ in dimensions $2np-2$ and $2np-1$, connected by a Bockstein. By naturality of the Bockstein, ν^* is an isomorphism in dimension $2np-2$, so h^* is an isomorphism in dimension $2np-2$. Thus h is degree r on the bottom cell, where r is prime to p, so the self-map $r\iota$ of S^{2np-1} is an equivalence. Composing h with the homotopy inverse of $\Omega(r\iota)$ and applying Lemma 4.3 show that the homotopy fiber of h is S^{2n-1}. Therefore $\mathcal{F} = S^{2n-1}$, and we have a homotopy fibration sequence $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} L$. Thus L is a delooping of W_n, the fiber of E^2, so L is BW_n. □

Theorem 1.1 now follows from Theorem 4.1, Lemma 4.2 and Theorem 4.5.

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